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El problema de Grünbaum–Hadwiger–Ramos para asignaciones de masa

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T E S I S

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Contents

Resumen

En la presente tesis aplicamos métodos de topología algebraica a un problema de geometría discreta. Para ser más precisos, el problema en cuestión es un problema de particiones de medidas, mientras que los métodos utilizados incluyen sucesiones espectrales y teoría de índice cohomológica ideal-valuada.

En el capítulo 1 introducimos la terminología necesaria y el problema clásico de particiones de medidas conocido como el *problema de Grünbaum–Hadwiger–Ramos para masas.* Presentamos también algo de historia referente a dicho problema, así como los resultados más relevantes obtenidos en los ultimos años. Finalmente, motivados por el reciente trabajo de Patrick Schnider [1], presentamos una extensión del problema clásico a las llamadas *asignaciones de masa.*

En los siguientes tres capítulos hablamos de las herramientas que vamos a utilizar para probar la extensión propuesta del problema de Grünbaum–Hadwiger–Ramos. En el capítulo 2 describimos el *método de la función de prueba*, el cual provee un puente entre la geometría y la topología. La idea principal de este método es reescribir nuestro problema geométrico en términos topológicos, para luego resolverlo usando técnicas de topología algebraica. Lo que sigue es introducir dichas técnicas. En el capítulo 3 presentamos una breve introducción de la *sucesión espectral de Leray–Serre* asociada a una fibración. Esta sucesión espectral, entre otras cosas, nos permite obtener información del anillo de cohomología del espacio total de la fibración, así como del correspondiente homomorfismo inducido en cohomología. Luego, en el capítulo 4, introducimos la *teoría de índice de Fadell–Husseini* y presentamos algunas de sus propiedades más importantes. Es en esta parte donde el uso de la sucesión espectral del Leray–Serre se vuelve esencial para los cálculos. La teoría de índice que presentamos va a ser el ingrediente clave para resolver nuestro problema topológico.

Finalmente, usando los métodos y técnicas introducidas en los capítulos anteriores, en el capítulo 5 probamos nuestro resultado principal, el *problema de Grünbaum–Hadwiger–Ramos para asignaciones de masa.* Este es un trabajo en conjunto con Pavle V. M. Blagojević, Michael C. Crabb and Aleksandra S. Dimitrijević Blagojević.

Palabras clave: Particiones de medidas, funciones equivariantes, construcción de Borel, sucesiones espectrales, índice de Fadell-Husseini.

Abstract

In this thesis we apply methods from algebraic topology to a problem in discrete geometry. To be more precise, the question involves a mass partition problem, whereas the methods include spectral sequences and a cohomological ideal-valued index theory.

In Chapter 1 we introduce some necessary terminology and the classical mass partition problem known as the *Grünbaum–Hadwiger–Ramos problem for masses*. We provide some history around said problem, as well as the most relevant results obteined in the last few years. Finally, motivated by the recent work of Patrick Schnider [1], we present an extension of this classical mass partition problem to the so-called mass assignments.

In the following three chapters we talk about the tools we use to prove the proposed extension of the Grünbaum–Hadwiger–Ramos mass partition problem. In Chapter 2 we describe the *configuration space/test map scheme*, which provides a bridge between geometry and topology. The idea is to rephrase the geometric problem in topological terms to solve it using techniques from algebraic topology. What follows then is to introduce such techniques. In Chapter 3 we present a brief introduction of the *cohomological Leray–Serre spectral sequence* associated to a fibration. Particularly, this spectral sequence allows to obtain information about and in some cases fully calculate the cohomology ring of the total space of the fibration, as well as the induced homomorphism in cohomology. Next, in Chapter 4 we introduce the *Fadell–Husseini index theory* and some of its most important properties. Here the Leray–Serre spectral sequence is essential for all the computations. This ideal-valued index theory is the key ingredient to solve our topological problem.

Finally, using the methods and techniques introduced in the previous chapters, in Chapter 5 we prove our main result, the *Grünbaum–Hadwiger–Ramos problem for mass assignments.* This is a joint work with Pavle V. M. Blagojević, Michael C. Crabb and Aleksandra S. Dimitrijević Blagojević.

Chapter 1

Introduction

In this chapter, besides providing some important terminology, we introduce the classical *Grünbaum–Hadwiger–Ramos hyperplane mass partition problem*, pointing out the progress that has been made on it. Also, at the end of the chapter, we present the main problem of this thesis, a new version of the Grünbaum–Hadwiger–Ramos problem using some new objects called mass assignments.

1.1 Terminology

A *mass* is a finite Borel measure on a Euclidean space that vanishes on each affine hyperplane. Examples of masses in $\mathbb{R}^d$ are: measures given by the d-dimensional volume of a proper convex body, measures induced by lengths of interval on a moment curve in $\mathbb{R}^d$, and measures given by a finite collection of pairwise disjoint balls.

Let X be a locally compact Hausdorff space, and let $M_+(X)$ denote the set of all finite Borel measures on X. For a definition of the Borel measure on a topological space consult for example [2, Def. 2.15]. The *weak topology* on $M_+(X)$ is defined to be the minimal topology such that for every bounded and upper semi-continuous function $f\colon X \longrightarrow \mathbb{R}$, the induced function $M_+(X) \longrightarrow \mathbb{R}$, $\nu \longmapsto \int f d\nu$, is upper semi-continuous. Here minimality is considered with respect to the inclusion of families of (open) subsets of X. In the case when $X = \mathbb{R}^\ell$ we denote by $M'_+(\mathbb{R}^\ell) \subseteq M_+(\mathbb{R}^\ell)$ the subspace of all masses on $\mathbb{R}^\ell$. For more details about spaces of measures and related notions consult [3].

Let $G_\ell(\mathbb{R}^d)$, $0 \leq \ell \leq d$, denotes the Grassmann manifold of all ℓ-dimensional linear subspaces of $\mathbb{R}^d$. Consider the following fiber bundle

$$M'_+(\mathbb{R}^\ell) \longrightarrow \mathcal{M}'_+(\ell, d) \xrightarrow{\pi} G_\ell(\mathbb{R}^d), \tag{1.1}$$

where the total space is given by

$$\mathcal{M}'_+(\ell, d) := \{(L, \nu) \mid L \in G_\ell(\mathbb{R}^d) \text{ and } \nu \in M'_+(L)\}$$

and the map π by $(L, \nu) \longmapsto L$.

Definition 1.1.1. A *mass assignment* μ on $G_\ell(\mathbb{R}^d)$ is a cross-section of the fiber bundle (1.1), which assigns to each subspace $L \in G_\ell(\mathbb{R}^d)$ a mass μ^L on L.

Examples of mass assignments on $G_\ell(\mathbb{R}^d)$ are: projections of masses in $\mathbb{R}^d$ to ℓ-dimensional linear subspaces and volumes of intersections of proper convex body in $\mathbb{R}^d$ with ℓ-dimensional linear subspaces. Mass assignments have been recently studied by Schnider [1], and by Soberón and Axelrod-Freed [4].

Let $v \in S^{d-1}$ be a unit vector in $\mathbb{R}^d$, and let $a \in \mathbb{R}$. The *oriented affine hyperplane* $H(v;a)$ in $\mathbb{R}^d$, oriented by v at distance a from the origin (in direction v), determines the associated affine hyperplane

$$H_{v;a} := \{x \in \mathbb{R}^d \colon \langle x, v\rangle = a\},$$

and, in addition, two closed half-spaces which are denoted by

$$H^0_{v;a} := \{x \in \mathbb{R}^d \colon \langle x, v\rangle \geq a\} \qquad \text{and} \qquad H^1_{v;a} := \{x \in \mathbb{R}^d \colon \langle x, v\rangle \leq a\}.$$

In particular, the following equalities hold: $H_{v;a} = H_{-v;-a}$, $H^0_{v;a} = H^1_{-v;-a}$ and $H^1_{v;a} = H^0_{-v;-a}$.

A *k-arrangement* $\mathcal{H}$ in $\mathbb{R}^d$ is an ordered collection of k oriented affine hyperplanes $\mathcal{H} = \big(H(v_1;a_1), \dots, H(v_k;a_k)\big)$. The *orthant* determined by the k-arrangement $\mathcal{H}$ and the element $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_2^k = \{0,1\}^k$ of the abelian group $\mathbb{Z}_2^k$ is the following intersection of closed half-spaces

$$\mathcal{O}^{\mathcal{H}}_\alpha = H^{\alpha_1}_{v_1;a_1} \cap \dots \cap H^{\alpha_k}_{v_k;a_k}.$$

A k-arrangement $\mathcal{H}$ *equiparts* a collection of masses $\mathcal{M} = (\mu_1, \dots, \mu_j)$ if for every element $\alpha \in \mathbb{Z}_2^k$ and every $r \in \{1, \dots, j\}$ holds:

$$\mu_r(\mathcal{O}^{\mathcal{H}}_\alpha) = \tfrac{1}{2^k}\mu_r(\mathbb{R}^d). \tag{1.2}$$

This can be achieved only in the case when $k \leq d$ for the following reason: Let us denote by $O(d,k)$ the maximum number of non-empty orthants determined by a k-arrangement in $\mathbb{R}^d$. First, we will prove that

$$O(d,k) = \sum_{j=0}^{d} \binom{k}{j}. \tag{1.3}$$

We proceed by induction on the dimension d and the number of hyperplanes k. Considering $d = 1$ and k point in it, we get a division of a line into $k+1$ pieces, so (1.3) holds. Notice that the formula is also correct for $k = 0$ and all $d \geq 1$, which represent the whole space with no hyperplanes

Suppose now that we are in dimension d, we have $k-1$ hyperplanes, and we insert another one. Since we are considering the maximum number of non-empty orthants, the $k-1$ previous hyperplanes divide the newly inserted hyperplane H into $O(d-1,k-1)$ pieces. Each such $(d-1)$-dimensional orthant within H divides one of dimension d into exactly two part. This means that the total increase in the number of orthants caused by inserting H is thus $O(d-1,k-1)$. We obtain then the following recurrence,

$$O(d,k) = O(d,k-1) + O(d-1,k-1). \tag{1.4}$$

In this way, considering the initial conditions ($d = 1$ and $k = 0$), the recurrence (1.4) determines all the values of $O(\cdot,\cdot)$. Let us now assume, as an induction hypothesis, that

$O(d, k-1)$ and $O(d-1, k-1)$ satisfies the formula (1.3). It remains just to verify if the recurrence (1.4) also satisfies (1.3). We have that

$$\begin{aligned} O(d,k) &= O(d,k-1) + O(d-1,k-1) \\ &= \sum_{j=0}^{d} \binom{k-1}{j} + \sum_{j=0}^{d-1} \binom{k-1}{j} \\ &= \binom{k-1}{0} + \left[\binom{k-1}{0} + \binom{k-1}{1}\right] + \cdots + \left[\binom{k-1}{d-1} + \binom{k-1}{d}\right] \\ &= \sum_{j=0}^{d} \binom{k}{j} \end{aligned}$$

Finally, for $d < k$,

$$O(d,k) = \sum_{j=0}^{d} \binom{k}{j} < \sum_{j=0}^{k} \binom{k}{j} = 2^k.$$

This means that, assuming $d < k$, in an d-dimensional real vector space no k hyperplanes can define 2^k non-empty orthants. For that reason it is always silently assumed that the number of hyperplanes we consider does not exceed the dimension of the ambient space. For more details about the last argument see [5, Prop. 6.1.1].

1.2 The Grünbaum–Hadwiger–Ramos problem for masses

The study of mass partition problems by affine hyperplanes started with a classical result, the so called ham sandwich theorem, conjectured by Hugo Steinhaus [6, Problem 123], and proved by Karol Borsuk in 1938; for details about the history see [7]. The ham sandwich theorem states that for any collection of d masses living in a d-dimensional Euclidean space there exists an affine hyperplane which equiparts the collection, that is, cuts each of the masses into two equal parts.

A few decades later Branko Grünbaum in his paper [8] asked the following question: *Is it possible to equipart a single mass in $\mathbb{R}^d$ by a d-arrangment?* He noted that, while the answer in the case of a line is obviously positive, the positive answer for the case of the plane follows directly from the ham sandwich theorem. The positive answer to the Grünbaum's question in the case $d = 3$ was given by Hugo Hadwiger [9] in 1966 as a consequence of his result: For any collection of two masses in $\mathbb{R}^3$ there exists a 2-arrangement which equiparts the collection. In 1984 David Avis [10] showed that in every dimension $d \geq 5$ there is a mass which cannot be equiparted by a d-arrangement. The case of dimension 4, to this very day, is still open, meaning that we would like to know if it is possible to equipart one mass in $\mathbb{R}^4$ with a 4-arrangement.

In 1996 Edgar Ramos [11] proposed the following extension of the Grünbaum hyperplane mass partition problem.

The Grünbaum–Hadwiger–Ramos mass partition problem. *Determine the minimal dimension $d = \Delta(j,k)$ of a Euclidean space $\mathbb{R}^d$ such that for every collection of j masses in $\mathbb{R}^d$ there exists a k-arrangement equiparting the collection of masses.*

In particular, the ham sandwich theorem is equivalent to the equality $\Delta(d,1) = d$, while the results of Grünbaum and Hadwiger imply that $\Delta(1,2) = 2$, $\Delta(2,2) = 3$ and $\Delta(1,3) = 3$. Based on the ideas of Avis, Ramos derived the following lower bound for the function $\Delta(j,k)$:

$$\tfrac{2^k-1}{k} j \leq \Delta(j,k).$$

The lower bound transformed into the following conjecture.

The Ramos conjecture. $\Delta(j,k) = \lceil \frac{2^k-1}{k} j \rceil$ *for every* $j \geq 1$ *and* $k \geq 1$.

An upper bound for the function $\Delta(j,k)$ was obtained in 2006 by Peter Mani-Levitska, Siniša Vrećica & Rade Živaljević in [12, Thm. 39]:

$$\Delta(j,k) \;\leq\; j + (2^{k-1} - 1)2^{\lfloor \log_2 j \rfloor}.$$

The only instance in which lower and upper bounds coincide is in the case when $k = 2$ and $j = 2^{t+1} - 1 \geq 1$.

Over the years, using variety of methods from equivariant algebraic topology, different groups of authors studied the conjecture of Ramos. Despite considerable effort the conjecture has been confirmed rigorously only in a few special cases; for more details on the history and discussion of solution methods consult a critical review [13], and for the currently best known results see [14].

1.3 An extension of the classical partition problem

The problem we consider in this thesis is the following extension of the Grünbaum–Hadwiger–Ramos problem to the mass assignments.

A 4-tuple of natural numbers (d, ℓ, j, k), where $1 \leq \ell \leq d$, is called *mass assignment admissible* if for every collection of j mass assignments $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ on the Grassmann manifold $G_\ell(\mathbb{R}^d)$ there exists a vector subspace $L \in G_\ell(\mathbb{R}^d)$ and a k-arrangement $\mathcal{H}^L$ in L which equiparts the collection of j masses $(\mu_1^L, \ldots, \mu_j^L)$. Observe, that (d, ℓ, j, k) is mass assignment admissible only when $k \leq \ell$. The case $\ell = d$ coincides with the classical Grünbaum–Hadwiger–Ramos problem.

Main Problem. *Determine all mass assignment admissible* 4*-tuples.*

Patrick Schnider, in his recent publication [1, Thm. 2], showed that any 4-tuple of the form $(d, \ell, d, 1)$, with $1 \leq \ell \leq d$, is mass assignment admissible.

In this work we prove the following general algebraic criterion from the assignment admissibility which further on yields multiple corollaries. For the statement of the theorem we introduce the following truncated polynomial ring

$$R_{d,\ell,k} := \mathbb{F}_2[x_1, \ldots, x_k, w_1, \ldots, w_\ell, \overline{w}_1, \ldots, \overline{w}_{d-\ell}]/I_{d,\ell} \tag{1.5}$$

where $\deg(x_1) = \cdots = \deg(x_k) = 1$, $\deg(w_s) = s$, $\deg(\overline{w}_r) = r$ for $1 \leq s \leq \ell$, $1 \leq r \leq d - \ell$, and $I_{d,\ell}$ is the ideal generated by the following d polynomials

$$\sum_{s=\max\{0,r+\ell-d\}}^{\min\{r,\ell\}} w_s \cdot \overline{w}_{r-s}, \qquad 1 \leq r \leq d,$$

in variables $w_1, \dots, w_\ell, \overline{w}_1, \dots, \overline{w}_{d-\ell}$. Actually, the polynomials that generate the ideal $I_{d,\ell}$ are exactly the d relations which are derived from the equality

$$(1 + w_1 + \cdots + w_\ell)(1 + \overline{w}_1 + \cdots + \overline{w}_{d-\ell}) = 1.$$

Note that the ring $R_{d,\ell,k}$ is isomorphic with the cohomology ring $H^*(\mathrm{B}(\mathbb{Z}_2^k) \times G_\ell(\mathbb{R}^d); \mathbb{F}_2)$, and can also be seen as the polynomial ring over the ring $H^*(G_\ell(\mathbb{R}^d); \mathbb{F}_2)$, that is

$$R_{d,\ell,k} \cong (\mathbb{F}_2[w_1, \dots, w_\ell, \overline{w}_1, \dots, \overline{w}_{d-\ell}]/I_{d,\ell})[x_1, \dots, x_k].$$

The main result of this work is the following theorem which we prove in Section 5.3.1 based on the configuration space/test map scheme developed in Chapter 2, and the computations done in Section 5.2.

Theorem 1.3.1. *Let $d \geq 1$, $k \geq 1$, $j \geq 1$ and $\ell \geq 1$ be integers. A 4-tuple of natural numbers (d, ℓ, j, k) where $1 \leq \ell \leq d-1$ is mass assignment admissible if the element*

$$e_{k,j} := \prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^k \setminus \{(0,\dots,0),(1,\dots,0),\dots,(0,\dots,1)\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j$$

of the ring $R_{d,\ell,k}$ is not contained in the ideal

$$\mathcal{I}_{d,\ell,k,j} := \Big\langle \sum_{s=0}^{\ell} x_r^s \, w_{\ell-s} \; : \; 1 \leq r \leq k \Big\rangle,$$

where w_0 is assumed to be 1.

As the first consequence of Theorem 1.3.1 we recover the ham sandwich type result of Schnider [1, Thm. 2].

Corollary 1.3.2. *Let $d \geq 2$ be an integer. Every 4-tuple of the form $(d, \ell, d, 1)$, where $1 \leq \ell \leq d$, is mass assignment admissible.*

Proof. In the case when $\ell = d$ the admissibility of $(d, d, 1, d)$ is just the classical ham sandwich theorem. Thus, the proof which we present is novel when for $1 \leq \ell \leq d-1$.

Since we are in the situation where $k = 1$ and $j = d$, then $e_{1,d} = x_1^{d-1}$ and the ideal $\mathcal{I}_{d,\ell,1,d}$ is the principal ideal generated by the polynomial $p := \sum_{s=0}^{\ell} x_1^s \, w_{\ell-s}$. According to the Theorem 1.3.1 the 4-tuple $(d, \ell, d, 1)$ is mass assignment admissible if $e_{1,d} \notin \mathcal{I}_{d,\ell,1,d}$, or equivalently $p \nmid e_{1,d}$. This means that

$$\sum_{s=0}^{\ell} x_1^s \, w_{\ell-s} \nmid x_1^{d-1} \tag{1.6}$$

in the ring $R_{d,\ell,1} := \mathbb{F}_2[x_1, w_1, \dots, w_\ell, \overline{w}_1, \dots, \overline{w}_{d-\ell}]/I_{d,\ell}$. Therefore, by verifying the claim of the relation (1.6) we complete the proof of the corollary.

The ambient ring $R_{d,\ell,1}$ can also be seen as a polynomial ring in one variable x_1 over the ring $\mathbb{F}_2[w_1, \dots, w_\ell, \overline{w}_1, \dots, \overline{w}_{d-\ell}]/I_{d,\ell}$, that is

$$R_{d,\ell,1} \cong (\mathbb{F}_2[w_1, \dots, w_\ell, \overline{w}_1, \dots, \overline{w}_{d-\ell}]/I_{d,\ell})[x_1].$$

Here we are slightly abusing a notation: the ideal $I_{d,\ell}$ is always considered in the appropriate ring. For this reason multiplication by x_1 is a monomorphism.

Let us assume that the relation (1.6) does not hold, that is

$$x_1^{d-1} = \Big(\sum_{s=0}^{\ell} x_1^s\, w_{\ell-s}\Big)\cdot\Big(\sum_{s=0}^{d-\ell-1} x_1^s\, u_{d-\ell-1-s}\Big)$$

for some coefficients $u_0,\dots,u_{d-\ell-1}\in\mathbb{F}_2[w_1,\dots,w_\ell,\overline{w}_1,\dots,\overline{w}_{d-\ell}]/I_{d,\ell}$, where $u_0=1$. Thus, assuming that $w_i=u_i=0$ for all $i<0$, we have that

$$\begin{aligned} x_1^{d-1} =&(u_0w_0)x_1^{d-1}+(u_0w_1+u_1w_0)x_1^{d-2}+\cdots+\\ &(u_0w_{d-\ell-1}+u_1w_{d-\ell-2}+\cdots+u_{d-\ell-1}w_0)x_1^{\ell}+\\ &(u_0w_{d-\ell}+u_1w_{d-\ell-1}+\cdots+u_{d-\ell-1}w_1)x_1^{\ell-1}+\cdots \end{aligned}$$

Consequently, we get the following equalities:

$$\begin{aligned} u_0w_0&=1,\\ u_0w_1+u_1w_0&=0,\\ &\ \vdots\\ u_0w_{d-\ell-1}+u_1w_{d-\ell-2}+\cdots+u_{d-\ell-1}w_0&=0,\\ u_0w_{d-\ell}+u_1w_{d-\ell-1}+\cdots+u_{d-\ell-1}w_1&=0, \end{aligned}$$

in the ring of coefficients $\mathbb{F}_2[w_1,\dots,w_\ell,\overline{w}_1,\dots,\overline{w}_{d-\ell}]/I_{d,\ell}$. From the first $d-\ell$ equations we deduce that $u_0=\overline{w}_0,\dots,u_{d-\ell-1}=\overline{w}_{d-\ell-1}$, because

$$\sum_{s=\max\{0,r+\ell-d\}}^{\min\{r,\ell\}} w_s\cdot\overline{w}_{r-s},\qquad 1\le r\le d,$$

are the generators of the ideal $I_{d,\ell}$. Then the last equality yields a contradiction:

$$0=u_0w_{d-\ell}+u_1w_{d-\ell-1}+\cdots+u_{d-\ell-1}w_1=\overline{w}_{d-\ell}\neq 0.$$

Indeed, the relation (1.6) holds. □

In order to state a consequence of Corollary 1.3.2 we introduce a special type of a mass assignment. Let $s\colon G_\ell(\mathbb{R}^d)\longrightarrow E(\gamma_\ell^d)$ be a section of the tautological vector bundle γ_ℓ^d over $G_\ell(\mathbb{R}^d)$. For a positive real number $\varepsilon>0$ the section s defines a mass assignment μ_s given by $L\longmapsto B_L(s(L),\varepsilon)$. Here $B_L(s(L),\varepsilon)$ denotes the Euclidean closed ball in L with center at $s(L)$ and radius ε, or in other words the mass induced by this ball. Since any closed Euclidean ball is cut into halves of equal volume by an affine hyperplane if and only if this hyperplane passes through the center of the ball, we get the following statement as a direct consequence of Corollary 1.3.2.

