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Bounded Topology and Trace Ideals on ω

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que para optar por el grado de

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presenta

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Resumen

El trabajo se divide en dos partes. En la primera de ellas se explora de una manera topológica algunas propiedades combinatorias de los ideales sobre ω. Para tal motivo introducimos una topología, a la que llamamos *topología acotada* (Definición 3.2.3), la cual es más fina que la topología métrica heredada de 2^ω y cuyas características están estrechamente relacionadas a propiedades de acotamiento del ideal, en especial con los conjuntos débilmente acotados y los fuertemente no acotados (Definición 3.1.1).

Salvo en P-ideales no magros, esta topología es distinta a la usual (Teorema 3.3.10) y para los P-ideales analíticos coincide con la topología dada por las submedidas estudiadas por S. Solecki en [41] y [42] (Teorema 3.3.20). Mostraremos cómo algunos conceptos combinatorios se relacionan con las propiedades de esta topología, dando algunos resultados paralelos entre ambos. Veremos que el espacio dado por el ideal $\mathrm{Fin} \times \emptyset$ presenta algunas cualidades interesantes (Proposición 3.3.13 y debajo).

Esta parte del trabajo fue motivada en gran parte por una conjetura de A. Louveau y B. Veličković en [34] (Conjetura 3.4.1). Sobre ésta, damos una solución parcial (Corolario 3.4.13) en una clase de ideales que abarca a los P-ideales. Damos un ejemplo de un subideal de $\mathrm{Fin} \times \mathrm{Fin}$ al que llamamos *ideal triangular* (Definición 3.4.7) que testifica que la conjetura aún sigue abierta y motiva a una nueva (Conjetura 3.4.14). También daremos una clasificación de los F_σ-ideales usando su topología acotada (Teorema 3.5.10), que si bien no es exhaustiva, creemos que está cerca de serlo (Conjetura 3.5.11).

En la otra parte del trabajo, estudiamos a los ideales traza (Definición 4.0.4) que fueron introducidos por J. Brendle y S. Yatabe en [7]. Mostraremos cómo algunas propiedades del ideal original se traducen a propiedades de su traza. Definimos también nuevos cardinales y estudiamos su relación con los usuales (resumidos en el diagrama bajo Corolario 4.1.4). Al final, damos ejemplos de ideales traza y algunas de sus propiedades.

Palabras clave. Teoría de conjuntos. Ideales analíticos. Orden de Tukey. Espacios topológicos. Combinatoria.

ABSTRACT

This work consists of two main parts. In the first one, we will delve into a topological approach for the combinatorics of an ideal on ω. In order to do that, we introduce the *bounded topology* (Definition 3.2.3), defined for ideals on ω which is finer than the usual one, and whose properties are closely related with the combinatorics of the ideal, specially with its weakly bounded and strongly unbounded sets (Definition 3.1.1).

Excluding the non-meager P-ideals, the bounded topology is distinct from the inherited metric topology (Theorem 3.3.10), and for analytic P-ideals it is equal to the topology introduced by S. Solecki in [41] and [42] (Theorem 3.3.20). We will study how combinatorial concepts for an ideal are related to properties of its bounded topology, giving some parallel results. Along the way, we found that the bounded topology of $\mathrm{Fin} \times \emptyset$ has certain interesting properties (Proposition 3.3.13 and below).

This part was largely motivated by a conjecture given by A. Louveau and B. Veličković in [34] (Conjecture 3.4.1). We present a partial answer (Corollary 3.4.13) for ideals with a weaker property than being an P-ideal. We also introduce a subideal of $\mathrm{Fin} \times \mathrm{Fin}$, namely *the triangular ideal* (Definition 3.4.7), which does not has this property and motivates a new speculation (Conjecture 3.4.14). We will also use the bounded topology to give a classification among F_σ-ideals (Theorem 3.5.10), although it is not exhaustive, we expect to be close to being (Conjecture 3.5.11).