Corollary 1.3.3. *Let $d\ge 2$ and $1\le\ell\le d-1$ be integers. For every collection of d sections $s_1,\dots,s_d\colon G_\ell(\mathbb{R}^d)\longrightarrow E(\gamma_\ell^d)$ of the tautological vector bundle γ_ℓ^d over $G_\ell(\mathbb{R}^d)$, there exists a subspace $L\in G_\ell(\mathbb{R}^d)$ and an affine hyperplane H in L such that $s_1(L)\in H,\dots,s_d(L)\in H$.*

While Corollary 1.3.3 is an easy consequence of Corollary 1.3.2 and does not use much information about the Grassmann manifold $G_\ell(\mathbb{R}^d)$, one can deduce more by using some additional information about the Stiefel–Whitney classes of γ_ℓ^d. More precisely, the so called intersection lemma [15, Lem. 4.3] in combination with the fact that $w_\ell(\gamma_\ell^d)^{d-\ell} \neq 0$ does not vanish, see [16, Lem. 1.2], yields the following fact: For every collection of $d-\ell$ section $s_1, \dots, s_{d-\ell}$ of γ_ℓ^d there exists a subspace $L \in G_\ell(\mathbb{R}^d)$ with the property that $s_1(L) = \cdots = s_{d-\ell}(L)$. In particular, the points $s_1(L), \dots, s_{d-\ell}(L), s_{d-\ell+1}(L), \dots s_d(L)$ lie on a hyperplane in L.

Like in the case of the classical Grünbaum–Hadwiger–Ramos problem, it is clear that no 4-tuple (d, ℓ, j, k), where $k \geq \ell + 1$, can be mass assignment admissible. Simply, as we mentioned before, in an ℓ-dimensional real vector space no k hyperplanes can define 2^k non-empty orthants. In particular, Theorem 1.3.1 implies the following algebraic fact.

Corollary 1.3.4. *Let $d \geq 1$, $k \geq 1$, $j \geq 1$ and $\ell \geq 1$ be integers with $1 \leq \ell \leq d-1$ and $k \geq \ell + 1$. Then the element*

$$e_{k,j} := \prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^k \setminus \{(0,\dots,0),(1,\dots,0),\dots,(0,\dots,1)\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j$$

of the ring $R_{d,\ell,k}$ is contained in the ideal

$$\mathcal{I}_{d,\ell,k,j} := \Big\langle \sum_{s=0}^{\ell} x_r^s \, w_{\ell-s} \ : \ 1 \leq r \leq k \Big\rangle.$$

Even though this fact is a direct consequence of the assumption $k \geq \ell + 1$ and the proof of Theorem 1.3.1, in Section 5.3.2 we give an independent and direct argument.

The major consequence of Theorem 1.3.1 is the following numerical criterion for a 4-tuple (d, ℓ, j, k) to be mass assignment admissible. The proof of this result is given in Section 5.3.3.

Theorem 1.3.5. *Let $d \geq 2$, $k \geq 1$, $j \geq 1$, $\ell \geq 1$ and $t \geq 0$, $r \geq 0$ be integers with $1 \leq k \leq \ell \leq d$. If $j = 2^t + r$ with $0 \leq r \leq 2^t - 1$, and $d \geq 2^{t+k-1} + r$, then the 4-tuple (d, ℓ, j, k) is mass assignment admissible.*

An interesting observation is that the condition for the 4-tuple (d, ℓ, j, k), $1 \leq k \leq \ell \leq d-1$, to be admissible given by Theorem 1.3.5 does not depend on ℓ whatsoever. Is this an artefact of the proof method or maybe an intrinsic property of the problem?

Finally, in Table 1.1 we compare the result of Theorem 1.3.5 with the corresponding know results for the classical Grünbaum–Hadwiger–Ramos mass partition problem for some concrete choices of parameters (d, ℓ, j, k). For that we recall the known equalities

$$\Delta(2^t + 1, 2) = 3 \cdot 2^{t-1} + 1 \qquad \text{and} \qquad \Delta(2^{t+1} - 1, 2) = 3 \cdot 2^t - 1,$$

where $t \geq 2$. For more details of these two results see for example [13].

Admissible 4-tuples		
$(\Delta(j,k), \Delta(j,k), j, k)$		$(d, \ell, j = 2^t + r, k)$
$\Delta(2,2) = 3$	$\Rightarrow$ $(3, 3, 2, 2)$	$(8, 3, 4, 2)$ (considering $t = 2$ and $r = 0$)
$\Delta(1,3) = 3$	$\Rightarrow$ $(3, 3, 1, 3)$	$(9, 3, 3, 3)$ (considering $t = 1$ and $r = 1$)
$\Delta(5,2) = 8$	$\Rightarrow$ $(8, 8, 5, 2)$	$(11, 8, 7, 2)$ (considering $t = 2$ and $r = 3$)
$\Delta(9,2) = 14$	$\Rightarrow$ $(14, 14, 9, 2)$	$(17, 14, 15, 2)$ (considering $t = 3$ and $r = 7$)
$\Delta(7,2) = 11$	$\Rightarrow$ $(11, 11, 7, 2)$	$(23, 11, 15, 2)$ (considering $t = 3$ and $r = 7$)
$\Delta(15,2) = 23$	$\Rightarrow$ $(23, 23, 15, 2)$	$(47, 23, 31, 2)$ (considering $t = 4$ and $r = 15$)

Table 1.1: Comparison between the classical Grünbaum–Hadwiger–Ramos mass partition problem and Theorem 1.3.5.

As can be appreciated in Table 1.1, the solutions of the Grünbaum–Hadwiger–Ramos problem for mass assignments may consider more masses than one can hope for in the classical case. This means that, in the extension of the classical problem, we can equipart more masses with the same number of hyperplanes in the appropriate Grassmann manifold.

What follows now is to introduce the methods and techniques we use to prove Theorem 1.3.1.

Chapter 2

The configuration space/test map scheme.

The *configuration cpace/test map scheme* (***CS/TM-scheme***) is a very useful and general method for proving combinatorial or geometric facts. It was developed in numerous research papers over the years and formalized by R. Živaljević in [17, 18]. Such method provides a bridge between the problem itself and a topological question. The main idea is to reduce the problem to the question about the non-existence of a particular equivariant map. Let us describe briefly how it works in 3 easy steps.

Step 1: Given a geometric or combinatorial problem $\mathcal{P}$, an associated ***configuration space*** $X_{\mathcal{P}}$ is the set of all possible candidates to be a solution of $\mathcal{P}$. The space $X_{\mathcal{P}}$, which is actually a topological space, parametrizes configurations of geometric objects (like arrangements of points, lines, flags, etc.) or combinatorial structures (like trees, graphs, partitions, etc.) which represent each of the possible solutions. The selection of an appropriate configuration space is very often the crucial point of the application of the CS/TM-scheme. Its construction is often based on a variety of combinatorial and geometrical ideas.

Step 2: Having defined the space of all possible candidates to be a solution of $\mathcal{P}$, we need to determine when an element $p \in X_{\mathcal{P}}$ is a solution to our problem. Let

$$f\colon X_{\mathcal{P}} \longrightarrow V_{\mathcal{P}}$$

be a continuous map, called ***test map***, from the configuration space $X_{\mathcal{P}}$ into the ***test space*** $V_{\mathcal{P}}$, which tests if the candidate $p \in X_{\mathcal{P}}$ is a solution of $\mathcal{P}$ or not. This can be done considering a subspace $Z_{\mathcal{P}} \subset V_{\mathcal{P}}$, where $p \in X_{\mathcal{P}}$ is a solution of $\mathcal{P}$ if and only if $f(p) \in Z_{\mathcal{P}}$. Usually $V_{\mathcal{P}} \cong \mathbb{R}^n$, while $Z_{\mathcal{P}}$ is the origin $\{0\} \subset V_{\mathcal{P}}$.

Step 3: The last ingredient in the CS/TM-scheme is a group G of symmetries of $\mathcal{P}$ which acts on $X_{\mathcal{P}}$ and $V_{\mathcal{P}}$, keeping the subspace $Z_{\mathcal{P}}$ G-invariant. Moreover, the test map f is G-equivariant, i.e, $f(g \cdot p) = g \cdot f(p)$ for every $g \in G$ and $p \in X_{\mathcal{P}}$.

Since the condition for an element $p \in X_{\mathcal{P}}$ to be a solution of $\mathcal{P}$ is that $f(p) \in Z_{\mathcal{P}}$, proving that the induced map $f\colon X_{\mathcal{P}} \to V_{\mathcal{P}} \setminus Z_{\mathcal{P}}$ does not exist guarantee that our problem $\mathcal{P}$ has a solution. This last part is where we usually use tools of algebraic topology. For a more detailed introduction of CS/TM-scheme see [19].

Examples 2.0.1. Let us present some well known examples in which we apply CS/TM-scheme:

1. *(Equilateral triangles in a compact subspace [19, Example 21.1.1])* Let d be a metric on $\mathbb{R}^2$ that induces the same topology as the usual Euclidean metric, and $\Gamma \subset \mathbb{R}^2$ be a compact subspace. Consider the problem of finding equilateral triangles in Γ, i.e., triples (x, y, z) of distinct point in Γ such that $d(x, y) = d(y, z) = d(z, x)$.

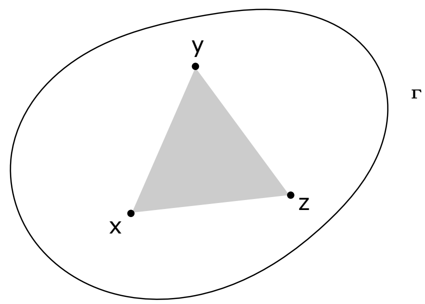

Figure 2.1: Equilateral triangles in a compact subspace.

Let us apply the CS/TM-scheme to this problem. Since we are looking for equilateral triangles, there are some cases which we can exclude from the space of all possible solutions, for example degenerated triangles (x, y, z) such that at least one of the numbers $d(x, y)$, $d(y, z)$ or $d(z, x)$ is zero (this illustrate the fact that there are several possibilities for a configuration space associated to a problem). Our choice of configuration space is $X = \Gamma^3 \setminus \Delta$, where $\Delta = \{(x, x, x) \in \Gamma^3 \mid x \in \Gamma\}$, and our test map $f\colon X \to \mathbb{R}^3$ is given by

$$f(x, y, z) = \big(d(x, y), d(y, z), d(z, x)\big).$$

The group of symmetries which acts on the configuration space X, the test space $\mathbb{R}^3$, and make the test map equivariant, is the group of all permutations of 3 elements $\mathfrak{S}_3$.

Notice that a triangle (x, y, z) is equilateral if and only if $\big(d(x, y), d(y, z), d(z, x)\big) \in Z$, where $Z = \{(u, u, u) \in \mathbb{R}^3 \mid u \in \mathbb{R}\}$. Then our problem will have a solution if and only if $\operatorname{im}(f) \cap Z \neq \emptyset$.

2. *(Ham sandwich theorem for measures [20, Theorem 3.1.1])* The informal statement that gave the ham sandwich its name is the following: *For every sandwich made of ham, cheese and bread, there is a planar cut that simultaneously bisects the ham, the cheese, and the bread.*

There is a formal version of the ham sandwich theorem, in terms of measures, which we will work with. Let $\mu_1, \mu_2, \ldots, \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each μ_i. Then there exists a hyperplane H such that

$$\mu_i(H^+) = \frac{1}{2}\mu_i(\mathbb{R}^d) \quad \text{for} \quad i = 1, 2, \ldots, d,$$

where H^+ denotes one of the half-spaces defined by H. Here the value of each measure represents the amount of one of the ingredients.

The CS/TM-scheme here works as follows: For every point $v = (v_0, v_1, \ldots, v_d)$ in S^d we assign the half-space

$$H_v^+ := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid v_1 x_1 + \ldots + v_d x_d \leq v_0\}.$$

In the cases that v is of the form $(\pm 1, 0, \ldots, 0)$ note that

$$H^+_{(1,0,\ldots,0)} := \mathbb{R}^d \quad \text{and} \quad H^+_{(-1,0,\ldots,0)} := \emptyset.$$

Let us define now a continuous map $f\colon S^d \to \mathbb{R}^d$ given by

$$f(v) = \left(\mu_1(H_v^+) - \frac{\mu_1(\mathbb{R}^d)}{2}, \ldots, \mu_d(H_v^+) - \frac{\mu_d(\mathbb{R}^d)}{2}\right),$$

which since antipodal points correspond to opposite half-spaces, f is $\mathbb{Z}_2$-equivariant. Notice that the hyperplane that we are looking for is contained in $f^{-1}(0)$. Then, using the CS/TM-scheme, we have to prove that the induced $\mathbb{Z}_2$-equivariant map $f\colon S^d \to S^{d-1}$ does not exist. To finish the proof, the non-existence of the map f comes from the famous Borsuk–Ulam theorem.

In Chapter 5 we will rephrase our main problem as a topological one using the CS/TM-scheme just described.

Chapter 3

The Leray–Serre spectral sequence.

In this chapter we give some background of (cohomological) spectral sequences necessary for understanding the computations of Fadell–Husseini index theory developed in Chapter 4. For more details about spectral sequences see [21], [22] and [23].

3.1 What is a spectral sequence?

Spectral sequences are a useful technique in algebraic topology traditionally applied to compute (co)homology and homotopy groups of spaces. Intuitively, we can think of a spectral sequence as a book consisting of a sequence of pages, each of which is a two-dimensional array of Abelian groups. On each page there are maps between the groups, and these maps form chain complexes. The (co)homology groups of these chain complexes are precisely the groups that appear on the next page. The desired computation is codified on the "last page".

Generally speaking we define a spectral sequence as follows:

Definition 3.1.1. A ***(cohomological) spectral sequence*** is a collection of Abelian groups and homomorphisms,

$$\{E_r^{p,q},\ d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}\},$$

indexed by integers p, q, r, satisfying the following conditions:

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0 \qquad \text{and} \qquad E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{im} d_r^{p+r,q-r+1}.$$

The conditions mentioned before mean that the $(r+1)$-th sheet $E_{r+1}^{p,q}$, usually called the E_{r+1}-term, is the cohomology of $(E_r^{p,q}, d_r^{p,q})$. For an illustration of the E_r-term see Figure 3.1. We are interested in first quadrant spectra sequences, that is, spectral sequences for which $E_r^{p,q} = 0$ when $p < 0$ or $q < 0$.

With this general definition, consider the Abelian group $E_r^{p,q}$ for $r > \max\{p, q+1\}$. Here, since $q + 1 - r < 0$ and $p - r < 0$, the differential $d_r^{p,q}$ becomes trivial. Thus $E_{r+1}^{p,q} = E_r^{p,q}$ and, continuing in the same way, $E_{r+k}^{p,q} = E_r^{p,q}$ for $k \geq 1$. We denote this common group by

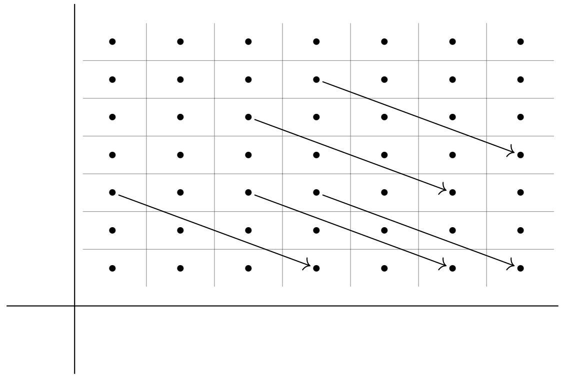

Figure 3.1: Illustration of the E_r-term.

$E_\infty^{p,q}$. A spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ is said to ***converge*** to a graded Abelian group H^* if there is a filtration[1] by subgroups

$$0 = F^{n+1}H^n \subseteq F^n H^n \subseteq F^{n-1}H^n \subseteq \cdots \subseteq F^1 H^n \subseteq F^0 H^n = H^n$$

for each $H^n \subset H^*$ such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}. \tag{3.1}$$

This means that H^n, or at least plenty of information about it, is codified in the diagonal $E_\infty^{p,q}$, where $p+q=n$. Moreover, by (3.1), H^n can be achieved via a finite number of extensions. In case there is only one pair (p,q) such that $E_\infty^{p,q} \neq 0$, then $H^n = E_\infty^{p,q}$.

Even though "finding" H^* inside $E_\infty^{*,*}$ is a not a piece of cake, in some cases the computation is an achievable task. The simplest case occurs when a finite number of steps complete the computation. A spectral sequence is said to ***collapse*** at the n-th term if $d_r^{p,q} = 0$ for all $p, q \geq 0$ and $r \geq n$. Of course the immediate consequence of collapse at the n-th term is that

$$E_n^{*,*} \cong E_{n+1}^{*,*} \cong \cdots \cong E_\infty^{*,*},$$

so H^* is now codified in $E_n^{*,*}$.

3.2 Leray–Serre spectral sequence of a fibration.

Before introducing the spectral sequence of a fibration, let us start by presenting the spectral sequence of a filtered topological space. For that, let R be a commutative ring with unit, and consider an increasing filtration of a topological space X,

$$0 \subset X_0 \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X_n \subset \cdots \subset X,$$

[1] A *(decreasing) filtration* F^* of an Abelian group A is a family of subgroups $\{F^i A\}_{i\in\mathbb{Z}}$ such that

$$\cdots \subseteq F^{i+1}A \subseteq F^i A \subseteq F^{i-1}A \subseteq \cdots \subseteq A.$$

In case we change the Abelian group A by a topological space, then the filtration will consist of topological subspaces.

for which we have information about the singular cohomology of the pairs (X_p, X_{p-1}). Defining $F^n H^*(X)$ as the kernel of the map $H^*(X;R) \to H^*(X_{n-1};R)$ induced by the inclusión, there is a first quadrant spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ with

$$\begin{aligned} E_1^{p,q} &= H^{p+q}(X_p, X_q; R), \\ d_1^{p,q} &= \delta^* \colon H^{p+q}(X_p, X_{p-1}; R) \to H^{p+q+1}(X_{p+1}, X_p; R), \end{aligned}$$

and converging to $H^*(X;R)$. Here the homomorphism δ^* belongs to the cohomology sequence of the triple (X_{p+1}, X_p, X_{p-1}). Notice that the explicit formulas for E_1 and d_1 allow us to completely ignore the zeroth terms. A similar situation is going to happen when we consider fibrations instead of filtered topological spaces.

The spectral sequence for a filtration can be used to derive an spectral sequence of a fibration. We recall that a fibration is a continuous map $p\colon E \to B$ satisfying the homotopy lifting property with respect to all space Y (see [24, Chapter 4]). It follows that if B is path-connected, then all the fibers $p^{-1}(b)$, for $b \in B$, are homotopy equivalent. Hence we can speak of *the fiber* and denote it by F.

Let us consider a locally trivial fibration

$$\xi = \big(E,\ B,\ E \xrightarrow{p} B,\ F\big),$$

where B is a path-connected CW-complex with skeletons $B^{(p)}$. This implies that B is equipped with a filtration by skeleta. We can induce then a filtration on E by letting $J^s = p^{-1}(B^{(s)})$, that is, the subspace of E that lies over the s-squeleton of B.

$$\begin{array}{ccccccccccc} \emptyset & \subset & J^0 & \subset \cdots \subset & J^{s-1} & \subset & J^s & \subset \cdots \subset & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \emptyset & \subset & B^{(0)} & \subset \cdots \subset & B^{(s-1)} & \subset & B^{(s)} & \subset \cdots \subset & B \end{array}$$

We can use now this filtration to obtain a spectral sequence. The spectral sequence associated to the filtration $\{J^s\}$ is called ***the Leray–Serre spectral sequence of the fibration*** ξ.

Let us begin with the calculation of the terms E_1 and E_2. Consider first the simplest case of a trivial fibration $B \times F \to B$. Here, since $(J^s, J^{s-1}) = (B^{(s)}, B^{(s-1)}) \times F$, by the Künneth theorem (see [25, Chapter 3]) we get that

$$\begin{aligned} E_1^{p,q} &= H^{p+q}(J^p, J^{p-1}; R) \\ &= H^{p+q}((B^{(p)}, B^{(p-1)}) \times F; R) \\ &\cong H^p\big((B^{(p)}, B^{(p-1)}); H^q(F;R)\big) \\ &= \mathcal{C}^p\big(B; H^q(F;R)\big). \end{aligned}$$

In the case of an arbitrary fibration we need to consider a possibly non-trivial twisting of the fiber and the base space inside the total space. This means that, for $b_0, b_1 \in B$, the isomorphism between $H^n(p^{-1}(b_0); R)$ and $H^n(p^{-1}(b_1); R)$ may depend on a path $\gamma\colon I \to B$ joining the points b_0 and b_1. For that we use coefficients in a bundle of groups given by a collection of groups

$$\mathcal{H}^*(F;R) = \big\{H^*(p^{-1}(b); R) \mid b \in B\big\},$$

together with a collection of isomorphisms

$$\big\{J_\gamma\colon H^*(p^{-1}(b_1);R) \to H^*(p^{-1}(b_0);R) \mid \gamma \in \pi_1(B,b_0,b_1)\big\},$$

where $\pi_1(B,b_0,b_1)$ is the set of homotopy classes of paths in B joining b_0 and b_1. We call $\mathcal{H}^*(F;R)$ a ***system of local coefficients on*** B ***induced by*** F. In the case that the isomorphism between $H^n(p^{-1}(b_0);R)$ and $H^n(p^{-1}(b_1);R)$ does not depend on any path γ, like the example of the trivial bundle, we call the system of local coefficients simple, or simply that there are no local coefficients. The following result describes the initial terms E_1 and E_2 of the spectral sequence associated to an arbitrary fibration.

Proposition 3.2.1. *In the spectral sequences of a fibration* $F \hookrightarrow E \xrightarrow{p} B$*,*

1. $E_1^{p,q} = \mathcal{C}^p\big(B;\mathcal{H}^q(F;R)\big)$.
2. $d_1^{p,q} = \delta\colon \mathcal{C}^p\big(B;\mathcal{H}^q(F;R)\big) \to \mathcal{C}^{p+1}\big(B;\mathcal{H}^q(F;R)\big)$.
3. $E_2^{p,q} = H^p(B;\mathcal{H}^q(F;R))$.

Here $H^p(B;\mathcal{H}^q(F;R))$ represent the cohomology of B with local coefficients in the cohomology of F. In case that the system of local coefficients is simple,

$$H^p(B;\mathcal{H}^q(F;R)) \cong H^p(B;H^q(F;R))$$

and therefore we recover the usual cohomology in the E_2-term. For an algebraic definition of cohomology with local coefficients see [25, Chapter 3.H]. The E_2-term can be pictured as in Figure 3.2.

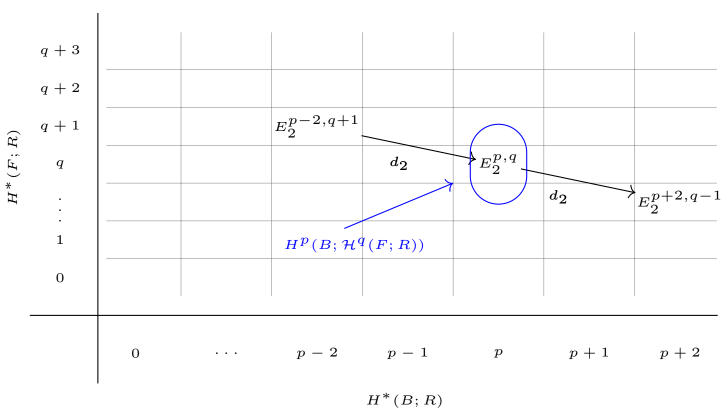

Figure 3.2: E_2-term of a spectral sequence.

The following result summarizes the previous discussion about the spectral sequence associated to a fibration:

Theorem 3.2.2 (The cohomology Leray–Serre spectral sequence)**.** *Let* R *be a commutative ring with unit. Given a fibration* $F \hookrightarrow E \xrightarrow{p} B$*, where* B *is path connected CW-complex, there is a first quadrant spectral sequence* $\{E_r^{p,q}, d_r^{p,q}\}$ *with*

$$E_2^{p,q} \cong H^p(B;\mathcal{H}^q(F;R)),$$

and converging to $H^*(E;R)$*.*

Remark 3.2.3. Introducing some convenient hypothesis, the Leray–Serre spectral sequence can be seen to take a manageable form. Suppose that $F \hookrightarrow E \xrightarrow{p} B$ is a fibration with B path-connected and the system of local coefficients on B induced by F simple. Assuming coefficient on a field $\mathbb{F}$, by the Universal Coefficient Theorem we get that the E_2-term of the associated spectral sequence looks as follows:

$$E_2^{p,q} \cong H^p(B;\mathbb{F}) \otimes H^q(F;\mathbb{F}).$$

Also, since the spectral sequence converges to $H^*(E;\mathbb{F})$, and $\mathbb{F}$ is a field,

$$H^n(E;\mathbb{F}) \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

3.3 Additional properties of the Leray–Serre spectral sequence.

In order to compute differentials of a spectral sequence, as many as possible, we briefly introduce two important properties that will help us with the challenge.

3.3.1 Multiplicative structure.

The Leray–Serre spectral sequence for cohomology becomes much more powerful when cup products are brought into the picture. The way in which the cup products come into play is by providing a multiplication for the spectral sequence. In other words, $\{E_r^{p,q},\}$ can be furnished with bilinear products

$$E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'},$$

for $1 \leq r \leq \infty$ satisfying the following properties:

1. Each differential d_r is a derivation, satisfying the *Leibniz rule*

$$d_r^{p+p',q+q'}(\alpha\beta) = d_r^{p,q}(\alpha)\beta + (-1)^{p+q}\alpha d_r^{p',q'}(\beta),$$

   for $\alpha \in E_r^{p,q}$ and $\beta \in E_r^{p',q'}$. This implies that the multiplication $E_r^{p,q} \times E_r^{p,q} \to E_r^{p+p',q+q'}$ induces a product $E_{r+1}^{p,q} \times E_{r+1}^{p,q} \to E_{r+1}^{p+p',q+q'}$, and this is the multiplication for E_{r+1}.

2. The multiplication in E_2 coincides with the multiplication in the cohomology of B with coefficients in the system of local coefficients induced by the cohomology of F.

3. The multiplication in E_∞ is adjoint to the multiplication in $H^*(E;R)$ in the following sense: If $a \in F^pH^m(E;R)$ and $b \in F^qH^n(E;R)$, then $ab \in F^{p+q}H^{m+n}(E;R)$, and if $\alpha \in E_\infty^{p,m-p}$, $\beta \in E_\infty^{q,n-q}$ and $\gamma \in E_\infty^{p+q,m+n-p-q}$ are represented by a, b and ab, then $\gamma = \alpha\beta$.

As we saw in Remark 3.2.3, under some convenient conditions the computations on the spectral sequence becomes more manageable. That is also the case with the multiplicative structure. Suppose that the system of local coefficients is simple, and that B and F are path-connected. Then the multiplicative structure on

$$E_2^{p,0} \cong H^p(B;H^0(F;R)) \cong H^p(B;R)$$

and

$$E_2^{0,q} \cong H^0(B; H^q(F;R)) \cong H^q(F;R)$$

correspond to the cup product on $H^p(B;R)$ and $H^q(F;R)$ respectively. We will see later that the structure in the zeroth row and zeroth column plays an important role in our computations.

This multiplicative structure in favorable cases allows many more differentials to be computed purely formally.

3.3.2 Homomorphism of spectral sequences.

Following with the idea of computing as many differentials as possible, constructing homomorphisms between spectral sequences represent an important tool to make the work easier.

Let (E, B, F, p) and (E', B', F', p') be two locally trivial fibrations with connected CW bases B and B'. Given a fiber-preserving continuous map $f\colon E \to E'$, there is a map $g\colon B \to B'$ such that

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B' \end{array}$$

is commutative. For every point $x \in B$, the map f induces a map h of the fiber $p^{-1}(x)$ into the fiber $(p')^{-1}(g(x))$, which induces homomorphisms $h^*\colon H^q(F;R) \to H^q(F';R)$ possibly depending on the choice of x. Also, by the cellular approximation theorem and the homotopy lifting property, we can assume that f and g are compatible with the filtrations.

Now, let $\{E_r^{p,q}, d_r^{p,q}\}$ and $\{'E_r^{p,q}, 'd_r^{p,q}\}$ be the associated spectral sequences of (E, B, F, p) and (E', B', F', p') respectively. Then there is a collection of homomorphisms

$$f^* = \{(f^*)_r^{p,q}\colon E_r^{p,q} \to {}'E_r^{p,q}\}$$

that commute with the differentials,

$${}'d_r^{p,q} \circ (f^*)_r^{p,q} = (f^*)_r^{p+r,q-r+1} \circ d_r^{p,q},$$

and satisfy the following conditions:

1. The homomorphism $(f^*)_2^{p,q}$ coincides with the cohomology homomorphism induced by the maps g and h (considering possibly local coefficients).

2. The homomorphism $(f^*)_{r+1}^{p,q}$ is the cohomology homomorphism induced by $(f^*)_r^{p,q}$.

3. The map $(f^*)_\infty^m\colon \bigoplus_{p+q=m} E_\infty^{p,q} \to \bigoplus_{p+q=m} {}'E_\infty^{p,q}$ is induced by the map $f^*\colon H^m(E;R) \to H^m(E';R)$.

The collection $f^* = \{(f^*)_r^{p,q}\}$ is called a ***homomorphism of spectral sequences***. Some times we use the notation

$$\{E_r(E, B, F)\} \xrightarrow{f^*} \{E_r(E', B', F')\}$$

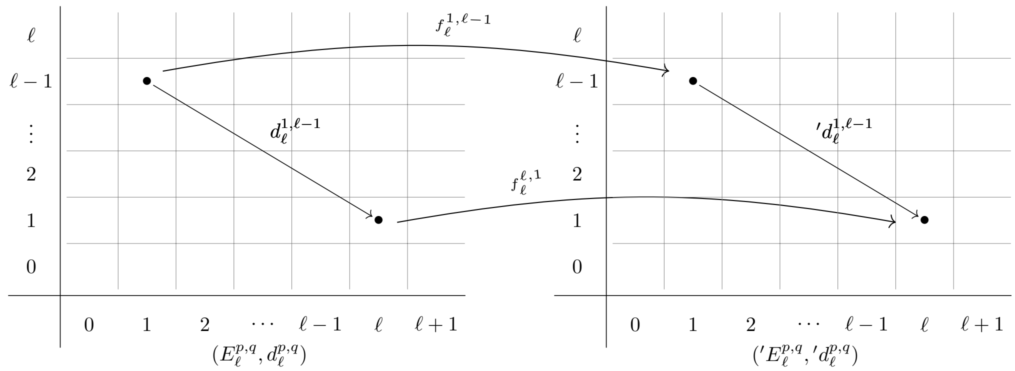

Figure 3.3: Homomorphism between spectral sequences.

to refer to a homomorphism of spectral sequences. In a few words, the previous discussion means that the spectral sequence described in Theorem 3.2.2 is natural with respect to fiber-preserving maps of fibrations. For an illustration of a homomorphism of spectral sequences see Figure 3.3.

There is an obvious but important property of homomorphism of spectral sequences that we cannot overlook. If for some r the homomorphism $E_r \to {'E_r}$ belonging to a homomorphism of spectral sequences is an isomorphism, then so are all the homomorphisms $E_s \to {'E_s}$ with $s > r$ (including $s = \infty$). Moreover, if two homomorphisms of spectral sequences coincide on E_r for some r, then they coincide on E_s for all $s > r$.

Finally, we present the result that justify why we use spectral sequences in this thesis.

Proposition 3.3.1. *Consider a locally trivial fibration $F \hookrightarrow E \xrightarrow{\pi} B$ with B path-connected and such that the system of local coefficients on B induced by F is simple. Then the compositions*

$$H^p(B;R) = E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_p^{p,0} \twoheadrightarrow E_{p+1}^{p,0} = E_\infty^{p,0} \subset H^p(E;R)$$

and

$$H^q(E;R) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F;R)$$

are the homomorphisms

$$\pi^* : H^p(B;R) \longrightarrow H^p(E;R) \quad \textit{and} \quad i^* : H^q(E;R) \longrightarrow H^q(F;R)$$

respectively.

Proof. Consider the diagram of fibrations

$$\begin{array}{ccccc}
F & \xrightarrow{id} & F & \longrightarrow & \mathrm{pt} \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i} & & \downarrow \\
F & \xrightarrow{i} & E & \xrightarrow{\pi} & B \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle id} \\
\mathrm{pt} & \longrightarrow & B & \xrightarrow{id} & B
\end{array}$$

By the naturality of the Leray–Serre spectral sequence, we get induced homomorphisms of spectral sequences

$$\{E_r(B,B,\mathrm{pt})\} \xrightarrow{\pi^*} \{E_r(E,B,F)\} \xrightarrow{i^*} \{E_r(F,\mathrm{pt},F)\}. \tag{3.2}$$

Notice that $E_2(F,\mathrm{pt},F)$ consists of the column $E_2^{0,*} = H^*(F;R)$ and so collapse at the E_2-term. Also $E_2(B,B,\mathrm{pt})$ is a single row, $E_2^{*,0} = H^*(B;R)$, and collapse at the same term. Then the homomorphisms of spectral sequences in (3.2) project $H^0(B;H^*(F;R))$ onto $H^*(F;R)$, and inject $H^*(B;R)$ into $H^*(B;H^0(F;R))$ at E_2. Composing such maps with the corresponding inclusions and projections given by the convergence of the spectral sequence $\{E_r(E,B,F)\}$ we get the following maps:

$$\begin{array}{c} H^*(B;R) \\ \big\downarrow \\ H^*(B;H^0(F;R)) = E_2^{*,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p,0} \subset H^p(E;R) \end{array}$$

and

$$\begin{array}{c} H^q(E;R) \twoheadrightarrow E_\infty^{0,q} \subset \cdots \subset E_2^{0,q} = H^0(B;H^q(F;R)) \\ \big\downarrow \\ H^*(F;R). \end{array}$$

Finally, since the homomorphisms of spectral sequences in (3.2) converge to $\pi^*\colon H^*(B;R) \to H^*(E;R)$ and $i^*\colon H^*(E;R) \to H^*(F;R)$ respectively at E_∞, the proof is completed considering the following commutative diagrams

$$\begin{array}{ccc} H^*(B;R) & \xrightarrow{\pi^*} & \\ \big\downarrow & & \searrow \\ E_2^{*,0} \twoheadrightarrow \cdots & \twoheadrightarrow & E_\infty^{p,0} \subset H^p(E;R) \end{array} \qquad \begin{array}{ccc} H^*(E;R) \twoheadrightarrow E_\infty^{0,q} \subset \cdots & \subset & E_2^{0,q} \\ \searrow^{i^*} & & \big\downarrow \\ & & H^*(F;R). \end{array}$$

□

Examples of how to use these two properties, as well as Proposition 3.3.1, are presented in Chapters 4 and 5.

3.4 A convenient trick to deduce differentials.

In [26] Albrecht Dold presented a very useful result to deduce some differentials of the Leray–Serre spectral sequence associated to a particular kind of fiber bundles. We conclude this chapter by introducing such result.

Let $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$ be vector bundles of rank n and m over the same paracompact space B, and let $f\colon S(E) \to E'$ an odd map, where $S(E)$ is the total space of the sphere bundle associated to π, such a that

$$\begin{array}{ccc} S(E) & \xrightarrow{\ f\ } & E' \\ & {\scriptstyle s\pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & B & \end{array}$$

commutes. Let us define $Z_f = \{z \in S(E) \mid f(z) = 0\}$, where 0 stands for the zero section of π', and the projection maps

$$S(E) \longrightarrow \bar{S}(E) = S(E)/\mathbb{Z}_2 \qquad \text{and} \qquad Z_f \longrightarrow \bar{Z}_f = Z_f/\mathbb{Z}_2,$$

where we are considering the fibrewise antipodal action.

Cohomology H^* is understood in the Čech sense [25, Sec. 3.3][2] and $H^*(B;\mathbb{F}_2)[x]$ is the polynomial ring over $H^*(B;\mathbb{F}_2)$ in one indeterminate x of degree 1. Since the antipodal action is fixed point free in $S(E)$ and Z_f, the projection maps $S(E) \to \bar{S}(E)$ and $Z_f \to \bar{Z}_f$ are 2-sheeted covering maps. Their characteristic classes, denoted by u and u_0 respectively, can be replaced by the indeterminate x and obtain an homomorphism of $H^*(B;\mathbb{F}_2)$-algebras

$$\sigma\colon H^*(B;\mathbb{F}_2)[x] \longrightarrow H^*(\bar{S}(E);\mathbb{F}_2) \longrightarrow H^*(\bar{Z}_f;\mathbb{F}_2)$$

given by $x \longmapsto u \longmapsto u_0$. Dold proved the following result:

Theorem 3.4.1 (Dold's argument)**.** *If $q(x) \in H^*(B;\mathbb{F}_2)[x]$ is such that $\sigma(q(x)) = 0$, then*

$$q(x)W(\pi';x) = W(\pi;x)q'(x)$$

for some $q'(x) \in H^*(B;\mathbb{F}_2)[x]$, where $W(\pi;x) = \sum_{j=0}^{n} w_j(\pi) \otimes x^{n-j}$ is the Stiefel-Whitney polynomial associated to π (similarly $W(\pi';x)$).

The last theorem means that, under the last conditions, $W(\pi;x)$ divides $q(x)W(\pi';x)$. We show the effectiveness of this theorem in the following remark.

Remark 3.4.2. Under the same hypothesis, consider the $\mathbb{Z}_2$-equivariant fiber bundles

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & S(E) \\ & & \big\downarrow {\scriptstyle s\pi} \\ & & B \end{array} \qquad\qquad \begin{array}{ccc} \{0\} & \hookrightarrow & B \times \{0\} \\ & & \big\downarrow {\scriptstyle proj_1} \\ & & B \end{array}$$

where $\mathbb{Z}_2$ acts antipodally on $S(E)$ and trivially on B. Let $f\colon S(E) \longrightarrow B \times \{0\}$ be a $\mathbb{Z}_2$-equivariant map given by $f(e) = (\pi(e), 0)$, such that the following diagram commutes:

$$\begin{array}{ccc} S(E) & \xrightarrow{\ f\ } & B \times \{0\} \\ & {\scriptstyle s\pi}\searrow \quad \swarrow{\scriptstyle proj_1} & \\ & B & \end{array}$$

Notice that $Z_f = S(E)$ and $W(proj_1, x) = 1$. If we consider $q(x)$ as the image of the transgression map $d_n^{0,n-1}$ of the Leray–Serre spectral sequence associated to the sphere bundle

$$S^{n-1} \hookrightarrow E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S(E) \xrightarrow{id_{E\mathbb{Z}_2} \times_{\mathbb{Z}_2} s\pi} E\mathbb{Z}_2 \times_{\mathbb{Z}_2} B = \mathrm{B}\mathbb{Z}_2 \times B,$$

[2] In case that the topological space is homotopy equivalent to a CW-complex, then the Čech cohomology is naturally isomorphic to the singular cohomology.

then

$$\begin{aligned}\sigma \circ q(x) &= \sigma \circ d_n^{0,n-1}(z) \\ &= \big(id_{E\mathbb{Z}_2} \times_{\mathbb{Z}_2} s\pi\big) \circ d_n^{0,n-1}(z) \\ &= 0,\end{aligned}$$

where $H^*(S^{n-1};\mathbb{F}_2) = \mathbb{F}_2[z]/\langle z^2\rangle$. Finally, by Theorem 3.4.1,

$$q(x) = d_n^{0,n-1}(z) = \sum_{j=0}^{n} w_j(\pi) \otimes x^{n-j}.$$

Chapter 4

The Fadell–Husseini index

In this chapter we introduce the main tool of equivariant topology for this work, known as the *Fadell–Husseini index*. To define and study this particular ideal-valued index theory we first recall some facts about the Borel construction of an arbitrary G-space. In addition to important properties of the index, we present some preliminary calculations that are essential for the upcoming results in the next chapters.

4.1 Definition and basic properties.

In 1988 Edward Fadell and Sufian Husseini, in their seminal paper [27], introduced a notion of the ideal-value index theory, a covariant functor $\mathrm{Index}_G(\cdot; R)$ from the category of topological G-spaces into the partially ordered set, seen as a category, of all ideal in the cohomology of ring $H^*(\mathrm{B}G; R)$ ordered by inclusion. Here $\mathrm{B}G$ denotes the classifying space of the group G, and R denotes a commutative ring with unit. This means in particular that if X and Y are G-spaces, and there is a continuous G-equivariant map $X \longrightarrow Y$, then $\mathrm{Index}_G(X; A) \supseteq \mathrm{Index}_G(Y; A)$.

In this chapter we use a slight extension of the original notion of the ideal-valued index theory to the category of all continuous G-equivariant maps from G-spaces to the fixed space B equipped with the trivial G-action. More precisely, let G be a finite group and let $\mathrm{E}G \to \mathrm{B}G$ be the universal $G-$bundle over the classifying space $\mathrm{B}G$. For a $G-$space X we define the *Borel construction* of X with respect to the action of G as the quotient space $\mathrm{E}G \times X/\backsim$, where $(e, x) \backsim (eg^{-1}, gx)$. Since the Borel construction is functorial, every G-equivariant map $\rho\colon X \longrightarrow Y$ induces a $G-$equivariant morphism between the corresponding Borel constructions

$$\rho_G := \mathrm{id} \times_G \rho\colon \mathrm{E}G \times_G X \to \mathrm{E}G \times_G Y,$$

given by $\rho_G(e, x) = (e, \rho(x))$. This gives rise to the following definition:

Definition 4.1.1. Let G be a finite group, and let R be a commutative ring with unit. For a fixed topological space B with trivial G-action, and a G-equivariant map $\rho\colon X \longrightarrow B$, the *Fadell–Husseini index* of ρ with coefficients in R is defined to be the kernel ideal of the induced map ${\rho_G}^*$,

$$\begin{aligned}\mathrm{Index}^B_G\left(\rho; R\right) &:= \ker\left({\rho_G}^*\colon H^*(\mathrm{E}G \times_G B; R) \longrightarrow H^*(\mathrm{E}G \times_G X; R)\right)\\ &= \ker\left({\rho_G}^*\colon H^*_G(B; R) \longrightarrow H^*_G(X; R)\right).\end{aligned}$$

Here $H^*_G(\cdot)$ stands for the equivariant cohomology defined as the cohomology of the Borel construction $\mathrm{E}G \times_G X$ associated to the $G-$space X. In the case when B is a point, $\rho\colon X \longrightarrow B$ is just the constant map and we recover the original definition of the ideal-valued index of the $G-$space X. For that reason we simplify the notation and write

$$\mathrm{Index}^B_G\big(\rho; R\big) = \mathrm{Index}^{pt}_G\big(X; R\big) = \mathrm{Index}_G\big(X; R\big).$$

Let us present some of the essential properties of the index introduced and proved in [28, 27].