In the second part, we study the trace ideals (Definition 4.0.4) introduced by J. Brendle and S. Yatabe in [7]. We prove how some properties of the original ideal are related with its trace ideal. We define related cardinals and show its relation with the usual ones (summarized in the diagram below Corollary 4.1.4). Finally, we present examples of trace ideals and some of their properties.

Keywords. Set theory. Analytic ideal. Tukey order. Topological spaces. Combinatorics.

Dedicatoria

Al profesor y amigo Fernando Hernández Hernández, quién en más de una ocasión me recordó lo maravilloso que es continuar por este camino, quién dedicó parte de su tiempo y esfuerzo en apoyarme, quién fue mi tutor en el amplio sentido de la palabra.

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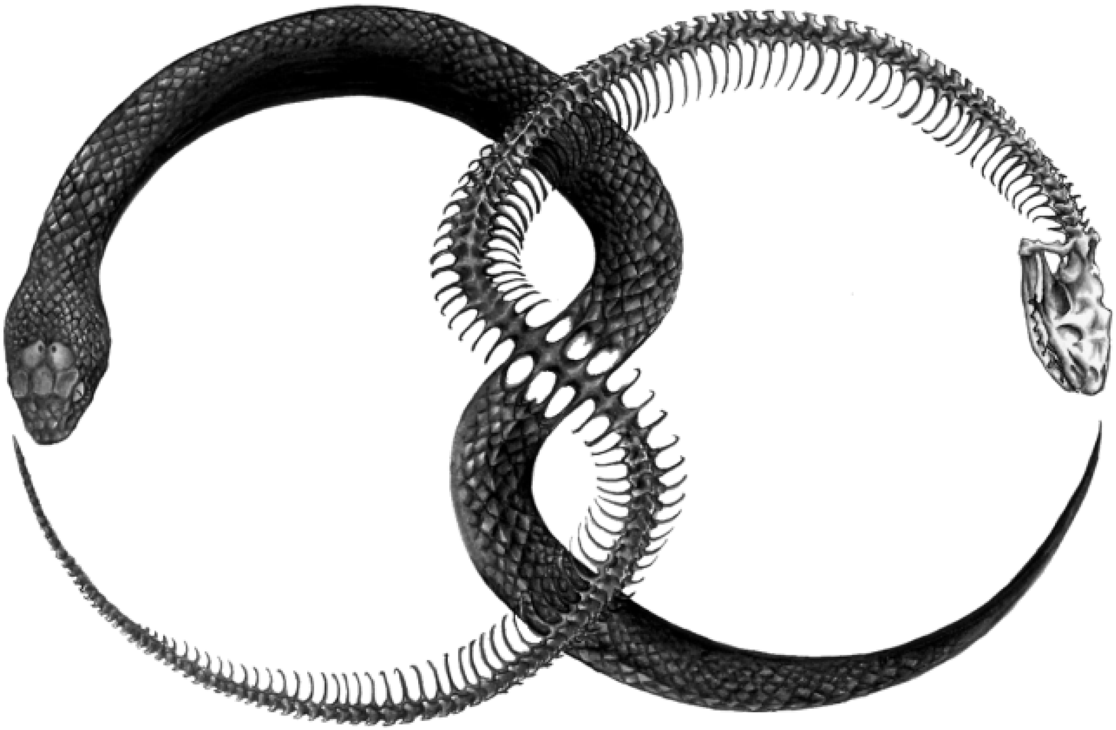
A la Universidad Nacional Autónoma de México, a la Universidad Michoacana de San Nicolás de Hidalgo, al Posgrado Conjunto en Ciencias Matemáticas.

A las circunstancias.



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Contents

Chapter 1

Preliminaries

This work will be framed by the standard set theory axiom system known as ZFC (the Zermelo-Fraenkel axiomatic system with and the Axiom of Choice). All the bases of ZFC and the standard notation for it can be consulted in [8], [21] and [33], among others. In this chapter we will give some basics definitions that will be used throughout this work.