Lemma 4.1.2 (*Monotonicity*)**.** *If $\rho\colon X \longrightarrow B$ and $\nu\colon Y \longrightarrow B$ are $G-$equivariant maps, and $f\colon X \longrightarrow Y$ is a $G-$equivariant map such that $\rho = \nu \circ f$, then*

$$\mathrm{Index}^B_G\big(\rho; R\big) \supseteq \mathrm{Index}^B_G\big(\nu; R\big).$$

Proof. Consider the following commutative diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle\rho}\searrow & \swarrow{\scriptstyle\nu} \\ & B & \end{array} \qquad \begin{array}{ccc} \mathrm{E}G\times_G X & \xrightarrow{\ f_G\ } & \mathrm{E}G\times_G Y \\ & {\scriptstyle\rho_G}\searrow \quad \swarrow{\scriptstyle\nu_G} & \\ & \mathrm{E}G\times_G B & \end{array}$$

and the diagram resulting from applying the cohomology functor

$$\begin{array}{ccc} H^*(\mathrm{E}G\times_G X; R) & \xleftarrow{\ f^*_G\ } & H^*(\mathrm{E}G\times_G Y; R) \\ & {\scriptstyle\rho^*_G}\nwarrow \quad \nearrow{\scriptstyle\nu^*_G} & \\ & H^*(\mathrm{E}G\times_G B; R). & \end{array}$$

Since $\rho^*_G = f^*_G \circ \nu^*_G$, then $ker(\rho^*_G) \supseteq ker(\nu^*_G)$. ☐

Lemma 4.1.3 (*Additivity*)**.** *If $(X_1 \cup X_2, X_1, X_2)$ is an excisive triple of G-spaces and $\rho\colon X_1 \bigcup X_2 \longrightarrow B$ is a $G-$equivariant map, then*

$$\mathrm{Index}^B_G(\rho|_{X_1}; R) \cdot \mathrm{Index}^B_G(\rho|_{X_2}; R) \subseteq \mathrm{Index}^B_G(\rho; R).$$

Proof. Let us consider an element $x_1 \in \mathrm{Index}^B_G\big(\rho|_{X_1}; R\big)$, and let us define $x'_1 := \rho^*_G(x_1) \in H^*_G(X; R)$ and the inclusion map $i_1\colon X_1 \hookrightarrow X$. By the uniqueness of ρ_G up to homotopy, we get the following commutative diagram:

$$\begin{array}{ccc} H^*_G(B; R) & \xrightarrow{\ \rho^*_G\ } & H^*_G(X; R) \\ & {\scriptstyle(\rho|_{X_1})^*_G}\searrow & \downarrow{\scriptstyle i^*_1} \\ & & H^*_G(X_1; R), \end{array}$$

where since $(\rho|_{X_1})^*_G(x_1) = 0$, then $i^*_1(x'_1) = 0$. Now using the long exact sequence of the pair (X, X_1),

$$H^*_G(X_1; R) \xleftarrow{i^*_1} H^*_G(X; R) \xleftarrow{j^*_1} H^*_G(X, X_1; R),$$

there is an element y_1 in $H^*_G(X;X_1;R)$ such that $j_1^*(y_1) = x_1'$. In the same way, for an element $x_2 \in \mathrm{Index}^B_G\big(\rho|_{X_2};R\big)$, there is a y_2 in $H^*_G(X;X_2;R)$ such that $j_2^*(y_2) = x_2'$. Then, by the cup product

$$\cup\colon H^*_G(X,X_1;R)\otimes H^*_G(X,X_2;R) \longrightarrow H^*_G(X,X_1\cup X_2;R) = H^*_G(X,X;R) = 0,$$

$y_1 \cup y_2 = 0$ and therefore $x_1' \cup x_2' = j_1^*(y_1)\cup j_2^*(y_2) = (\bar{j}\circ j)^*(y_1\cup y_2) = 0$, where

$$\begin{array}{ccc} (X,\emptyset) & \xrightarrow{\quad j_i \quad} & (X,X_i) \\ & {}_{j}\searrow \quad \nearrow_{\bar{j}} & \\ & (X,X_1\cap X_2). & \end{array}$$

Finally, since $\rho^*_G(x_1\cup x_2) = \rho^*_G(x_1)\cup\rho^*_G(x_2) = x_1'\cup x_2' = 0$, $x_1\cup x_2 \in \mathrm{Index}^B_G(\rho;R)$. □

Let $\rho_1\colon X_1 \to B_1$ and $\rho_2\colon X_2 \to B_2$ be equivariant maps over G_1 and G_2 respectively, and consider the G-equivariant map $\rho := \rho_1\times\rho_2\colon X_1\times X_2 \to B_1\times B_2$ with $G = G_1\times G_2$. The map ρ induces a homomorphism

$$\rho_G\colon (EG_1\times EG_2)\times_G (X_1\times X_2) \to (EG_1\times EG_2)\times_G (B_1\times B_2)$$

which can be identified with

$$\rho_{1G_1}\times\rho_{2G_2}\colon (EG_1\times_{G_1} X_1)\times(EG_2\times_{G_2} X_2) \to (EG_1\times_{G_1} B_1)\times(EG_2\times_{G_2} B_2).$$

In the case that $H^*_{G_i}(B_i;\mathbb{F})$ and $H^*_{G_i}(X_i;\mathbb{F})$ are $\mathbb{F}$-modules over some field $\mathbb{F}$, ρ^*_G can be identified with

$$\rho_1{}^*_{G_1}\otimes\rho_2{}^*_{G_2}\colon H^*_{G_1}(B_1;\mathbb{F})\otimes H^*_{G_2}(B_2;\mathbb{F}) \to H^*_{G_1}(X_1;\mathbb{F})\otimes H^*_{G_2}(X_2;\mathbb{F})$$

via the Künneth formula for field coefficients [29]. We obtain then the following result:

Proposition 4.1.4. *The Fadell–Husseini index of the G-equivariant map*

$$\rho = \rho_1\times\rho_2\colon X_1\times X_2 \to B_1\times B_2,$$

with $G = G_1\times G_2$, is given by

$$\mathrm{Index}^{B_1\times B_2}_{G1\times G_2}\big(\rho;\mathbb{F}\big) = \mathrm{Index}^{B_1}_{G1}\big(\rho_1;\mathbb{F}\big)\otimes H^*_{G_2}(B_2;\mathbb{F}) + H^*_{G_1}(B_1;\mathbb{F})\otimes \mathrm{Index}^{B_2}_{G_2}\big(\rho_2;\mathbb{F}\big).$$

Proof. To simplify the notation we omit the coefficients of the cohomology rings. First, by the right exactness of $\_\otimes H^*_{G_2}(B_2)$ we get that the sequence induced by $\rho_1{}^*_{G_1}$,

$$\ker\rho_1{}^*_{G_1}\otimes H^*_{G_2}(B_2) \hookrightarrow H^*_{G_1}(B_1)\otimes H^*_{G_2}(B_2) \xrightarrow{\rho_1{}^*_{G_1}\otimes\mathrm{id}} \mathrm{im}\,\rho_1{}^*_{G_1}\otimes H^*_{G_2}(B_2), \tag{4.1}$$

is exact. Meanwhile, by the same logic, using the right exactness of $\mathrm{im}\,\rho_1{}^*_{G_1}\otimes\_$ we get that the sequence induced by $\rho_2{}^*_{G_2}$,

$$\mathrm{im}\,\rho_1{}^*_{G_1}\otimes\ker\rho_2{}^*_{G_2} \hookrightarrow \mathrm{im}\,\rho_1{}^*_{G_1}\otimes H^*_{G_2}(B_2) \xrightarrow{\mathrm{id}\otimes\rho_2{}^*_{G_2}} \mathrm{im}\,\rho_1{}^*_{G_1}\otimes\mathrm{im}\,\rho_2{}^*_{G_2}, \tag{4.2}$$

is also exact. Using now the maps in (4.1) and (4.2), we obtain the following commutative diagram

$$\begin{array}{ccccc} & & \operatorname{im} {\rho_1}^*_{G_1} \otimes H^*_{G_2}(B_2) & & \\ & \overset{{\rho_1}^*_{G_1} \otimes \mathrm{id}}{\nearrow} & & \overset{\mathrm{id} \otimes {\rho_2}^*_{G_2}}{\searrow} & \\ H^*_{G_1}(B_1) \otimes H^*_{G_2}(B_2) & & \xrightarrow[{\rho_1}^*_{G_1} \otimes {\rho_2}^*_{G_2}]{} & & \operatorname{im} {\rho_1}^*_{G_1} \otimes \operatorname{im} {\rho_2}^*_{G_2} \end{array}$$

from which we get that ${\rho_1}^*_{G_1} \otimes {\rho_2}^*_{G_2} = [\mathrm{id} \otimes {\rho_2}^*_{G_2}] \circ [{\rho_1}^*_{G_1} \otimes \mathrm{id}]$. Thus, the kernel of the map ${\rho_1}^*_{G_1} \otimes {\rho_2}^*_{G_2}$ restricted to its image is given by

$$\begin{aligned} \ker({\rho_1}^*_{G_1} \otimes {\rho_2}^*_{G_2}) &= \Big[\ker({\rho_1}^*_{G_1} \otimes \mathrm{id})\Big] \oplus \Big[({\rho_1}^*_{G_1} \otimes \mathrm{id})^{-1}\big(\ker(\mathrm{id} \otimes {\rho_2}^*_{G_2})\big)\Big] \\ &= \Big[\ker {\rho_1}^*_{G_1} \otimes H^*_{G_2}(B_2)\Big] \oplus \Big[({\rho_1}^*_{G_1} \otimes \mathrm{id})^{-1}\big(\operatorname{im} {\rho_1}^*_{G_1} \otimes \ker {\rho_2}^*_{G_2}\big)\Big] \\ &= \Big[\ker {\rho_1}^*_{G_1} \otimes H^*_{G_2}(B_2)\Big] \oplus \Big[H^*_{G_1}(B_1) \otimes \ker {\rho_2}^*_{G_2}\Big]. \end{aligned}$$

Finally, since $\operatorname{im} {\rho_1}^*_{G_1}$ and $H^*_{G_2}(X_2)$ are free $\mathbb{F}$-modules, in particular flat, there is an embedding

$$\operatorname{im} {\rho_1}^*_{G_1} \otimes \operatorname{im} {\rho_2}^*_{G_2} \hookrightarrow \operatorname{im} {\rho_1}^*_{G_1} \otimes H^*_{G_2}(X_2) \hookrightarrow H^*_{G_1}(X_1) \otimes H^*_{G_2}(X_2)$$

and therefore we have that the sequence

$$\begin{aligned} 0 \to \big(\ker {\rho_1}^*_{G_1} \otimes H^*_{G_2}(B_2)\big) \oplus \big(H^*_{G_1}(B_1) \otimes \ker {\rho_2}^*_{G_2}\big) \hookrightarrow \\ H^*_{G_1}(B_1) \otimes H^*_{G_2}(B_2) \xrightarrow{{\rho_1}^*_{G_1} \otimes {\rho_2}^*_{G_2}} H^*_{G_1}(X_1) \otimes H^*_{G_2}(X_2) \end{aligned}$$

is exact. □

From Proposition 4.1.4 we get 2 important corollaries.

Corollary 4.1.5. *Under the same hypothesis,*

1. *If we set $X_2 = B_2 = pt$ in Proposition 4.1.4, then*

$$\mathrm{Index}^{B_1}_{G1 \times G_2}\big(\rho_1; \mathbb{F}\big) = \mathrm{Index}^{B_1}_{G1}\big(\rho_1; \mathbb{F}\big) \otimes H^*(\mathrm{B}G_2; \mathbb{F}),$$

   where $G_1 \times G_2$ acts on X_1 and B_1 by $(g_1, g_2) \cdot x = g_1 \cdot x$.

2. *Consider $B_1 = B_2 = pt$ in Proposition 4.1.4. If $H^*(BG_1; \mathbb{F}) = \mathbb{F}[x_1, \dots, x_k]$ and $H^*(BG_2; \mathbb{F}) = \mathbb{F}[y_1, \dots, y_l]$ are polynomial rings over $\mathbb{F}$, and let $\mathrm{Index}^{pt}_{G_1}\big(X_1; \mathbb{F}\big) = \langle f_1, \dots, f_m \rangle$ and $\mathrm{Index}^{pt}_{G_2}\big(X_2; \mathbb{F}\big) = \langle g_1, \dots, g_n \rangle$, then*

$$\mathrm{Index}^{pt}_{G1 \times G_2}\big(X_1 \times X_2; \mathbb{F}\big) = \langle f_1, \dots, f_m, g_1, \dots, g_n \rangle.$$

   is the ideal generated by the polynomials f_i and g_i.

Corollary 4.1.5(2) provides a relation among the three indices $\mathrm{Index}^{pt}_{G1 \times G_2}\big(X_1 \times X_2; \mathbb{F}\big)$, $\mathrm{Index}^{pt}_{G1}\big(X_1; \mathbb{F}\big)$ and $\mathrm{Index}^{pt}_{G_2}\big(X_2; \mathbb{F}\big)$, under suitable conditions.

Examples 4.1.6. We present now some useful examples that we use in Chapter 5:

1. Consider the antipodal $\mathbb{Z}_2$-action on the sphere S^n. We are interested in the Fadell–Husseini index of S^n with coefficients in the field of two elements $\mathbb{F}_2$. That is:

$$\text{Index}^{\text{pt}}_{\mathbb{Z}_2}\left(S^n;\mathbb{F}_2\right) = ker\Big(H^*(B\mathbb{Z}_2;\mathbb{F}_2) \xrightarrow{p^*} H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n;\mathbb{F}_2)\Big),$$

where $H^*(\mathrm{B}\mathbb{Z}_2;\mathbb{F}_2) = \mathbb{F}_2[x]$ is the polynomial ring with one generator in dimension 1. In order to calculate the index $\text{Index}^{\text{pt}}_{\mathbb{Z}_2}\left(S^n;\mathbb{F}_2\right)$ we consider the fiber bundle

$$S^n \hookrightarrow E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n \longrightarrow \mathrm{B}\mathbb{Z}_2$$

induced, via the Borel construction, by the constant $\mathbb{Z}_2$-equivariant map $S^n \to \text{pt}$. The associated Leray–Serre spectral sequence has E_2-term given by

$$E_2^{p,q} = H^p(\mathbb{Z}_2;\mathcal{H}^q(S^n;\mathbb{F}_2)). \tag{4.3}$$

Since S^n is simply connected there are no local coefficients and the E_2-term of the spectral sequence simplifies and becomes

$$E_2^{p,q} = H^p(\mathbb{Z}_2;H^q(S^n;\mathbb{F}_2)) \cong H^p(\mathbb{Z}_2;\mathbb{F}_2)\otimes H^q(S^n;\mathbb{F}_2).$$

In addition, all the differentials of the spectral sequence satisfy the Leibniz rule. Notice first that, because the first differential appears in the E_{n+1}-term, $E_{n+1}^{*,*} = E_2^{*,*}$. Also, since S^n is a free $\mathbb{Z}_2$-space, $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n \simeq S^n/\mathbb{Z}_2$ and then

$$H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n;\mathbb{F}_2) \cong H^*(S^n/\mathbb{Z}_2;\mathbb{F}_2).$$

In particular $H^i(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n;\mathbb{F}_2) = 0$ for all $i > n$. All these properties, together with the information exposed in Chapter 3, give us a very good idea of how the corresponding spectral sequence looks like. For an illustration of the associated Serre spectral sequence see Figure 4.1. Finally, by Proposition 3.3.1,

$$\text{Index}^{\text{pt}}_{\mathbb{Z}_2}\left(S^n;\mathbb{F}_2\right) = \langle x^{n+1}\rangle \subset \mathbb{F}_2[x].$$

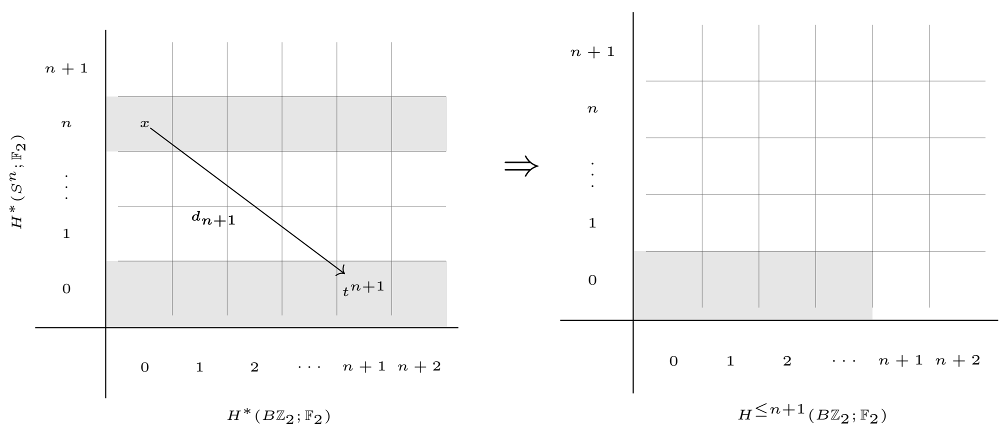

Figure 4.1: Transition between the E_{n+1}-term and the E_{n+2}-term of the spectral sequence 4.3.

2. The previous example can be extended via Corollary 4.1.5 to the product action of $\mathbb{Z}_2^k$ on $S^{n_1} \times \cdots \times S^{n_k}$. As a result of this we get that the Fadell–Husseini index of $S^{n_1} \times \cdots \times S^{n_k}$ with coefficients in $\mathbb{F}_2$ is given by
$$\mathrm{Index}^{\mathrm{pt}}_{\mathbb{Z}_2^k}\big(S^{n_1} \times \cdots \times S^{n_k}; \mathbb{F}_2\big) = \langle x_1^{n_1+1}, \ldots, x_k^{n_k+1}\rangle \subset H^*(\mathrm{B}\mathbb{Z}_2^k; \mathbb{F}_2),$$
where $H^*(\mathrm{B}\mathbb{Z}_2^k; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_k]$ and $\deg(x_1) = \cdots = \deg(x_k) = 1$.

3. Let X and B be $G-$spaces, with the action of G on B trivial, and let
$$\pi_2 \colon X \times B \longrightarrow B$$
be the projection onto the second factor. We are interested in the Fadell–Husseini index of π_2 with coefficients in the field with two elements $\mathbb{F}_2$. That is:
$$\mathrm{Index}^{B}_{G}\big(\pi_2; \mathbb{F}_2\big) := \ker\Big((\mathrm{id} \times_G \pi_2)^* \colon H^*\big(\mathrm{E}G \times_G B; \mathbb{F}_2\big) \longrightarrow H^*\big(\mathrm{E}G^k \times_G (X \times B); \mathbb{F}_2\big)\Big).$$
Since π_2 is a trivial fiber bundle and the G-action on B is trivial we see that induced map
$$\mathrm{id} \times_G \pi_2 \colon \mathrm{E}G \times_G \big(X \times B\big) \longrightarrow \mathrm{E}G \times_G B$$
can be transformed into the following product map
$$\mathrm{id} \times_G \pi_2 = u \times \mathrm{id} \colon \big(\mathrm{E}G \times_G X \times B \longrightarrow \mathrm{B}G \times B\big).$$
Here, the map $u \colon \mathrm{E}G \times_G X \longrightarrow \mathrm{B}X$ is induced, via the Borel construction, by the constant G-equivariant map $X \longrightarrow \mathrm{pt}$. In particular,
$$\mathrm{Index}^{\mathrm{pt}}_{G}\big(X; \mathbb{F}_2\big) = \ker\big(u^* \colon H^*(\mathrm{B}G; \mathbb{F}_2) \longrightarrow H^*(\mathrm{E}G \times_G X; \mathbb{F}_2)\big).$$
Consequently, the induced map in cohomology $(\mathrm{id} \times_G \pi_2)^* = (u \times \mathrm{id})^*$, after application of the Künneth formula for field coefficients [29], becomes the tensor product homomorphism
$$u^* \otimes \mathrm{id} \colon H^*(\mathrm{B}G; \mathbb{F}_2) \otimes H^*(B; \mathbb{F}_2) \longrightarrow H^*(\mathrm{E}G \times_G X; \mathbb{F}_2) \otimes H^*(B; \mathbb{F}_2).$$
Thus,
$$\begin{aligned}\mathrm{Index}^{B}_{G}\big(\pi_2; \mathbb{F}_2\big) &= \ker((\mathrm{id} \times_G \pi_2)^*) \\ &= \ker(u^* \otimes \mathrm{id}) \\ &= \mathrm{Index}^{\mathrm{pt}}_{G}\big(X; \mathbb{F}_2\big) \otimes H^*(B; \mathbb{F}_2).\end{aligned}$$

4.2 The index of sphere representations

We would like to know how to compute the index of a sphere that is not equipped with the antipodal $\mathbb{Z}_2$-action. For that we have the following two propositions:

Proposition 4.2.1. *Let U, V be two G-representations and let $S(U), S(V)$ be the associated G-spheres. Let R be a ring with unit and assume that $H^*(S(U); R)$ and $H^*(S(V); R)$ are trivial G-modules. If*
$$\mathrm{Index}^{pt}_{G}\big(S(U); R\big) = \langle f\rangle \quad \text{and} \quad \mathrm{Index}^{pt}_{G}\big(S(V); R\big) = \langle g\rangle,$$
then
$$\mathrm{Index}^{pt}_{G}\big(S(U \oplus V); R\big) = \langle f \cdot g\rangle \subset H^*(BG; R).$$

In the case of the group $\mathbb{Z}_2^k$, all the irreducible representations are 1-dimensional. Every such representation is identified with a group homomorphism (character) $\chi\colon \mathbb{Z}_2^k \to \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{0,1\}$ is an additive group. This homomorphism is completely determined by the values on generators $\{\varepsilon_1, \dots, \varepsilon_k\}$ of $\mathbb{Z}_2^k$, that is, by the *0-1 vector* $\big(\chi(\varepsilon_1), \dots, \chi(\varepsilon_k)\big)$. For the 0-1 vector $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_2^k$ let $V_\alpha = \langle v_\alpha \rangle \subset \mathbb{R}^{2^k}$ denote the 1-dimensional real $\mathbb{Z}_2^k$-representation defined by:

$$\omega \cdot v_\alpha := (-1)^{\omega_1\alpha_1} \cdots (-1)^{\omega_k\alpha_k} v_\omega = (-1)^{\omega_1\alpha_1+\dots+\omega_k\alpha_k} v_\alpha,$$

for $\omega = (\omega_1, \dots, \omega_k) \in \mathbb{Z}_2^k$. Then there is an isomorphism of $\mathbb{Z}_2^k$-representations:

$$\mathbb{R}^{\mathbb{Z}_2^k} \cong \bigoplus_{\alpha \in \mathbb{Z}_2^k} V_\alpha.$$

Proposition 4.2.2. *1. Let V be the 1-dimensional $\mathbb{Z}_2^k$-representation with the associated 0-1 vector $(\alpha_1, \dots \alpha_k) \in \mathbb{Z}_2^k$. Then*

$$\mathrm{Index}^{pt}_{(\mathbb{Z}_2)^k}\big(S(V); \mathbb{F}_2\big) = \langle \alpha_1 x_1 + \ldots + \alpha_k x_k \rangle \subset \mathbb{F}_2[x_1, \dots, x_k].$$

2. Let U be an n-dimensional $\mathbb{Z}_2^k$-representation with a decomposition $U \cong V_1 \oplus \ldots \oplus V_n$ in 1-dimensional $\mathbb{Z}_2^k$-representations $V_1, \dots, V_n$. If $(\alpha_{1,i}, \dots, \alpha_{k,i}) \in \mathbb{Z}_2^k$ is the associated 0-1 vector of V_i, then by Proposition 4.2.1

$$\mathrm{Index}^{pt}_{(\mathbb{Z}_2)^k}\big(S(U); \mathbb{F}_2\big) = \Big\langle \prod_{i=1}^{n} (\alpha_{1,i} x_1 + \ldots + \alpha_{k,i} x_k) \Big\rangle.$$

Example 4.2.3. For $k = 2$, the coordinate index set for $\mathbb{R}^4$ is $(00, 01, 10, 11)$. Then

$$\begin{array}{ll} v_{00} = (0,0,0,0) & v_{01} = (0,1,0,1) \\ v_{10} = (0,0,1,1) & v_{11} = (0,1,1,0). \end{array}$$

Now, if $V_{10} = \langle v_{10} \rangle$, $V_{01} = \langle v_{01} \rangle$ and $V_{11} = \langle v_{11} \rangle$ are the 1-dimensional real $\mathbb{Z}_2^2$-representations introduced before, then by Proposition 4.2.2

$\mathrm{Index}^{\mathrm{pt}}_{\mathbb{Z}_2^2}\big(S(V_{10}); \mathbb{F}_2\big) = \langle x_1 \rangle$, $\mathrm{Index}^{\mathrm{pt}}_{\mathbb{Z}_2^2}\big(S(V_{01}); \mathbb{F}_2\big) = \langle x_2 \rangle$ and $\mathrm{Index}^{\mathrm{pt}}_{\mathbb{Z}_2^2}\big(S(V_{11}); \mathbb{F}_2\big) = \langle x_1 + x_2 \rangle$,

where $H^*(\mathrm{B}\mathbb{Z}_2^2; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2]$ with $\deg(x_1) = \deg(x_2) = 1$.