1.1 Set Theory

For a pair of sets A, B its *symmetric difference* is the set $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

For a set X, a *relation on* X is a set $R \subseteq X \times X$. A relation R on X is an *order* (more specifically, a *partial order on* X) if it is *reflexive* $((\forall\, x \in X)\ \ (x, x) \in R)$, *antisymmetric* $((\forall\, x, y \in X)\ \ ((x, y) \in R\ \wedge\ (y, x) \in R)\ \to\ x = y))$ and *transitive* $((\forall\, x, y, z \in X)\ \ ((x, y) \in R \wedge (y, z) \in R) \to (x, z) \in R)$, in this case we say that the pair (X, R) (or simply X when the order is known) is an *ordered set*. We write xRy instead of $(x, y) \in R$. An order $\leq$ on X is a *directed order* if any pair of elements in X has an *upper bound* (that is, $(\forall\, x, y \in X)\,(\exists\, z \in X)\ \ x, y \leq z$). For an ordered set $(X, \leq)$, a subset $C \subseteq X$ is $\leq$-*cofinal* if $(\forall\, x \in X)\,(\exists\, c \in C)\ \ x \leq c$, and $A \subseteq X$ is an $\leq$-*antichain* if $(\forall\, x, y \in A)\ \ x \neq y\ \to\ (x \not\leq y)$ and $(y \not\leq x)$. In general, when there is no ambiguity on the order, we omit the prefix in the concepts, then for example we simply write *cofinal* instead of $\leq$-cofinal.

An order $\leq$ on X is a *well-order* if any non-empty subset of X has a minimum element (that is, $(\forall\, A \subseteq X)\ \ (A = \emptyset \vee ((\exists\, a \in A)\,(\forall\, b \in A)\ \ a \leq b)))$. For a set X the relation $\in_X$, given by $a \in_X b$ if and only if $a \in b$ or $a = b$ for $a, b \in X$ is an order on X. An ordered set $(X, \in_X)$ is an *ordinal number* (or simply an *ordinal*) if $\in_X$ is a well-order and $(\forall\, y)\ \ y \in X \ \rightarrow\ y \subseteq X$. For a pair of ordinals α, β we say that $\alpha < \beta$ if $\alpha \in \beta$, and $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$. Every well-ordered set is isomorphic to an unique ordinal number, and by the Axiom of Choice any set can be well-ordered (see [8, Chapter 4]). An ordinal κ is a *cardinal number* (or simply a *cardinal*) if for any ordinal $\alpha < \kappa$ there is no a bijection between α and κ.

The set of finite cardinal numbers is the *set of natural numbers*, denoted by ω. A set X is *equipotent* to another set Y if there is a bijection among them, by the previous discussion, any set X is equipotent to an unique cardinal number, the *cardinality of* X, and it is denoted by $|X|$. A set equipotent to ω is called *countable*. The cardinal of ω is denoted by ω_0, $\aleph_0$ or by ω itself. The first non countable cardinal is denoted by ω_1. The cardinality of $\mathcal{P}\,(\omega)$ (which is the same cardinal from the set of real numbers $\mathbb{R}$) is $2^{\aleph_0}$, usually denoted by $\mathfrak{c}$.

A *sequence* on a set X is a map from ω to X, which is denoted by $\{x_n : n \in \omega\}$. For a pair of sets X, Y, the set of all maps from Y to X is denoted by X^Y. Via the characteristic functions, the set $\mathcal{P}\,(X)$ is equipotent to 2^X. For a set X and a cardinal κ, $X^{<\kappa}$ denotes the set of all maps from some cardinal $\lambda < \kappa$ to X; $[X]^\kappa$ denotes the set of all subsets of X with cardinality equal to κ and $[X]^{<\kappa}$ denotes the set $\bigcup \big\{[X]^\lambda : \lambda < \kappa\big\}$.