4.3 The index of a sphere bundle.

Consider the tautological bundle over the Grassmann manifold $G_\ell(\mathbb{R}^d)$,

$$\xi := \gamma_\ell^d = \big(E(\gamma_\ell^d),\ G_\ell(\mathbb{R}^d),\ E(\gamma_\ell^d) \xrightarrow{\pi} G_\ell(\mathbb{R}^d),\ \mathbb{R}^\ell\big),$$

and the associated sphere bundle of ξ:

$$S\xi := S\gamma_\ell^d = \big(E(S\xi),\ E(S\xi) \xrightarrow{S\pi} G_\ell(\mathbb{R}^d),\ S^{\ell-1}\big).$$

The antipodal action on $S^{\ell-1}$ induces a $\mathbb{Z}_2$-action on $E(S\xi)$ which makes $S\pi$ $\mathbb{Z}_2$-equivariant. Having said that, we prove the Fadell–Husseini index of $S\pi$.

Proposition 4.3.1. *The Fadell–Husseini index of* $S\pi$ *with respect to the introduced* $\mathbb{Z}_2$ *action is given by*

$$\operatorname{Index}_{\mathbb{Z}_2}^{G_\ell(\mathbb{R}^d)}\big(S\pi;\mathbb{F}_2\big)=\Big\langle\sum_{j=0}^{\ell}x^j\otimes w_{\ell-j}(\gamma_\ell^d)\Big\rangle,$$

where $H^*(B\mathbb{Z}_2;\mathbb{F}_2)=\mathbb{F}_2[x]$ *with* $deg(x)=1$.

Proof. Recall that the Fadell–Husseini index of $S\pi$ is described as follows:

$$\operatorname{Index}_{\mathbb{Z}_2}^{G_\ell(\mathbb{R}^d)}\big(S\pi;\mathbb{F}_2\big):=\ker\Big((\mathrm{id}\times_{\mathbb{Z}_2}S\pi)^*\colon H^*\big(\mathrm{B}\mathbb{Z}_2\times G_\ell(\mathbb{R}^d);\mathbb{F}_2\big)\longrightarrow H^*\big(\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2}E(S\xi);\mathbb{F}_2\big)\Big).$$

In order to calculate the index $\operatorname{Index}_{\mathbb{Z}_2}^{G_\ell(\mathbb{R}^d)}\big(S\pi;\mathbb{F}_2\big)$ consider the Borel construction bundle induce by $S\pi$,

$$\mathrm{id}\times_{\mathbb{Z}_2}S\pi\colon \mathrm{E}\mathbb{Z}_2\times_{\mathbb{Z}_2}E(S\xi)\to \mathrm{E}\mathbb{Z}_2\times_{\mathbb{Z}_2}G_\ell(\mathbb{R}^d),$$

where since $\mathbb{Z}_2$ acts trivially on $G_\ell(\mathbb{R}^d)$, $\mathrm{E}\mathbb{Z}_2\times_{\mathbb{Z}_2}G_\ell(\mathbb{R}^d)\simeq \mathrm{B}\mathbb{Z}_2\times G_\ell(\mathbb{R}^d)$. The associated Leray–Serre spectral sequence has E_2-term given by

$$E_2^{p,q}=H^p\big(\mathrm{B}\mathbb{Z}_2\times G_\ell(\mathbb{R}^d);\mathcal{H}^q(S^{\ell-1};\mathbb{F}_2)\big),$$

where the local coefficient system is determined by the fundamental group of $S^{\ell-1}$. Since $\pi_1(S^{\ell-1})=0$, there are no local coefficients and the E_2-term simplifies to

$$\begin{aligned}E_2^{p,q}&=H^p\big(\mathrm{B}\mathbb{Z}_2\times G_\ell(\mathbb{R}^d);H^q(S^{\ell-1};\mathbb{F}_2)\big)\\&=H^p(\mathrm{B}\mathbb{Z}_2\times G_\ell(\mathbb{R}^d);\mathbb{F}_2)\otimes H^q(S^{\ell-1};\mathbb{F}_2).\end{aligned}$$

Also, because

$$H^*(S^{\ell-1};\mathbb{F}_2)\cong\mathbb{F}_2[\bar{y}]/\langle\bar{y}^2\rangle,\ \text{with}\ \deg(\bar{y})=\ell-1,$$

then $E_2^{*,*}\cong E_\ell^{*,*}$. This means that the first non-trivial differential appears on the E_ℓ-term and by Theorem 3.4.1 applied to the fiber bundle $id\times_{\mathbb{Z}_2}S\pi$ we have that

$$d_\ell^{0,\ell-1}(\bar{y})=\sum_{j=0}^{\ell}x^j\otimes w_{\ell-j}(\gamma_\ell^d),$$

where $w_0(\xi),\dots,w_\ell(\xi)$ are the Stiefel–Whitney classes of the tautological vector bundle γ_ℓ^d. Finally, since all the differentials satisfy the Leibniz rule, $d_\ell^{0,\ell-1}$ determine all the differentials of the associated spectral sequence and therefore the kernel of $(\mathrm{id}\times_{\mathbb{Z}_2}S\pi)^*$. □

We can extend the idea of Proposition 4.3.1 in the following way: Let us start by considering the inclusion $i_r\colon\mathbb{Z}_2\to\mathbb{Z}_2^k$ into the r-th summand. Then there is an action of $\mathbb{Z}_2^k$ on $S^{\ell-1}$ described as follows: all $\mathbb{Z}_2^k$ acts trivial except the r-th summand which acts antipodally on the sphere. Using the introduced $\mathbb{Z}_2^k$-action we get our last result of the chapter.

Proposition 4.3.2. *The Fadell–Husseini index of* $S\gamma_\ell^d$ *with respect to the* $\mathbb{Z}_2^k$ *action described in the previous paragraph is given by*

$$\operatorname{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(S\pi;\mathbb{F}_2\big)=\Big\langle\sum_{j=0}^{\ell}x_r^j\otimes w_{\ell-j}(\gamma_\ell^d)\Big\rangle,$$

where $H^*(B\mathbb{Z}_2^k;\mathbb{F}_2)=\mathbb{F}_2[x_1,\dots,x_k]$, *with* $\deg(x_1)=\cdots=\deg(x_k)=1$.

Proof. Given the inclusion map i_r, there is an induced map between the corresponding classifying spaces

$$\mathrm{B}\mathbb{Z}_2 \xrightarrow{\mathrm{B}(i_r)} \mathrm{B}\mathbb{Z}_2^k,$$

which induces a map between the Borel construction bundles

$$\begin{array}{ccc} E\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E\big(S\gamma_\ell^d\big) & \longleftarrow & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} E\big(S\gamma_\ell^d\big) \\ {\scriptstyle id\times_{\mathbb{Z}_2{}^k} S\pi}\downarrow & & \downarrow{\scriptstyle id\times_{\mathbb{Z}_2} S\pi} \\ B\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d) & \xleftarrow{B(i_r)\times \mathrm{id}} & B\mathbb{Z}_2 \times G_\ell(\mathbb{R}^d). \end{array}$$

This morphism of bundles induces a morphism of the corresponding Serre spectral sequences which on the zero column of the E_2-term is an isomorphism. If we focus on the spectral sequence associated to the bundle $id \times_{\mathbb{Z}_2} S\pi$, by Proposition 4.3.1 we get that the only transgression map that appears is given by

$$d_\ell^{0,\ell-1}(\bar{y}) = \sum_{j=0}^{\ell} x^j \otimes w_{\ell-j}(\gamma_\ell^d),$$

where
- $\langle \bar{y} \rangle = H^{\ell-1}(S^{\ell-1};\mathbb{F}_2) \cong E_2^{0,\ell-1} \cong E_\ell^{0,\ell-1} \cong \mathbb{F}_2$,
- $H^*(\mathrm{B}\mathbb{Z}_2;\mathbb{F}_2) = \mathbb{F}_2[x]$, with $\deg(x) = 1$, and
- $w_0(\xi), \dots, w_\ell(\xi)$ are Stiefel–Whitney classes of the tautological vector bundle γ_ℓ^d.

Then, by the commutativity of the differentials, we conclude that the differential $d_\ell^{0,\ell-1}$ of spectral sequences associated to $id \times_{\mathbb{Z}_2{}^k} S\pi$ is given by

$$d_\ell^{0,\ell-1}(y) = \sum_{j=0}^{\ell} x_r^j \otimes w_{\ell-j}(\gamma_\ell^d).$$

The proof is complete since all the differentials of the spectral sequence satisfy the Leibniz rule. □

Chapter 5

The Grünbaum–Hadwiger–Ramos problem for mass assignments

In this chapter we consider an extension of the classical Grünbaum–Hadwiger–Ramos mass partition problem to the so-called mass assignments.

Let d, ℓ, k and j be natural numbers with $1 \leq \ell \leq d-1$. Consider a collection of j mass assignments $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ on the Grassmann manifold $G_\ell(\mathbb{R}^d)$. Our aim is to find a linear subspace $L \in G_\ell(\mathbb{R}^d)$, and a k-arrangement $\mathcal{H}^L = (H_1^L, \ldots, H_k^L)$ in L, such that $\mathcal{H}^L$ equiparts the collection of masses $\mathcal{M}^L = (\mu_1^L, \ldots, \mu_j^L)$. That is, we are looking for a mass assignment admissible 4-tuple.

The proof of this new problem has two stages: First we rewrite the problem as a parametrized Borsuk–Ulam type question to derive an appropriate configuration space/test map scheme. As we mentioned in Chapter 2, this translate our partition problem to an equivariant one. Second, we use the ideal-valued index theory and the computations of Chapter 4 to solve the equivariant problem.

5.1 Our problem as a parametrized Borsuk–Ulam type question.

In order to derive an appropriate *CS/TM scheme* and make our topological methods work correctly, we consider an additional assumption on the first mass assignment μ_1. We assume that for every linear subspace $L \in G_\ell(\mathbb{R}^d)$ the associated mass μ_1^L has compact and connected support. This implies that for every direction in L there exists a unique oriented affine hyperplane, orthogonal to the direction, equiparting the mass μ_1^L. Or in other words, the space of all oriented affine hyperplanes in L which equiparts μ_1^L is homeomorphic to $S^{dim(L)-1}$.

In the remainder of this section we describe all the ingredients to apply the CS/TM scheme to our problem.

5.1.1 The configuration space.

For the configuration space associated to our problem, or in other words the space of all solution candidates, we take collections of k-arrangements in each of the linear space $L \in G_\ell(\mathbb{R}^d)$ such that each of the k affine hyperplanes in L equiparts the mass μ_1^L (given by the first mass assignment μ_1).

More precisely, let us first consider the tautological bundle on the Grassmann manifold $G_\ell(\mathbb{R}^d)$:

$$\xi := \gamma_\ell^d = \big(E(\gamma_\ell^d),\ B(\gamma_\ell^d) = G_\ell(\mathbb{R}^d),\ E(\gamma_\ell^d) \xrightarrow{\pi} B(\gamma_\ell^d),\ F(\gamma_\ell^d) = \mathbb{R}^\ell\big).$$

The assumption on the mass assignment μ_1 allows us to see the associated sphere bundle of ξ:

$$S\xi := S\gamma_\ell^d = \big(E(S\xi),\ B(S\xi) = G_\ell(\mathbb{R}^d),\ E(S\xi) \xrightarrow{S\pi} B(S\xi),\ F(S\xi) = S^{\ell-1}\big),$$

as the space of all oriented affine hyperplanes of linear subspaces $L \in G_\ell(\mathbb{R}^d)$ which equipart corresponding masses μ_1^L. In this way we already obtained a configuration space for the case $k = 1$.

For $k \geq 2$, that is for arrangements with more than one hyperplane, we proceed as follows. Consider the k-fold product bundle $(S\xi)^k$:

$$(S\xi)^k = (E(S\xi)^k,\ G_\ell(\mathbb{R}^d)^k,\ E(S\xi)^k \xrightarrow{q_k := (S\pi)^k} G_\ell(\mathbb{R}^d)^k,\ (S^{\ell-1})^k),$$

and take the pullback along the diagonal embedding $\Delta_k \colon G_\ell(\mathbb{R}^d) \longrightarrow G_\ell(\mathbb{R}^d)^k$:

$$\begin{array}{ccc} E\big(\Delta_k^*((S\xi)^k)\big) & \longrightarrow & E(S\xi)^k \\ \big\downarrow p_k & & \big\downarrow q_k \\ G_\ell(\mathbb{R}^d) & \xrightarrow{\Delta_k} & G_\ell(\mathbb{R}^d)^k. \end{array}$$

The space of all solution candidates, associated to the parameters (d, ℓ, j, k), is the total space of the pullback bundle:

$$\begin{aligned} \mathcal{C}(d,\ell,k) &:= E\big(\Delta_k^*((S\xi)^k)\big) \\ &= \{(L; v_1, \ldots, v_k) \mid L \in G_\ell(\mathbb{R}^d), v_i \in L,\ \|v_1\| = \cdots = \|v_k\| = 1\}. \end{aligned}$$

This means that $\mathcal{C}(d,\ell,k)$ is the total space of the fiber bundle $\Delta_k^*((S\xi)^k)$:

$$(S^{\ell-1})^k \longrightarrow \mathcal{C}(d,\ell,k) \xrightarrow{p_k} G_\ell(\mathbb{R}^d), \tag{5.1}$$

where the map p_k is given by $(L; v_1, \ldots, v_k) \longmapsto L$. Recall that by our assumption for every $(L; v_1, \ldots, v_k) \in \mathcal{C}(d,\ell,k)$ each of the vectors v_i parametrizes the oriented affine hyperplane in L orthogonal to v_i, oriented by v_i, which equiparts the mass μ_1^L.

The Weyl group $\mathfrak{S}_k^{\pm} = \mathbb{Z}_2^k \rtimes \mathfrak{S}_k$, also called the group of signed permutations, acts naturally, and fiberwise, on $\mathcal{C}(d,\ell,k)$ by

$$((\beta_1, \ldots, \beta_k) \rtimes \tau) \cdot (L; v_1, \ldots, v_k) = (L; (-1)^{\beta_1} v_{\tau^{-1}(1)}, \ldots, (-1)^{\beta_k} v_{\tau^{-1}(k)}),$$

for $((\beta_1, \ldots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ and $(L; v_1, \ldots, v_k) \in \mathcal{C}(d,\ell,k)$.

5.1.2 The test space.

Consider the real vector space $\mathbb{R}^{\mathbb{Z}_2^k}$ and its codimension 1 subspace:

$$U_k := \Big\{(y_\alpha)_{\alpha\in\mathbb{Z}_2^k} \in \mathbb{R}^{\mathbb{Z}_2^k} : \sum_{\alpha\in\mathbb{Z}_2^k} y_\alpha = 0\Big\}.$$

The structure of a real $\mathfrak{S}_k^{\pm}$-representation on the vectors space $\mathbb{R}^{\mathbb{Z}_2^k}$ is induced by: the element $((\beta_1,\dots,\beta_k)\rtimes\tau) \in \mathfrak{S}_k^{\pm}$ acts on the vector $(y_{(\alpha_1,\dots,\alpha_k)})_{(\alpha_1,\dots,\alpha_k)\in\mathbb{Z}_2^k} \in \mathbb{R}^{\mathbb{Z}_2^k}$ by permuting the indices as follows:

$$((\beta_1,\dots,\beta_k)\rtimes\tau)\cdot(\alpha_1,\dots,\alpha_k) = (\beta_1+\alpha_{\tau^{-1}(1)},\dots,\beta_k+\alpha_{\tau^{-1}(k)}).$$

Here the addition is assumed to be in $\mathbb{Z}_2$. With respect to this action, the subspaces U_k is a $\mathfrak{S}_k^{\pm}$-subrepresentation of $\mathbb{R}^{\mathbb{Z}_2^k}$.

Let us first consider $\mathbb{R}^{\mathbb{Z}_2^k}$ as an $\mathbb{Z}_2^k$-representation where $\mathbb{Z}_2^k \subseteq \mathbb{Z}_2^k \rtimes \mathfrak{S}_k = \mathfrak{S}_k^{\pm}$. For $\omega := (\omega_1,\dots,\omega_k) \in \mathbb{Z}_2^k$ we denote by V_ω the 1-dimensional real $\mathbb{Z}_2^k$-representation defined by:

$$\alpha\cdot v := (-1)^{\omega_1\alpha_1}\cdots(-1)^{\omega_k\alpha_k}v = (-1)^{\omega_1\alpha_1+\cdots+\omega_k\alpha_k}v$$

for $\alpha = (\alpha_1,\dots,\alpha_k) \in \mathbb{Z}_2^k$ and $v \in V_\omega$. Then there are isomorphisms of $\mathbb{Z}_2^k$-representations:

$$\mathbb{R}^{\mathbb{Z}_2^k} \cong \bigoplus_{\omega\in\mathbb{Z}_2^k} V_\omega \qquad \text{and} \qquad U_k \cong \bigoplus_{\omega\in\mathbb{Z}_2^k\backslash\{0\}} V_\omega. \tag{5.2}$$

Notice that, under the previous isomorphism, V_ω corresponds to the subspace of $\mathbb{R}^{\mathbb{Z}_2^k}$ generated by the vector with αth coordinate $(-1)^{\omega_1\alpha_1+\cdots+\omega_k\alpha_k}$.

In order to obtain a decomposition of $\mathbb{R}^{\mathbb{Z}_2^k}$, now as an $\mathfrak{S}_k^{\pm}$-representation, let us first partition $\mathbb{Z}_2^k$, as a set, into the disjoint union $\mathbb{Z}_2^k = A_0 \sqcup A_1 \sqcup\cdots\sqcup A_k$ where, for $0\le i\le k$, we define

$$A_i := \{(\omega_1,\dots,\omega_k)\in\mathbb{Z}_2^k : \omega_1+\cdots+\omega_k = i\}.$$

Addition is assumed to be in $\mathbb{Z}$. It is not hard to see that for every $0\le i\le k$ the direct sum $W_i := \bigoplus_{\omega\in A_i} V_\omega$ is a $\mathfrak{S}_k^{\pm}$-representation, and in addition there are isomorphisms of $\mathfrak{S}_k^{\pm}$-representations:

$$\mathbb{R}^{\mathbb{Z}_2^k} \cong W_0\oplus W_1\oplus\cdots\oplus W_k \qquad \text{and} \qquad U_k \cong W_1\oplus\cdots\oplus W_k.$$

We set $U_k' := W_2\oplus\cdots\oplus W_k$, and consequently $U_k \cong W_1 \oplus U_k'$.

The test space, associated to the parameters (d,ℓ,j,k), is the total space of the following trivial vector bundle π_2:

$$U_k'\oplus(U_k)^{\oplus j} \longrightarrow (U_k'\oplus(U_k)^{\oplus j-1})\times G_\ell(\mathbb{R}^d) \longrightarrow G_\ell(\mathbb{R}^d). \tag{5.3}$$

The reason for such a choice of the test space becomes clear in the next section and in Theorem 5.1.1. The group $\mathfrak{S}_k^{\pm}$ acts on the product $(U_k'\oplus(U_k)^{\oplus j-1})\times G_\ell(\mathbb{R}^d)$ diagonally where the action on the Grassmann manifold $G_\ell(\mathbb{R}^d)$ is assumed to be trivial.

5.1.3 The test map.

We recall that the parameters (d,ℓ,j,k) are fixed and in addition we have fixed a collection of j mass assignments $\mathcal{M}=(\mu_1,\dots,\mu_j)$ on the Grassmann manifold $G_\ell(\mathbb{R}^d)$. The test map associated to the collection $\mathcal{M}$ is the bundle map

$$\phi_{\mathcal{M}}\colon \mathcal{C}(d,\ell,k)\longrightarrow (U_k'\oplus(U_k)^{\oplus j-1})\times G_\ell(\mathbb{R}^d)$$

defined by

$$\begin{aligned}(L;v_1,\dots,v_k)\longmapsto\Big(&\big(\mu_1^L(H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k})-\tfrac{1}{2^k}\mu_1^L(L)\big)_{(\alpha_1,\dots,\alpha_k)\in\mathbb{Z}_2^k},\\ &\big(\big(\mu_i^L(H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k})-\tfrac{1}{2^k}\mu_i^L(L)\big)_{(\alpha_1,\dots,\alpha_k)\in\mathbb{Z}_2^k}\big)_{2\le i\le j};L\Big).\end{aligned}$$

The fact that

$$\big(\mu_1^L(H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k})-\tfrac{1}{2^k}\big)_{(\alpha_1,\dots,\alpha_k)\in\mathbb{Z}_2^k}\ \in U_k'$$

is a consequence of our assumption that for every $(L;v_1,\dots,v_k)\in\mathcal{C}(d,\ell,k)$ each of the vectors v_i parametrizes the oriented affine hyperplane in L orthogonal to v_i, oriented by v_i, which equiparts the mass μ_1^L. More precisely, let us consider $\omega^i=(\omega_1^i,\dots,\omega_k^i)\in\mathbb{Z}_2^k$ such that $\omega_i^i=1$ and $\omega_j^i=0$ for every $i\neq j$. By the isomorphism of $\mathbb{Z}_2^k$-representations (5.2), V_{ω^i} corresponds to the subspace of $\mathbb{R}^{\mathbb{Z}_2^k}$ generated by the vector with αth coordinate $(-1)^{\omega_i\alpha_i}$, which means that the αth coordinate is positive only when $\alpha_i=0$. To simplify the notation, let us denote by $\mathcal{O}^{\mathcal{H}}_{\alpha_1,\dots,\alpha_k}$ the intersection $H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k}$. Since each affine hyperplane H_{v_i} in L equiparts the mass μ_1^L, the inner product between $\big(\mu_1^L(H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k})-\frac{1}{2^k}\big)_{\alpha\in\mathbb{Z}_2^k}$ and the generator of V_{ω^i} is given by

$$\begin{aligned}\Big\langle\big(\mu_1^L(\mathcal{O}^{\mathcal{H}}_{\alpha_1,\dots,\alpha_k})-\tfrac{1}{2^k}\big)_{\alpha\in\mathbb{Z}_2^k},\big((-1)^{(\alpha,\omega^i)}\big)_{\alpha\in\mathbb{Z}_2^k}\Big\rangle&=\Big[\sum\mu_1^L(\mathcal{O}^{\mathcal{H}}_{\alpha_1,\dots,0,\dots,\alpha_k})+\tfrac{1}{2}\mu_1^L(L)\Big]\\&\quad-\Big[\sum\mu_1^L(\mathcal{O}^{\mathcal{H}}_{\alpha_1,\dots,1,\dots,\alpha_k})+\tfrac{1}{2}\mu_1^L(L)\Big]\\&=0,\end{aligned}$$

for every $1\le i\le k$. Finally, we conclude that

$$\big(\mu_1^L(H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k})-\tfrac{1}{2^k}\mu_1^L(L)\big)_{(\alpha_1,\dots,\alpha_k)\in\mathbb{Z}_2^k}\in W_1^{\perp}.$$

The test map $\phi_{\mathcal{M}}$ is (fiberwise) $\mathfrak{S}_k^{\pm}$-equivariant map with respect to the already introduced actions of $\mathfrak{S}_k^{\pm}$ on the configuration space $\mathcal{C}(d,\ell,k)$ and on the test space $(U_k'\oplus(U_k)^{\oplus j-1})\times G_\ell(\mathbb{R}^d)$. The key property of the construction we have made is that *for the given collection of mass assignments $\mathcal{M}$ there exists a k arrangement in the linear subspace $L\in G_\ell(\mathbb{R}^d)$ which equiparts the collection of masses $\mathcal{M}^L$ if and only if $(L;0,0)\in$* $\mathrm{im}(\phi_{\mathcal{M}})\cap\big(U_k'\oplus U_k^{\oplus j-1}\times G_\ell(\mathbb{R}^d)\big)$. Consequently we have proved the following theorem.

Theorem 5.1.1. *Let d, ℓ, k and j be natural numbers with $1\le\ell\le d-1$. Assume that $\Delta_k^*((S\xi)^k)$ and π_2 are already introduced bundles in (5.1) and (5.3), and denote by $S\pi_2$ the associated sphere bundle.*

(a) *If (d,ℓ,j,k) is not a mass assignment admissible, then there exists a $\mathfrak{S}_k^{\pm}$-equivariant bundle map $\Delta_k^*((S\xi)^k)\longrightarrow S\pi_2$.*

(b) *If there in no $\mathfrak{S}_k^{\pm}$-equivariant bundle map $\Delta_k^*((S\xi)^k)\longrightarrow S\pi_2$, then (d,ℓ,j,k) is a mass assignment admissible.*

5.2 Fadell–Husseini index calculations.

From section 5.1 we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(d,\ell,k) & \xrightarrow{\phi_{\mathcal{M}}} & S(U_k' \oplus (U_k)^{\oplus j-1}) \times G_\ell(\mathbb{R}^d) \\ & {\scriptstyle \rho_k} \searrow \quad \swarrow {\scriptstyle S\pi_2} & \\ & G_t(\mathbb{R}^d) & \end{array} \tag{5.4}$$

associated to the parameters (d, ℓ, j, k). Here ρ_k and $S\pi_2$ are the introduced fiber bundles which define the configuration space and the test space respectively. Notice that, by Theorem 5.1.1, we are looking for the non-existence of the $\mathfrak{S}_k^{\pm}$-equivariant map $\phi_{\mathcal{M}}$ in 5.4. However, it is enough to prove that the restriction of $\phi_{\mathcal{M}}$ to the subgroup $\mathbb{Z}_2^k$ of the group of signed permutations $\mathfrak{S}_k^{\pm}$ does not exist. The non-existence of the $\mathbb{Z}_2^k$-equivariant map $\phi_{\mathcal{M}}$ can be established by using the properties of the Fadell–Husseini index introduced in Chapter 4. For that purpose, in this section we compute the Fadell–Husseini indices of the fiber bundles ρ_k and $S\pi_2$ with respect to the action of the subgroup $\mathbb{Z}_2^k \subset \mathfrak{S}_k^{\pm}$.

5.2.1 The Fadell–Husseini index of the test space.

Let us start computing the Fadell–Husseini index of the projection map

$$S\pi_2\colon S(U_k' \oplus (U_k)^{\oplus j-1}) \times G_\ell(\mathbb{R}^d) \longrightarrow G_\ell(\mathbb{R}^d)$$

with respect to the action of the subgroup $\mathbb{Z}_2^k \subseteq \mathfrak{S}_k^{\pm}$, and with coefficients in $\mathbb{F}_2$. This means we are looking for:

$$\begin{aligned} &\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(S\pi_2;\mathbb{F}_2\big) := \\ &\qquad \ker\Big((\mathrm{id}\times_{\mathrm{E}\mathbb{Z}_2^k}\pi_2)^*\colon H^*\big(\mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} G_\ell(\mathbb{R}^d);\mathbb{F}_2\big) \longrightarrow \\ &\qquad\qquad H^*\big(\mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} (S(U_k' \oplus (U_k)^{\oplus j-1} \times G_\ell(\mathbb{R}^d));\mathbb{F}_2\big)\Big). \end{aligned}$$

Since π_2 is a trivial fiber bundle, and the $\mathbb{Z}_2^k$-action on $G_\ell(\mathbb{R}^d)$ is trivial, by the example 3 in 4.1.6 and Proposition 4.2.2 we obtain the following result:

Theorem 5.2.1. *The Fadell–Husseini index of the projection map*

$$S\pi_2\colon S(U_k' \oplus (U_k)^{\oplus j-1}) \times G_\ell(\mathbb{R}^d) \longrightarrow G_\ell(\mathbb{R}^d)$$

with respect to the $\mathbb{Z}_2^k$ action described in 5.1.2 is given by

$$\begin{aligned} &\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(S\pi_2;\mathbb{F}_2\big) = \\ &\qquad \Big\langle \prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k\setminus\Gamma} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j \Big\rangle \otimes H^*(G_\ell(\mathbb{R}^d);\mathbb{F}_2), \end{aligned} \tag{5.5}$$

where $\Gamma = \{(0,\dots,0),(1,\dots,0),\dots,(0,\dots,1)\}$ and $H^*(B\mathbb{Z}_2^k;\mathbb{F}_2) = \mathbb{F}_2[x_1,\dots,x_k]$, with $\mid x_1 \mid = \cdots = \mid x_k \mid = 1$.

For the cohomology of the real Grassmann manifold we recall the classical result of Borel [30, p. 190], which gives a presentation of the cohomology with $\mathbb{F}_2$ coefficients in the form of the truncated polynomial ring as follows:

$$H^*(G_\ell(\mathbb{R}^d);\mathbb{F}_2) \cong \mathbb{F}_2[w_1,\dots,w_\ell,\overline{w}_1,\dots,\overline{w}_{d-\ell}]/I_{\ell,d},$$

where $\deg(w_i)=i$, $\deg(\overline{w}_j)=j$ for $1\le i\le \ell$, $1\le j\le d-\ell$, and $I_{\ell,d}$ is the ideal generated by the d graded components of the equality

$$(1+w_1+\dots+w_\ell)(1+\overline{w}_1+\dots+\overline{w}_{d-\ell})=1.$$

In other words, the ideal $I_{\ell,d}$ is generated by the polynomials:

$$\sum_{j=\max\{0,s+\ell-d\}}^{\min\{s,\ell\}} w_j\cdot\overline{w}_{s-j}, \qquad 1\le s\le d.$$

Here the generator w_i, for $1\le i\le\ell$, can be identify with the Stiefel–Whitney classes of the canonical bundle γ_ℓ^d while the remaining generators $\overline{w}_i$, for $1\le i\le d-\ell$, can be identify with the dual Stiefel–Whitney classes of γ_ℓ^d.

5.2.2 The Fadell–Husseini index of the configuration space.

In Section 5.1.1 we have defined the configuration space as the total space of the pull-back bundle

$$\begin{array}{ccc} \mathcal{C}(d,\ell,k)=E\big(\Delta_k^*((S\xi)^k)\big) & \longrightarrow & E(S\xi)^k \\ \downarrow & & \downarrow{\scriptstyle q_k} \\ G_\ell(\mathbb{R}^d) & \xrightarrow{\ \Delta_k\ } & G_\ell(\mathbb{R}^d)^k. \end{array}$$

where Δ_k is the diagonal embedding. More precisely,

$$\begin{aligned}\mathcal{C}(d,\ell,k) &= E\big(\Delta_k^*((S\xi)^k)\big)\\ &=\{(L;v_1,\dots,v_k)\mid L\in G_\ell(\mathbb{R}^d), v_i\in L,\ \|v_1\|=\dots=\|v_k\|=1\}.\end{aligned}$$

In this section we determine the Fadell–Husseini index of the map

$$p_k\colon \mathcal{C}(d,\ell,k)\longrightarrow G_\ell(\mathbb{R}^d)$$

with respect to the action of the subgroup $\mathbb{Z}_2^k\subseteq\mathfrak{S}_k^{\pm}$ and with coefficients in the field with two elements $\mathbb{F}_2$. In other words, we describe:

$$\begin{aligned}&\operatorname{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k;\mathbb{F}_2\big) := \\ &\qquad \ker\Big((\mathrm{id}\times_{\mathbb{Z}_2^k}p_k)^*\colon H^*\big(\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}G_\ell(\mathbb{R}^d);\mathbb{F}_2\big)\longrightarrow \\ &\qquad\qquad H^*\big(\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}\mathcal{C}(d,\ell,k);\mathbb{F}_2\big)\Big).\end{aligned}$$

The computation of the index $\operatorname{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k;\mathbb{F}_2\big)\subseteq H^*\big(\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}G_\ell(\mathbb{R}^d);\mathbb{F}_2\big)$ is done in two steps. First, we describe the index

$$\operatorname{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)^k}\big(q_k;\mathbb{F}_2\big)\subseteq H^*\big(\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}G_\ell(\mathbb{R}^d)^k;\mathbb{F}_2\big),$$

and then show that

$$\operatorname{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k;\mathbb{F}_2\big)=(\Delta_k)^*\big(\operatorname{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)^k}\big(q_k;\mathbb{F}_2\big)\big).$$

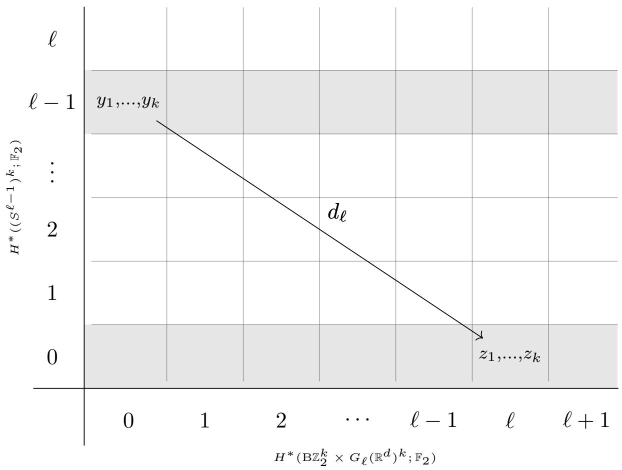

Figure 5.1: The E_ℓ-term of the spectral sequence (5.6).

Index of q_k

The Fadell–Husseini index of the map $q_k\colon E(S\xi)^k \longrightarrow G_\ell(\mathbb{R}^d)^k$ is, by definition, the kernel:

$$\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(q_k;\mathbb{F}_2\big) := \ker\Big((\mathrm{id}\times_{\mathbb{Z}_2^k} q_k)^*\colon H^*\big(\mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} G_\ell(\mathbb{R}^d)^k;\mathbb{F}_2\big) \longrightarrow H^*\big(\mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E(S\xi)^k;\mathbb{F}_2\big)\Big).$$

In oder to identify the index $\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(q_k;\mathbb{F}_2\big)$ we consider the fiber bundle

$$(S^{\ell-1})^k \longrightarrow \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E(S\xi)^k \longrightarrow \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} G_\ell(\mathbb{R}^d)^k = \mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d)^k.$$

The associated Serre spectral sequence has E_2-term given by

$$E_2^{i,j} = H^i(\mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d)^k; \mathcal{H}^j((S^{\ell-1})^k;\mathbb{F}_2)),$$

where the local coefficient system is determined by the action of the fundamental group of the base space. Since we are considering coefficients in the field $\mathbb{F}_2$, the fundamental group

$$\pi_1(\mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d)^k) \cong \pi_1(B\mathbb{Z}_2^k) \times \pi_1(G_\ell(\mathbb{R}^d))^k \cong \mathbb{Z}_2^k \times \mathbb{Z}_2^{\oplus k}$$

acts trivially on the cohomology of the fiber $H^j((S^{\ell-1})^k;\mathbb{F}_2)$, and consequently the E_2-term of the spectral sequence simplifies and becomes

$$E_2^{i,j} \cong H^i(\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d)^k;\mathbb{F}_2) \otimes H^j((S^{\ell-1})^k;\mathbb{F}_2). \tag{5.6}$$

In addition, all the differentials of the spectral sequence satisfy the Leibniz rule.

Let us denote the cohomology of the fiber $(S^{\ell-1})^k$ as follows

$$H^*((S^{\ell-1})^k;\mathbb{F}_2) \cong H^*(S^{\ell-1};\mathbb{F}_2)^{\otimes k} \cong \mathbb{F}_2[y_1]/(y_1^2) \otimes \cdots \otimes \mathbb{F}_2[y_k]/(y_k^2) \cong \mathbb{F}_2[y_1,\cdots,y_k]/(y_1^2,\cdots,y_k^2)$$

where $\deg(y_1) = \cdots = \deg(y_k) = \ell - 1$.

Now we determine the first possible non-trivial differentials d_ℓ by finding its values on the generators of the cohomology ring of the fiber, these are $z_1 = d_\ell(y_1), \ldots, z_k = d_\ell(y_k)$. For that, consider the morphism of bundles

$$\begin{array}{ccc} E(S\xi)^k & \longrightarrow & E(S\xi) \\ \big\downarrow{\scriptstyle q_k} & & \big\downarrow \\ G_\ell(\mathbb{R}^d)^k & \xrightarrow{\ g_r\ } & G_\ell(\mathbb{R}^d) \end{array}$$

induced by the projection $g_r \colon G_\ell(\mathbb{R}^d)^k \longrightarrow G_\ell(\mathbb{R}^d)$ of the r-th factor, for $1 \leq r \leq k$. This morphism induces yet another morphism of the bundles

$$\begin{array}{ccc} \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E(S\xi)^k & \longrightarrow & \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E(S\xi) \\ \big\downarrow & & \big\downarrow \\ \mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d)^k & \xrightarrow{\ \mathrm{id} \times g_r\ } & \mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d), \end{array}$$

because it respects the action of $\mathbb{Z}_2^k$. The new morphism of now Borel Constructions, in turn, induces a morphism of the corresponding Serre spectral sequences which on the level of fibers, the zero column, is a monomorphism. Furthermore, it is also a monomorphism on the zero row of the E_2 and consequently E_ℓ-term. Thus,

$$z_r = (\mathrm{id} \times g_r)^*(z),$$

and so we turn our attention to the Serre spectral sequence associated to the fiber bundle

$$S^{\ell-1} \longrightarrow \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E(S\xi) \longrightarrow \mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d).$$

In particular, we want to determine $z = d_\ell(y)$, where $y \in H^*(S^{\ell-1}; \mathbb{F}_2)$ is the generator. For an illustration of this morphism of spectral sequences see Figure 5.2.

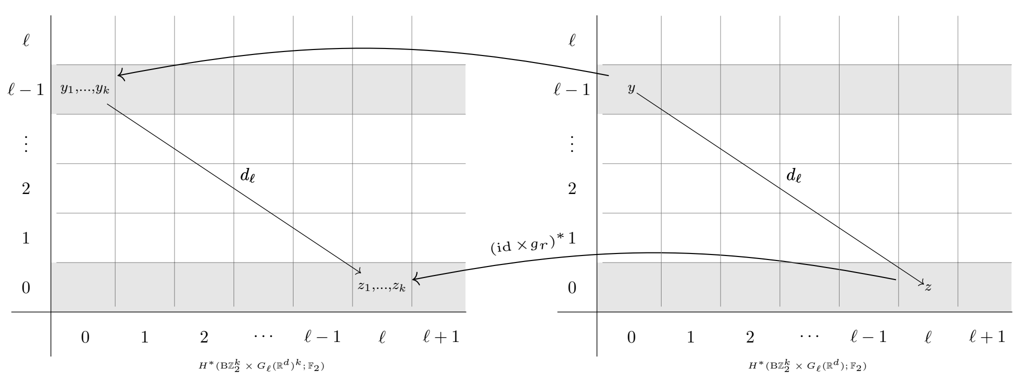

Figure 5.2: Morphism of spectral sequences induced by the projection g_r.

Now consider the inclusion $\mathbb{Z}_2 \xrightarrow{i_r} \mathbb{Z}_2^k$ into the r-th summand. Then there is an induced map between corresponding classifying spaces

$$\mathrm{B}\mathbb{Z}_2 \xrightarrow{B(i_r)} \mathrm{B}\mathbb{Z}_2^k,$$

which further on induces a map between corresponding Borel construction bundles

$$\begin{array}{ccc} \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E(S\xi) & \longleftarrow & \mathrm{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} E(S\xi) \\ \downarrow & & \downarrow \\ \mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d) & \xleftarrow{\mathrm{B}(i_r)\times \mathrm{id}} & \mathrm{B}\mathbb{Z}_2 \times G_\ell(\mathbb{R}^d). \end{array}$$

This morphism of bundles induces a morphism of the corresponding Serre spectral sequences which on the zero column of the E_2-term is an isomorphism; for an illustration see Figure 5.3.

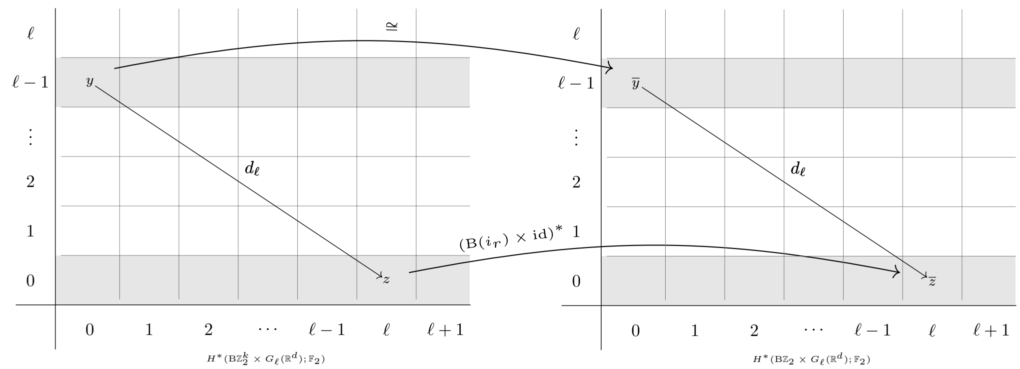

Figure 5.3: Morphism between spectral sequences induced by $\mathrm{B}(i_r)$.

From the classical work of Albrecht Dold presented in Section 3.4, applied to the Serre spectral sequence associated with fiber bundle

$$S^{\ell-1} \longrightarrow \mathrm{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} E(S\xi) \longrightarrow \mathrm{B}\mathbb{Z}_2 \times G_\ell(\mathbb{R}^d)\ ,$$

we have that

$$\overline{z} := d_\ell(\overline{y}) = \sum_{j=0}^{\ell} x^j \otimes w_{\ell-j}(\xi)$$
$$\in H^*(\mathrm{B}\mathbb{Z}_2 \times G_\ell(\mathbb{R}^d);\mathbb{F}_2) \cong H^*(\mathrm{B}\mathbb{Z}_2;\mathbb{F}_2) \otimes H^*(G_\ell(\mathbb{R}^d);\mathbb{F}_2),$$

where
- $\overline{y} \in H^{\ell-1}(S^{\ell-1};\mathbb{F}_2) \cong E_2^{0,\ell-1} \cong E_\ell^{0,\ell-1} \cong \mathbb{F}_2$ is the generator,
- $\overline{z} \in H^\ell(\mathrm{B}\mathbb{Z}_2 \times G_\ell(\mathbb{R}^d);\mathbb{F}_2) \cong \bigoplus_{a=0}^{\ell} H^a(\mathrm{B}\mathbb{Z}_2;\mathbb{F}_2) \otimes H^{\ell-a}(G_\ell(\mathbb{R}^d);\mathbb{F}_2)$,
- $H^*(\mathrm{B}\mathbb{Z}_2;\mathbb{F}_2) = \mathbb{F}_2[x]$, with $\deg(x) = 1$, and
- $w_0(\xi), \dots, w_\ell(\xi)$ are Stiefel–Whitney classes of the tautological vector bundle γ_ℓ^d.