A *partition* of a set X is a family $\mathcal{P} \subseteq \mathcal{P}\,(X)$ of non-empty sets which are *pairwise disjoint* (that is, $(\forall\, P, Q \in \mathcal{P})\ \ P \neq Q\ \rightarrow\ P \cap Q = \emptyset$) and such that $\bigcup \mathcal{P} = X$. A partition $\{P_n : n \in \omega\}$ of ω is an *interval partition* if any piece of it is an *interval* of ω (that is, $(\forall\, n \in \omega)\, (\exists\, a_n, b_n \in \omega)\ \ P_n = [a_n, b_n] = \{k \in \omega : a_n \leq k \leq b_n\}$). Let $A, B \subseteq X$, then $A \subseteq^* B$ if there is a set $F \in [X]^{<\omega}$ such that $A \subseteq B \cup F$, and $A =^* B$ if both $A \subseteq^* B$ and $B \subseteq^* A$. A family $\mathcal{A} \subseteq X$ is an *almost disjoint family* if $(\forall\, A, B \in \mathcal{A})\ \ A \neq B\ \rightarrow\ A \cap B =^* \emptyset$.

For a infinite set X, a non-empty family $\mathcal{I} \subseteq \mathcal{P}\,(X)$ is an *ideal* (or more specifically an *ideal on* X) if it is closed under finite unions ($(\forall\, A, B \in \mathcal{I})\ \ A \cup B \in \mathcal{I}$) and subsets ($(\forall\, A, B \in \mathcal{P}\,(X))\ \ A \subseteq B\ \wedge\ B \in \mathcal{I}\ \rightarrow\ A \in \mathcal{I}$). We will always consider ideals such that $X \notin \mathcal{I}$ and $[X]^{<\omega} \subseteq \mathcal{I}$. For $\mathcal{A} \subseteq \mathcal{P}\,(X)$, the *ideal generated by* $\mathcal{A}$ is the ideal $\langle \mathcal{A} \rangle = \{Y \subseteq X : (\exists\, F \in [\mathcal{A}]^{<\omega})\ \ Y \subseteq \bigcup F\}$. A family $\mathcal{B} \subseteq \mathcal{I}$ is a *basis for* $\mathcal{I}$ if it is a $\subseteq$-cofinal subset of the ideal $\mathcal{I}$. An ideal $\mathcal{I}$ is an *σ-ideal* if is closed under countable unions, and is an *P-ideal* if for every countable subfamily $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ there exists $A \in \mathcal{I}$ with $(\forall\, n \in \omega)\, A_n \subseteq^* A$ (such set A is a *pseudo-union* for the family $\{A_n : n \in \omega\}$). The family of *positive sets* of the ideal $\mathcal{I}$ is defined by $\mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$. An ideal $\mathcal{I}$ on X is *isomorphic* to an ideal $\mathcal{J}$ on Y, denoted by $\mathcal{I} \cong \mathcal{J}$, if there exists $A \in \mathcal{I}, B \in \mathcal{J}$ and a bijection $f : X \setminus A \rightarrow Y \setminus B$ such that for all $C \subseteq X$, $C \setminus A \in \mathcal{I}$ if and only if $f[C \setminus A] \in \mathcal{J}$.

We use the notation $(\forall^\infty n \in \omega)\ \varphi(n)$ for $(\exists N \in \omega)(\forall n \in \omega)\ n \geq N \ \rightarrow\ \varphi(n)$ and $(\exists^\infty n \in \omega)\ \varphi(n)$ for $(\exists X \in [\omega]^\omega)(\forall n \in X)\ \varphi(n)$, for a suitable property for natural numbers $\varphi(n)$.

1.2 Topology

Notions and related results can be consulted in [12], [13] and [22]. For a set X, a family $\tau \subseteq \mathcal{P}(X)$ is a *topology* (specifically a *topology on* X) if $\emptyset, X \in \tau$, it is closed under arbitrary unions (that is, $(\forall \mathcal{A} \subseteq \tau)\ \bigcup \mathcal{A} \in \tau$) and finite intersections (that is, $(\forall U, V \in \tau)\ U \cap V \in \tau$). $\tau = \mathcal{P}(X)$ is the *discrete* topology. For a topology τ on X, the pair (X, τ) (or simply X when the topology is known) is a *topological space*. Let τ_a, τ_b be a pair of topologies on X, we say that τ_a is *coarser* than τ_b (or τ_b is *finer* than τ_a) if $\tau_a \subseteq \tau_b$. If τ is a topology on X and $Y \subseteq X$ then $\tau{\restriction}_Y = \{U \cap Y : U \in \tau\}$ is the *induced topology on* Y and $(Y, \tau{\restriction}_Y)$ is a *subspace of* (X, τ). For the rest of the subsection, fix a topological space (X, τ).