Consequently,

$$z = \sum_{j=0}^{\ell} x_r^j \otimes w_{\ell-j}(\xi) \in H^*(\mathrm{B}\mathbb{Z}_2^k;\mathbb{F}_2) \otimes H^*(G_\ell(\mathbb{R}^d);\mathbb{F}_2).$$

Recall that we have already fixed the notation $H^*(\mathrm{B}\mathbb{Z}_2^k;\mathbb{F}_2)=\mathbb{F}_2[x_1,\cdots,x_k]$ where $\deg(x_1)=\cdots=\deg(x_k)=1$. Furthermore, for $1\le r\le k$ we get that

$$z_r \;=\; \sum_{j=0}^{\ell} x_r^j \otimes (1\otimes\cdots\otimes w_{\ell-j}(\xi)\otimes\cdots\otimes 1) \;\in\; H^*(\mathrm{B}\mathbb{Z}_2^k;\mathbb{F}_2)\otimes H^*(G_\ell(\mathbb{R}^d);\mathbb{F}_2)^{\otimes k}.$$

If we set for $0\le i\le \ell$ and $1\le r\le k$ that

$$w_{i,r}:=1\otimes\cdots\otimes w_i(\xi)\otimes\cdots\otimes 1,$$

we can rewrite

$$z_r=\sum_{j=0}^{\ell} x_r^j\otimes w_{\ell-j,r}.$$

This, means that

$$\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)^k}\big(q_k;\mathbb{F}_2\big)=\langle z_1,\dots,z_k\rangle=\Big\langle \sum_{j=0}^{\ell} x_r^j\otimes w_{\ell-j,r}\;:\;1\le r\le k\Big\rangle. \tag{5.7}$$

Index of p_k

In this part we finally retrieve the Fadell–Husseini index of the map $p_k\colon \mathcal{C}(d,\ell,k)\longrightarrow G_\ell(\mathbb{R}^d)$ with respect to the action of $\mathbb{Z}_2^k$ and with coefficients in $\mathbb{F}_2$. In other words, we compute:

$$\begin{aligned}\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k;\mathbb{F}_2\big):=&\\ &\ker\Big((\mathrm{id}\times_{\mathbb{Z}_2^k}p_k)^*\colon H^*\big(\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}G_\ell(\mathbb{R}^d);\mathbb{F}_2\big)\longrightarrow\\ &\qquad\qquad H^*\big(\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}\mathcal{C}(d,\ell,k);\mathbb{F}_2\big)\Big).\end{aligned}$$

The pullback diagram

$$\begin{array}{ccc}\mathcal{C}(d,\ell,k)=E\big(\Delta_k^*((S\xi)^k)\big) & \longrightarrow & E(S\xi)^k\\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle q_k}\\ G_\ell(\mathbb{R}^d) & \xrightarrow{\;\Delta_k\;} & G_\ell(\mathbb{R}^d)^k,\end{array}$$

after applying the Borel construction, yields the following the pullback diagram

$$\begin{array}{ccc}\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}\mathcal{C}(d,\ell,k)=\mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}E\big(\Delta_k^*((S\xi)^k)\big) & \longrightarrow & \mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}E(S\xi)^k\\ \downarrow{\scriptstyle \mathrm{id}\times_{\mathbb{Z}_2^k}p_k} & & \downarrow{\scriptstyle \mathrm{id}\times_{\mathbb{Z}_2^k}q_k}\\ \mathrm{B}\mathbb{Z}_2^k\times G_\ell(\mathbb{R}^d) & \xrightarrow{\;\Delta_k\;} & \mathrm{B}\mathbb{Z}_2^k\times G_\ell(\mathbb{R}^d)^k.\end{array}$$

Furthermore, this morphism between the fiber bundles

$$(S^{\ell-1})^k\longrightarrow \mathrm{E}\mathbb{Z}_2^k\times_{\mathbb{Z}_2^k}\mathcal{C}(d,\ell,k)\longrightarrow \mathrm{B}\mathbb{Z}_2^k\times G_\ell(\mathbb{R}^d)$$

and

$$(S^{\ell-1})^k \longrightarrow \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} E(S\xi)^k \longrightarrow \mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d)$$

induces a morphism between their Serre spectral sequences. Since the fibers of the bundles are homeomorphic, and the (non-trivial) fundamental groups of both base spaces act trivially on the cohomology of the fibers, the E_2-terms of the spectral sequences are as follows:

$$E_2^{i,j}(\mathrm{id} \times_{\mathbb{Z}_2^k} p_k) \cong H^i(\mathrm{B}\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d); \mathbb{F}_2) \otimes H^j((S^{\ell-1})^k; \mathbb{F}_2)$$

and

$$E_2^{i,j}(\mathrm{id} \times_{\mathbb{Z}_2^k} q_k) \cong H^i(B\mathbb{Z}_2^k \times G_\ell(\mathbb{R}^d)^k; \mathbb{F}_2) \otimes H^j((S^{\ell-1})^k; \mathbb{F}_2).$$

Having in mind that the induced morphism of spectral sequences is an isomorphism on the 0-column of the E_2-term, and the differentials of both sequences satisfy Leibniz rule, we get

$$\begin{aligned}
\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k; \mathbb{F}_2\big) &= (\mathrm{id} \times \Delta_k)^*\big(\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)^k}\big(q_k; \mathbb{F}_2\big)\big) \\
&= \Big\langle \sum_{s=0}^{\ell} (\mathrm{id} \times \Delta_k)^*(x_r^s \otimes w_{\ell-s,r}) \;:\; 1 \le r \le k \Big\rangle \\
&= \Big\langle \sum_{s=0}^{\ell} x_r^s \otimes \Delta_k^*(w_{\ell-s,r}) \;:\; 1 \le r \le k \Big\rangle \\
&= \Big\langle \sum_{s=0}^{\ell} x_r^s \otimes w_{\ell-s} \;:\; 1 \le r \le k \Big\rangle.
\end{aligned}$$

In summary, we have obtained that

$$\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k; \mathbb{F}_2\big) = \Big\langle \sum_{s=0}^{\ell} x_r^s \otimes w_{\ell-s} \;:\; 1 \le r \le k \Big\rangle. \tag{5.8}$$

5.3 Proofs of the main results.

Finally, in this section we present the proofs of all the remaining results introduced in Chapter 1. At the beginning of each subsection we briefly recall the result that we prove.

5.3.1 Proof of Theorem 1.3.1

Let (d, ℓ, j, k) be a 4-tuple of natural numbers where $1 \le \ell \le d - 1$. As introduced in (1.5), we denote by $R_{d,\ell,k}$ the truncated polynomial ring

$$\mathbb{F}_2[x_1, \dots, x_k, w_1, \dots, w_\ell, \overline{w}_1, \dots, \overline{w}_{d-\ell}]/I_{d,\ell}$$

where $\deg(x_1) = \cdots = \deg(x_k) = 1$, $\deg(w_i) = i$, $\deg(\overline{w}_j) = j$ for $1 \le i \le \ell$, $1 \le j \le d - \ell$, and $I_{d,\ell}$ is the ideal generated by the polynomials

$$\sum_{j=\max\{0,s+\ell-d\}}^{\min\{s,\ell\}} w_j \cdot \overline{w}_{s-j}, \qquad 1 \le s \le d.$$

In addition, assume that

$$\prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k\setminus\Gamma} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j \;\notin\; \Big\langle \sum_{s=0}^{\ell} x_r^s \otimes w_{\ell-s} \;:\; 1 \le r \le k \Big\rangle, \tag{5.9}$$

where $\Gamma = \{(0,\dots,0),(1,\dots,0),\dots,(0,\dots,1)\}$. Now, we will prove that the 4-tuple (d,ℓ,j,k) is mass assignment admissible.

Let us assume opposite, that (d,ℓ,k,j) is not mass assignment admissible. From the configuration space/test map scheme, Theorem 5.1.1(a), we get an $\mathfrak{S}_k^{\pm}$-equivariant bundle map $\Delta_k^*((S\xi)^k) \longrightarrow S\pi_2$. Here the bundles $\Delta_k^*((S\xi)^k)$ and π_2 are defined in (5.1) and (5.3) respectively. Thus, according the the monotonicity property of the Fadell–Husseini index, the following inclusion must hold:

$$\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(\pi_2;\mathbb{F}_2\big) \;\subseteq\; \mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k;\mathbb{F}_2\big).$$

On the other hand, from (5.5) and (5.8), we have that

$$\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(\pi_2;\mathbb{F}_2\big) = \Big\langle \prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k\setminus\Gamma} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j \Big\rangle \otimes H^*(G_\ell(\mathbb{R}^d);\mathbb{F}_2),$$

and

$$\mathrm{Index}_{\mathbb{Z}_2^k}^{G_\ell(\mathbb{R}^d)}\big(p_k;\mathbb{F}_2\big) = \Big\langle \sum_{s=0}^{\ell} x_r^s \otimes w_{\ell-s} \;:\; 1 \le r \le k \Big\rangle.$$

Consequently,

$$\Big\langle \prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k\setminus\Gamma} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j \Big\rangle \otimes H^*(G_\ell(\mathbb{R}^d);\mathbb{F}_2) \;\subseteq\; \Big\langle \sum_{s=0}^{\ell} x_r^s \otimes w_{\ell-s} \;:\; 1 \le r \le k \Big\rangle.$$

In particular,

$$\prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k\setminus\Gamma} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j \in \Big\langle \sum_{s=0}^{\ell} x_r^s \otimes w_{\ell-s} \;:\; 1 \le r \le k \Big\rangle.$$

This is a contradiction with the assumption (5.9), and we can conclude that the 4-tuple (d,ℓ,j,k) is mass assignment admissible.

5.3.2 Proof of Corollary 1.3.4

Let $d \ge 1$, $k \ge 1$, $j \ge 1$ and $\ell \ge 1$ be integers with $1 \le \ell \le d$. Assume that $k \ge \ell + 1$. We will prove that

$$e_{k,j} \in \mathcal{I}_{d,\ell,k}. \tag{5.10}$$

We prove (5.10) in two different ways, where the first proof is direct but more complicated, while the second argument is elegant, but relies on an additional fact.

The first proof of (5.10): For simplicity we denote the generators of the ideal $\mathcal{I}_{d,\ell,k}$ by $\beta_r := \sum_{s=0}^{\ell} x_r^s w_{\ell-s}$, where $1 \leq r \leq k$. Therefore, $\mathcal{I}_{d,\ell,k} = \langle \beta_1, \dots, \beta_k \rangle$. In order to show (5.10) we will prove that

$$\sum_{i=1}^{\ell+1} \beta_i \prod_{1 \leq a < b \leq \ell+1, a \neq i, b \neq i} (x_a + x_b) \mid e_{k,j}. \tag{5.11}$$

For that we use the following technical claim.

Claim 5.3.1. *let $k \geq 2$ be an integer. Then in the polynomial ring $\mathbb{F}_2[x_1, \dots, x_k]$ the following equality holds*

$$\prod_{1 \leq a < b \leq k} (x_a + x_b) = \sum_{\pi \in \mathfrak{S}_k} x_{\pi(1)}^{k-1} x_{\pi(2)}^{k-2} \cdots x_{\pi(k)}^{0}. \tag{5.12}$$

Proof. The proof is by induction on $k \geq 2$. While the case $k = 2$ obvious, the case $k = 3$ follows by the following direct computation:

$$\begin{aligned}\prod_{1 \leq a < b \leq 3} (x_a + x_b) &= (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) \\ &= x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 \\ &= \sum_{\pi \in \mathfrak{S}_3} x_{\pi(1)}^2 x_{\pi(2)}^1 x_{\pi(3)}^0.\end{aligned}$$

Assume now that the equality (5.12) is true for $k = r - 1$, that is

$$\prod_{1 \leq a < b \leq r-1} (x_a + x_b) = \sum_{\pi \in \mathfrak{S}_{r-1}} x_{\pi(1)}^{r-2} x_{\pi(2)}^{r-3} \cdots x_{\pi(r-1)}^{0} =: \mathcal{A}.$$

Then

$$\begin{aligned}\prod_{1 \leq a < b \leq r} (x_a + x_b) &= \prod_{1 \leq a < b \leq r-1} (x_a + x_b) \cdot \prod_{i=1}^{r-1} (x_i + x_r) \\ &= \mathcal{A} \cdot \prod_{i=1}^{r-1} (x_i + x_r) \\ &= \mathcal{A} \cdot \left(x_r^{r-1} + \sum_{s=1}^{r-1} \sum_{\substack{c_1, \dots, c_s \in [r-1], \\ c_p \neq c_q \text{ for } p \neq q}} x_r^{r-1-s} x_{c_1} \cdots x_{c_s} \right),\end{aligned}$$

where $[r-1] := \{1, \dots, r-1\}$. Now, by the induction hypothesis, each summand of the previous expression can be written in the following way

$$\begin{aligned}
\mathcal{A}\cdot x_r^{r-1} &= \sum_{c_1,\dots,c_{r-1}\in[r-1]} x_r^{r-1}x_{c_1}^{r-2}x_{c_2}^{r-3}\cdots x_{c_{r-2}}^{1}x_{c_{r-1}}^{0},\\
\mathcal{A}\cdot \sum_{i=1}^{r-1} x_r^{r-2}x_i &= \sum_{c_1,\dots,c_{r-1}\in[r-1]} x_{c_1}^{r-1}x_{r}^{r-2}x_{c_2}^{r-3}\cdots x_{c_{r-2}}^{1}x_{c_{r-1}}^{0},\\
\mathcal{A}\cdot \sum_{c_1,c_2\in[r-1]} x_r^{r-3}x_{c_1}x_{c_2} &= \sum_{c_1,\dots,c_{r-1}\in[r-1]} x_{c_1}^{r-1}x_{c_2}^{r-2}x_{r}^{r-3}\cdots x_{c_{r-2}}^{1}x_{c_{r-1}}^{0},\\
&\ \ \vdots \qquad\qquad\qquad \vdots\\
\mathcal{A}\cdot \sum_{c_1,\dots,c_{r-1}\in[r-1]} x_{c_1}\cdots x_{c_{r-1}} &= \sum_{c_1,\dots,c_{r-1}\in[r-1]} x_{c_1}^{r-1}x_{c_2}^{r-2}x_{c_3}^{r-3}\cdots x_{c_{r-1}}^{1}x_{r}^{0},
\end{aligned}$$

because we are working in $\mathbb{F}_2[x_1,\dots,x_k]$ and therefore all the remaining summands, appearing in pairs, vanish. Here all the indices in all the sums are assumed to be pairwise distinct, that is $c_p \neq c_q$ for $1 \leq p < q < r-1$. Summing up these equalities together, we conclude that

$$\begin{aligned}
\prod_{1\leq a<b\leq r}(x_a+x_b) &= \mathcal{A}\cdot\Bigg(x_r^{r-1} + \sum_{s=1}^{r-1}\sum_{\substack{c_1,\dots,c_s\in[r-1],\\ c_p\neq c_q \text{ for } p\neq q}} x_r^{r-1-s}x_{c_1}\cdots x_{c_s}\Bigg)\\
&= \sum_{\pi\in\mathfrak{S}_r} x_{\pi(1)}^{r-1}x_{\pi(2)}^{r-2}\cdots x_{\pi(r)}^{0},
\end{aligned}$$

as claimed. This completes the induction. □

Using the claim we just proved, the equality (5.12), we get that

$$\sum_{i=1}^{k}\Big(\prod_{1\leq a<b\leq k, a\neq i, b\neq i}(x_a+x_b)\Big) = \sum_{i=1}^{k}\Big(\sum_{\substack{c_1,\dots,c_{k-1}\in[k]\backslash\{i\},\\ c_p\neq c_q \text{ for } p\neq q}} x_{c_1}^{k-2}x_{c_2}^{k-3}\cdots x_{c_{k-1}}^{0}\Big). \tag{5.13}$$

Now looking at the right hand since of (5.13) we observe that for each monomial

$$x_{c_1}^{k-2}x_{c_2}^{k-3}\cdots x_{c_{k-1}}^{0}, \qquad c_1,\dots,c_{k-1}\in[k]\backslash\{i\},$$

in the i-th summand of the right hand side there is another identical monomial

$$x_{c_1}^{k-2}x_{c_2}^{k-3}\cdots x_{i}^{0}, \qquad c_1,\dots,c_{k-2}, i\in[k]\backslash\{c_{k-1}\},$$

in the c_{k-1}-st summand of the right hand side. Consequently, the right hand side of (5.13) vanishes and we get that

$$\sum_{i=1}^{k}\Big(\prod_{1\leq a<b\leq k, a\neq i, b\neq i}(x_a+x_b)\Big) = 0. \tag{5.14}$$

Consider now an integer f such that $1 \leq f \leq \ell - 1 \leq k-2$ and the following polynomial

$$\sum_{i=1}^{k} x_i^f \Big(\prod_{1 \leq a < b \leq k, a \neq i, b \neq i} (x_a + x_b) \Big) = \sum_{i=1}^{k} x_i^f \Big(\sum_{\substack{c_1, \ldots, c_{k-1} \in [k] \setminus \{i\}, \\ c_p \neq c_q \text{ for } p \neq q}} x_{c_1}^{k-2} x_{c_2}^{k-3} \cdots x_{c_{k-1}}^0 \Big). \quad (5.15)$$

Again we see that for each monomial

$$x_i^f (x_{c_1}^{k-2} x_{c_2}^{k-3} \cdots x_{c_s}^f \cdots x_{c_{k-1}}^0), \qquad c_1, \ldots, c_{k-1} \in [k] \setminus \{i\},$$

in the i-th summand of the right hand side of (5.15) there exists identical monomial

$$x_{c_s}^f (x_{c_1}^{k-2} x_{c_2}^{k-3} \cdots x_i^f \cdots x_{c_{k-1}}^0), \qquad c_1, \ldots, c_{k-1} \in [k] \setminus \{i\},$$

in the c_s-th summand. Hence,

$$\sum_{i=1}^{k} x_i^f \Big(\prod_{1 \leq a < b \leq k, a \neq i, b \neq i} (x_a + x_b) \Big) = 0. \quad (5.16)$$

In the final step of the proof we transform the polynomial from the left hand side of the relation (5.11) as follows:

$$\begin{aligned} \sum_{i=1}^{\ell+1} \beta_i \prod_{1 \leq a < b \leq \ell+1, a \neq i, b \neq i} (x_a + x_b) &= \sum_{i=1}^{\ell+1} \prod_{1 \leq a < b \leq \ell+1, a \neq i, b \neq i} (x_a + x_b) \Big(\sum_{s=0}^{\ell} x_i^s w_{\ell - s} \Big) \\ &= \sum_{s=0}^{\ell} w_{\ell-s} \sum_{i=1}^{\ell+1} x_i^s \prod_{1 \leq a < b \leq \ell+1, a \neq i, b \neq i} (x_a + x_b) \\ &\overset{(5.14),(5.16)}{=} w_0 \sum_{i=1}^{\ell+1} x_i^\ell \prod_{1 \leq a < b \leq \ell+1, a \neq i, b \neq i} (x_a + x_b) \\ &\overset{(5.12)}{=} \prod_{1 \leq a < b \leq \ell+1} (x_a + x_b). \end{aligned}$$

Since, $\ell + 1 \leq k$ we have that

$$\sum_{i=1}^{\ell+1} \beta_i \prod_{1 \leq a < b \leq \ell+1, a \neq i, b \neq i} (x_a + x_b) \mid \prod_{1 \leq a < b \leq k} (x_a + x_b).$$

On the other hand,

$$\prod_{1 \leq a < b \leq k} (x_a + x_b) \mid e_{k,j},$$

and so

$$\sum_{i=1}^{\ell+1} \beta_i \prod_{1 \leq a < b \leq \ell+1, a \neq i, b \neq i} (x_a + x_b) \mid e_{k,j}.$$

Thus, we have proved that $e_{k,j} \in \mathcal{I}_{d,\ell,k}$, as claimed.

The second proof of (5.10)*: For the proof we use the following fact:*

Let $f(T) \in (\mathbb{F}_2[w_1,\dots,w_\ell,\overline{w}_1,\dots,\overline{w}_{d-\ell}]/I_{d,\ell})[T]$ *be an arbitrary monic polynomial of degree* $\ell < k$*, and set* $g(T) = T^{k-1-\ell}f(T)$*. Then,*

$$\prod_{1\le a<b\le k}(x_a+x_b) \in \langle f(x_1),\dots,f(x_k)\rangle. \tag{5.17}$$

Indeed, let us consider the determinant modulo 2 *of the Vandermonde matrix. Then, using the invariance of determinant with respect to the row operations and expansion of the determinant with respect to a row, we have that*

$$\begin{aligned}\prod_{1\le a<b\le k}(x_a+x_b) = \det\begin{bmatrix}1 & \cdots & 1\\ x_1 & \cdots & x_k\\ \vdots & \vdots\vdots\vdots & \vdots\\ x_1^{k-1} & \cdots & x_k^{k-1}\end{bmatrix} &= \det\begin{bmatrix}1 & \cdots & 1\\ x_1 & \cdots & x_k\\ \vdots & \vdots\vdots\vdots & \vdots\\ g(x_1) & \cdots & g(x_k)\end{bmatrix}\\ &= \sum_{1\le r\le k} D(x_1,\dots,\widehat{x_r},\dots,x_k)g(x_r)\\ &\in \ \langle g(x_1),\dots,g(x_k)\rangle \ \subseteq \ \langle f(x_1),\dots,f(x_k)\rangle\end{aligned}$$

where $D(x_1,\dots,\widehat{x_r},\dots,x_k)$ *denotes an appropriate minor, which is a polynomial in all variables but* x_r*.*

Now (5.10)*, the claim that* $e_{k,j} \in \mathcal{I}_{d,\ell,k} = \langle \beta_1,\dots,\beta_k\rangle$*, follows from the observation* $\prod_{1\le a<b\le k}(x_a+x_b) \mid e_{k,j}$ *and the relation* (5.17) *with* $f(T) = \sum_{s=0}^{\ell} w_{\ell-s}T^s$*.*

5.3.3 Proof of Theorem 1.3.5

The case (d,d,k,j) *is due to Mani-Levitska, Vrećica and Živaljević [12, Thm. 39]. Thus, let us assume that* $d \ge 2$*,* $k \ge 1$*,* $j \ge 1$*,* $\ell \ge 1$ *and* $t \ge 0$*,* $r \ge 0$ *are integers with* $1 \le k \le \ell \le d-1$*. Set* $j = 2^t + r$ *with* $0 \le r \le 2^t - 1$*, and in addition assume that* $d \ge 2^{t+k-1} + r$*.*

In order to prove that the 4*-tuple* (d,ℓ,j,k) *is mass assignment admissible we show that*

$$e_{k,j} \notin \mathcal{I}_{d,\ell,k} \subseteq R_{d,\ell,k} \cong (\mathbb{F}_2[w_1,\dots,w_\ell,\overline{w}_1,\dots,\overline{w}_{d-\ell}]/I_{d,\ell})[x_1,\dots,x_k],$$

which, according to Theorem 1.3.1, suffices.

Since the element $e_{k,j}$ *is actually a polynomial with coefficients only in* $\mathbb{F}_2$*, that is* $e_{k,j} \in \mathbb{F}_2[x_1,\dots,x_k] \subseteq R_{d,\ell,k}$*, we first analyse the intersection* $\mathbb{F}_2[x_1,\dots,x_k] \cap \mathcal{I}_{d,\ell,k}$ *of the subring* $\mathbb{F}_2[x_1,\dots,x_k]$ *of the ring* $R_{d,\ell,k}$ *and the ideal* $\mathcal{I}_{d,\ell,k}$*. For simplicity, as in the previous section, we denote the generators of the ideal* $\mathcal{I}_{d,\ell,k}$ *by*

$$\beta_r := \sum_{s=0}^{\ell} x_r^s\, w_{\ell-s}$$

where $1 \le r \le k$*. Hence,* $\mathcal{I}_{d,\ell,k} = \langle \beta_1,\dots,\beta_k\rangle$*.*

Lemma 5.3.2. $e_{k,j} \notin \langle \beta_1,\dots,\beta_k\rangle \subseteq R_{d,\ell,k}$ *if and only if* $e_{k,j} \notin \langle x_1^d,\dots,x_k^d\rangle \subseteq \mathbb{F}_2[x_1,\dots,x_k]$*.*

Proof. First, we observe that $\langle x_1^d, \ldots, x_k^d\rangle \subseteq \langle \beta_1, \ldots, \beta_k\rangle$. Indeed, for every $1 \le i \le k$ we have that

$$x_i^d = \Big(\sum_{r=0}^{d-\ell} x_i^r\,\overline{w}_{d-\ell-r}\Big)\cdot\Big(\sum_{s=0}^{\ell} x_i^s\,w_{\ell-s}\Big) = \Big(\sum_{r=0}^{d-\ell} x_i^r\,\overline{w}_{d-\ell-r}\Big)\,\beta_i \in \langle \beta_1, \ldots, \beta_k\rangle.$$

Consequently, by contraposition, we get that

$$e_{k,j} \notin \langle \beta_1, \ldots, \beta_k\rangle \implies e_{k,j} \notin \langle x_1^d, \ldots, x_k^d\rangle.$$

To prove the opposite implication, it suffices to show that the inclusion

$$\mathbb{F}_2[x_1, \ldots, x_k] \longrightarrow (\mathbb{F}_2[w_1, \ldots, w_\ell, \overline{w}_1, \ldots, \overline{w}_{d-\ell}]/I_{d,\ell})[x_1, \ldots, x_k]$$

induces a monomorphism

$$\mathbb{F}_2[x_1, \ldots, x_k]/\langle x_1^d, \ldots, x_k^d\rangle \longrightarrow (\mathbb{F}_2[w_1, \ldots, w_\ell, \overline{w}_1, \ldots, \overline{w}_{d-\ell}]/I_{d,\ell})[x_1, \ldots, x_k]/\langle \beta_1, \ldots, \beta_k\rangle. \quad (5.18)$$

Indeed, if $e_{k,j} \notin \langle x_1^d, \ldots, x_k^d\rangle$, then its class $e_{k,j} + \langle x_1^d, \ldots, x_k^d\rangle \neq 0$ is non-zero in the quotient ring $\mathbb{F}_2[x_1, \ldots, x_k]/\langle x_1^d, \ldots, x_k^d\rangle$. Hence, the injectivity of the map (5.18) would imply that the corresponding class $e_{k,j} + \langle \beta_1, \ldots, \beta_k\rangle$ is also non-zero, meaning that $e_{k,j} \notin \langle \beta_1, \ldots, \beta_k\rangle$.

First, let us observe that in the ring $R_{d,\ell,k}$ the following identity holds

$$x_i^{d-1} = \Big(\sum_{s=0}^{\ell} x_i^s\,w_{\ell-s}\Big)\,q + \big(a_{\ell-1}x_i^{\ell-1} + \cdots + a_1x_i + a_0\big)$$

where $q \in R_{d,\ell,k}$ is a polynomial of degree $d-1-\ell$, and $1 \le i \le k$. The coefficients $a_{\ell-1}, \cdots, a_0$ of the remainder in the previous equation can be explicitly computed, as shown in [31, Proof of Prop. 4.1 with $\eta = \gamma_\ell^d$ and ξ trivial]. In particular, for $0 \le r \le \ell - 1$:

$$a_r = \overline{w}_{d-r-1} + w_1\overline{w}_{d-r-2} + \cdots + w_{\ell-r-1}\overline{w}_{d-\ell}.$$

Since $\overline{w}_s = 0$ for every $s \ge d-\ell+1$ we have that $a_r = w_{\ell-r-1}\overline{w}_{d-\ell}$ and specially $a_{\ell-1} = \overline{w}_{d-\ell} \neq 0$. Thus,

$$\begin{aligned} x_i^{d-1} + \langle \beta_1, \ldots, \beta_k\rangle &= \beta_i q + \overline{w}_{d-\ell}(x_i^{\ell-1} + w_1x_i^{\ell-2} + \cdots + w_{\ell-2}x_i^1 + w_{\ell-1}) + \langle \beta_1, \ldots, \beta_k\rangle \\ &= \overline{w}_{d-\ell}(x_i^{\ell-1} + w_1x_i^{\ell-2} + \cdots + w_{\ell-2}x_i^1 + w_{\ell-1}) + \langle \beta_1, \ldots, \beta_k\rangle \\ &\neq \langle \beta_1, \ldots, \beta_k\rangle. \end{aligned}$$

Now, we show the injectivity of the map (5.18). Denote by I the kernel ideal of the map

$$\mathbb{F}_2[x_1, \ldots, x_k] \longrightarrow \mathbb{F}_2[x_1, \ldots, x_k]/\langle x_1^d, \ldots, x_k^d\rangle \longrightarrow (\mathbb{F}_2[w_10, \ldots, w_\ell, \overline{w}_1, \ldots, \overline{w}_{d-\ell}]/I_{d,\ell})[x_1, \ldots, x_k]/\langle \beta_1, \ldots, \beta_k\rangle.$$

In particular, $\langle x_1^d, \ldots, x_k^d\rangle \subseteq I$. Further on assume that

$$0 \neq p = \sum_{(r_1,\ldots,r_k)\in A\subseteq\{0,\ldots,d-1\}^k} a_{r_1,\ldots,r_k}\,x_1^{r_1}\cdots x_k^{r_k} \in I,$$

for some index set $A \subseteq \{0, \dots, d-1\}^k$. If

$$z = \min\{z : a_{r_1,\dots,r_{k-1},z} \neq 0\}$$

then the polynomial $p \cdot x_k^{d-1-z}$ has all monomials with the exponent in x_k at least $d-1$. Considering the summands in $p \cdot x_k^{d-1-z}$ with exponent in x_k exactly $d-1$, we obtain a new polynomial also contained in I. Continuing in this way along the variables x_{k-1}, x_{k-2}, all the way down to x_1, we get that $x_1^{d-1} \cdots x_k^{d-1} \in I$. In particular, this means that

$$\begin{aligned}\langle \beta_1, \dots, \beta_k \rangle &= x_1^{d-1} \cdots x_k^{d-1} + \langle \beta_1, \dots, \beta_k \rangle \\ &= \overline{w}_{d-\ell}^k \prod_{1 \leq i \leq k} \left(x_i^{\ell-1} + w_1 x_i^{\ell-2} + \cdots + w_{\ell-1} \right) + \langle \beta_1, \dots, \beta_k \rangle \\ &\neq \langle \beta_1, \dots, \beta_k \rangle\end{aligned}$$

because $1 \leq k \leq \ell$ and $\overline{w}_{d-\ell}^{\ell} \neq 0$ (see for example [16]). We reached the contradiction with the assumption of the existence of a polynomial p in the ideal I. Hence, the injectivity of the map (5.18) is confirmed and the proof of the lemma is completed. □

Using Lemma 5.3.2 we complete the proof of Theorem 1.3.5 by proving the following fact.

Lemma 5.3.3. $e_{k,j} \notin \mathcal{I}_{d,\ell,k}$.

Proof. According to Lemma 5.3.2 it suffices to show that $e_{k,j} \notin \langle x_1^d, \dots, x_k^d \rangle \subseteq \mathbb{F}_2[x_1, \dots, x_k]$.

First, we transform the polynomial $e_{k,j}$ in as follows:

$$\begin{aligned}e_{k,j} &= \prod_{i=1}^{k} x_i^{j-1} \cdot \prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^k \setminus \{(0,\dots,0),(1,\dots,0),\dots,(0,\dots,1)\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j \\ &= \frac{1}{x_1 \cdots x_k} \prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^k \setminus \{(0,\dots,0)\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k)^j \\ &= \frac{1}{x_1 \cdots x_k} \Big(\prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^k \setminus \{(0,\dots,0)\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k) \Big)^j \\ &= \frac{1}{x_1 \cdots x_k} \cdot \Delta_k^j,\end{aligned}$$

where

$$\Delta_k := \prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^k \setminus \{(0,\dots,0)\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k)$$

is the Dickson polynomial of maximal degree. Further on, the Dickson polynomial Δ_k can be presented as a polynomial in x_k by:

$$\begin{aligned}\Delta_k &= \prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^k \setminus \{(0,\dots,0)\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k) \\ &= \Delta_{k-1} x_k \prod_{(\alpha_1,\dots,\alpha_k) \in \mathbb{F}_2^{k-1} \setminus \{(0,\dots,0)\}} (\alpha_1 x_1 + \cdots + \alpha_{k-1} x_{k-1} + x_k) \\ &= \Delta_{k-1} x_k \Big(\sum_{i=0}^{k-1} D_{k-1,i} x_k^{2^i - 1} \Big) \qquad (5.19)\end{aligned}$$

where $D_{k-1,0}, \dots, D_{k-1,k-2}$ are Dickson polynomials in variables $x_1, \dots, x_{k-1}$. In particular, $D_{k-1,0} = \Delta_{k-1}$ and $D_{k-1,k-1} = 1$. For more details on Dickson polynomials consult for example [32].

Now, we start the proof of the claim $e_{k,j} \notin \langle x_1^d, \dots, x_k^d \rangle$ using induction on k. In the case $k = 1$ we have that $\Delta_1 = x_1$ and $d = 2^t + r = j$. Then our claim reduces to the obvious fact that $e_{i,j} = x_1^{j-1} \notin \langle x_1^j \rangle$. Let us assume, as an induction hypothesis, that

$$e_{k-1,j} \notin \langle x_1^{2^{t+k-2}+r}, \dots, x_{k-1}^{2^{t+k-2}+r} \rangle. \tag{5.20}$$

For the induction step we present the integer j in the binary form as:

$$j = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_a},$$

where $a \geq 1$ and $t = t_1 > t_2 > \cdots > t_a \geq 0$. In particular, $r = 2^{t_2} + \cdots + 2^{t_a}$. Now from (5.19) follows that

$$\Delta_k^j = \Delta_{k-1}^j x_k^j \Big(\sum_{i=0}^{k-1} D_{k-1,i}\, x_k^{2^i-1} \Big)^j = \Delta_{k-1}^j x_k^j \prod_{b=1}^{a} \Big(\sum_{i=0}^{k-1} D_{k-1,i}^{2^{t_b}}\, x_k^{(2^i-1)2^{t_b}} \Big). \tag{5.21}$$

A typical monomial in the expansion of the left hand side of the (5.21) is of the form

$$\begin{aligned} m &= \Delta_{k-1}^j x_k^j \cdot D_{k-1,i_1}^{2^{t_1}}\, x_k^{(2^{i_1}-1)2^{t_1}} \cdot D_{k-1,i_2}^{2^{t_2}}\, x_k^{(2^{i_2}-1)2^{t_2}} \cdots D_{k-1,i_a}^{2^{t_a}}\, x_k^{(2^{i_a}-1)2^{t_a}} \\ &= \Delta_{k-1}^j D_{k-1,i_1}^{2^{t_1}} D_{k-1,i_2}^{2^{t_2}} \cdots D_{k-1,i_a}^{2^{t_a}} x_k^E \end{aligned}$$

where $0 \leq i_1, \dots, i_a \leq k-1$, and

$$E = j + (2^{i_1+t_1} - 2^{t_1}) + (2^{i_2+t_2} - 2^{t_2}) + \cdots + (2^{i_a+t_a} - 2^{t_a}) = 2^{i_1+t_1} + \cdots + 2^{i_a+t_a}.$$

Observe that,

$$E = 2^{i_1+t_1} + 2^{i_2+t_2} + \cdots + 2^{i_a+t_a} = 2^{t_1+k-1} + 2^{t_2} + \cdots + 2^{t_a}$$

if and only if

$$2^{t_2}(2^{i_2} - 1) + \cdots + 2^{t_a}(2^{i_a} - 1) = 2^{t_1+i_1}(2^{k-1-i_1} - 1)$$

if and only if

$$i_1 = k-1 \quad \text{and} \quad i_2 = \cdots = i_a = 0,$$

because $t_1 > t_2 > \cdots > t_a \geq 0$. Thus, in the expansion of the polynomial Δ_k^j the (one and only) monomial of degree $2^{t_1+k-1} + r = 2^{t+k-1} + r$ in variable x_k is of the form $\Delta_{k-1}^{j+r}\, x_k^{2^{t_1+k-1}+r}$. In other words, in the expansion of the polynomial $e_{k,j}$ the (one and only) monomial of degree $2^{t_1+k-1} + r - 1 = 2^{t+k-1} + r - 1$ in variable x_k is of the form $\frac{1}{x_1 \cdots x_{k-1}} \cdot \Delta_{k-1}^{j+r}\, x_k^{2^{t+k-1}+r-1}$.

In the final step of the proof, depending on $j + r$, we discuss three cases.

(1) If $0 \leq r \leq 2^{t-1} - 1$, then $j + r = 2^t + 2r$ with $0 \leq 2r \leq 2^t - 2$. Then, from induction hypothesis (5.20), we have that $\Delta_{k-1}^{j+r} \notin \langle x_1^\delta, \dots, x_{k-1}^\delta \rangle$ where

$$\delta = 2^{t+k-2} + 2r \leq 2^{t+k-2} + 2^t - 2 + r < 2^{t+k-1} + r.$$

Consequently, $e_{k,j} = \frac{1}{x_1 \cdots x_k} \cdot \Delta_k^j$ has a (non-zero) monomial $x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}} x_k^{2^{t+k-1}+r-1}$ where $\alpha_i \leq \delta - 1 < 2^{t+k-1} + r - 1$ for all $1 \leq i \leq k-1$.

(2) If $r = 2^{t-1}$, then $j + r = 2^t + 2 \cdot 2^{t-1} = 2^{t+1}$. Now the induction hypothesis (5.20) implies that $\Delta_{k-1}^{j+r} \notin \langle x_1^\delta, \ldots, x_{k-1}^\delta \rangle$ where

$$\delta = 2^{t+k-1} < 2^{t+k-1} + r.$$

Hence, $e_{k,j} = \frac{1}{x_1 \cdots x_k} \cdot \Delta_k^j$ has a (non-zero) monomial $x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}} x_k^{2^{t+k-1}+r-1}$ where $\alpha_i \leq \delta - 1 < 2^{t+k-1} + r - 1$ for all $1 \leq i \leq k-1$.

(3) If $2^{t-1}+1 \leq r \leq 2^t - 1$, then $j+r = 2^t + 2r = 2^{t+1} + (2r - 2^t)$ with $2 \leq 2r - 2^t \leq 2^t - 1$. In this case the induction hypothesis (5.20) implies that $\Delta_{k-1}^{j+r} \notin \langle x_1^\delta, \ldots, x_{k-1}^\delta \rangle$ where

$$\delta = 2^{t+k-1} + r + (r - 2^t) < 2^{t+k-1} + r.$$

Thus, $e_{k,j} = \frac{1}{x_1 \cdots x_k} \cdot \Delta_k^j$ has a (non-zero) monomial $x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}} x_k^{2^{t+k-1}+r-1}$ where $\alpha_i \leq \delta - 1 < 2^{t+k-1} + r - 1$ for all $1 \leq i \leq k-1$.

Therefore, $e_{k,j} \notin \langle x_1^d, \ldots, x_k^d \rangle$, and induction step is completed.

This proof adds the missing argument in the proof of [12, Thm. 39] and corrects the final steps of the proof of [13, Thm. 3.2]. □

The proof of Theorem 1.3.5 is now complete.

Bibliography

[1] *Patrick Schnider. Ham-Sandwich cuts and center transversals in subspaces. In* 35th International Symposium on Computational Geometry*, volume 129 of* LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 56, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019.*

[2] *Walter Rudin.* Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.*

[3] *Flemming Topsøe.* Topology and measure*. Lecture Notes in Mathematics, Vol. 133. Springer-Verlag, Berlin-New York, 1970.*

[4] *Ilani Axelrod-Freed and Pablo Soberón. Bisections of mass assignments using flags of affine spaces, 2021.*

[5] *Jiří Matoušek.* Lectures on Discrete Geometry*. Graduate Texts in Mathematics. Springer New York, 2013.*

[6] *Daniel R. Mauldin, editor.* The Scottish Book*. Birkhäuser, Boston, Mass., 1981. Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.*

[7] *William A. Beyer and Andrew Zardecki. The early history of the ham sandwich theorem.* Amer. Math. Monthly*, 111(1):58–61, 2004.*

[8] *Branko Grünbaum. Partitions of mass-distributions and of convex bodies by hyperplanes.* Pacific J. Math.*, 10:1257–1261, 1960.*

[9] *Hugo Hadwiger. Simultane Vierteilung zweier Körper.* Arch. Math. (Basel)*, 17:274–278, 1966.*

[10] *David Avis. Nonpartitionable point sets.* Inform. Process. Lett.*, 19(3):125–129, 1984.*

[11] *Edgar A. Ramos. Equipartition of mass distributions by hyperplanes.* Discrete Comput. Geom.*, 15(2):147–167, 1996.*

[12] *Peter Mani-Levitska, Siniša Vrećica, and Rade Živaljević. Topology and combinatorics of partitions of masses by hyperplanes.* Adv. Math.*, 207(1):266–296, 2006.*

[13] *Pavle V. M. Blagojević, Florian Frick, Albert Haase, and Günter M. Ziegler. Topology of the Grünbaum-Hadwiger-Ramos hyperplane mass partition problem.* Trans. Amer. Math. Soc.*, 370(10):6795–6824, 2018.*

[14] *Pavle V. M. Blagojević, Florian Frick, Albert Haase, and Günter M. Ziegler. Hyperplane mass partitions via relative equivariant obstruction theory.* Doc. Math.*, 21:735–771, 2016.*

[15] *Pavle V. M. Blagojević, Benjamin Matschke, and Günter M. Ziegler. Optimal bounds for a colorful Tverberg-Vrećica type problem.* Adv. Math.*, 226(6):5198–5215, 2011.*

[16] *Howard L. Hiller. On the cohomology of real Grassmanians.* Trans. Amer. Math. Soc.*, 257(2):521–533, 1980.*

[17] *Rade Živaljević. User's guide to equivariant methods in combinatorics. volume 59, pages 114–130. Matematički institut SANU, 1996.*

[18] *Rade T. Živaljević. User's guide to equivariant methods in combinatorics. II. volume 64(78), pages 107–132. 1998. 50th anniversary of the Mathematical Institute, Serbian Academy of Sciences and Arts (Belgrade, 1996).*

[19] *Rade Živaljević. Topological methods in discrete geometry. In* Handbook of Discrete and Computational Geometry, Third Edition*. Taylor & Francis, 2017.*

[20] *Jiří Matoušek.* Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.*

[21] *John McCleary.* A user's guide to spectral sequences*. Number 58. Cambridge University Press, 2001.*

[22] *Anatolij Timofeevič Fomenko and Dmitrij B Fuks.* Homotopical topology*, volume 273. Springer, 2016.*

[23] *Allen Hatcher. Spectral sequences in algebraic topology. Book preprint. Available at https://pi.math.cornell.edu/ hatcher/SSAT/SSATpage.html.*

[24] *M. Aguilar, S. Gitler, and C. Prieto.* Algebraic Topology from a Homotopical Viewpoint*. Universitext. Springer New York, 2013.*

[25] *A. Hatcher, Cambridge University Press, and Cornell University. Department of Mathematics.* Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002.*

[26] *Albrecht Dold. Parametrized Borsuk-Ulam theorems.* Commentarii Mathematici Helvetici*, 63(1):275–285, 1988.*

[27] *Edward Fadell and Sufian Husseini. An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems.* Ergodic Theory Dynam. Systems*, 8* (Charles Conley Memorial Issue):73–85, 1988.*

[28] *Edward Fadell and Sufian Husseini. Relative cohomological index theories.* Advances in Mathematics*, 64(1):1–31, 1987.*

[29] *Glen E. Bredon.* Topology and Geometry*, volume 139 of* Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.*

[30] *Armand Borel. La cohomologie mod* 2 *de certains espaces homogènes.* Commentarii Mathematici Helvetici*, 27(1):165–197, 1953.*

[31] *M. C. Crabb and J. Jaworowski. Aspects of the Borsuk-Ulam theorem.* J. Fixed Point Theory Appl.*, 13(2):459–488, 2013.*

[32] *Clarence Wilkerson. A primer on the Dickson invariants. In* Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*, volume 19 of* Contemp. Math.*, pages 421–434. Amer. Math. Soc., Providence, RI, 1983.*