A set $U \subseteq X$ is an *open set* (resp. $F \subseteq X$ is a *closed set*) if $U \in \tau$ (resp. $X \setminus F \in \tau$). A set $C \subseteq X$ is *clopen* if it is open and closed. For $x \in X$, a set $N \subseteq X$ is a *neighborhood of* x if there is some open set U such that $x \in U \subseteq N$. For $x \in X$, the set of all neighborhoods of x is denoted by $\eta(x)$. For a set $A \subseteq X$ the *closure* of A is the closed set $\mathrm{cl}_X(A) = \bigcap\{F \subseteq X : F \text{ is closed and } A \subseteq F\}$ and the *interior* of A is the open set $\mathrm{int}_X(A) = \bigcup\{U \subseteq X : U \text{ is open and } U \subseteq A\}$ (when the space is know, we write $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ instead).

Fix $x \in X$. A family $\mathcal{V}_x \subseteq \eta(x)$ such that $(\forall N \in \eta(x))(\exists V \in \mathcal{V}_x)\ V \subseteq N$ is a *neighborhood basis at* x. A family $\mathcal{V} \subseteq \mathcal{P}(X)$ is a *neighborhood basis for* τ (or simply a *neighborhood basis*) if $\mathcal{V}_y = \mathcal{V} \cap \eta(y)$ is a neighborhood basis at y for all $y \in X$. A family $\mathcal{B}_x \subseteq \tau$ is a *local basis at* x if $(\forall N \in \eta(x))(\exists B \in \mathcal{B}_x)\ x \in B \subseteq N$. A family $\mathcal{B} \subseteq \tau$ is a *basis for* τ (or simply a *basis*) if it is a local basis at y for all $y \in X$. For $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\bigcup \mathcal{B} = X$, the *topology generated by* $\mathcal{B}$ is the topology given by the unions of intersections of finite subfamilies of $\mathcal{B}$.

A map $\rho : X \times X \to \mathbb{R}$ is a *metric* (or, and more generally, a *distance on* X) if it is null from a point to itself ($(\forall x \in X)\ \rho(x, x) = 0$), positive for distinct points ($(\forall x, y \in X)\ x \neq y \ \rightarrow\ \rho(x, y) > 0$), symmetric ($(\forall x, y \in X)\ \rho(x, y) = \rho(y, x)$) and it satisfies the *triangle inequality* ($(\forall x, y, z \in X)\ \rho(x, y) \leq \rho(x, z) + \rho(z, y)$). Let ρ be a metric on X. For $r > 0$ and $x \in X$ the *open ball centering at x and radius r* is the set $\mathcal{B}^\rho_r(x) = \{y \in X : \rho(x, y) < r\}$. The *induced topology by* ρ is the topology generated by $\mathcal{B} = \{\mathcal{B}^\rho_r(x) : r > 0,\ x \in X\}$ (in this case, the set $\mathcal{B}$ is a basis for its induced topology). The metric ρ is *compatible* with the space if its induced topology is equal to τ. A sequence $\{x_n : n \in \omega\} \subseteq X$ is a *Cauchy sequence* if for all $\varepsilon > 0$ there is some $n_\varepsilon \in \omega$ such that $(\forall\, n, m \geq n_\varepsilon)\ \ \rho(x_n, x_m) < \varepsilon$. A sequence $\{x_n : n \in \omega\} \subseteq X$ *converges* to some point $x \in X$ (denoted by $\{x_n : n \in \omega\} \to x$ or simply $x_n \to x$) if $(\forall\, N \in \eta(x))\,(\exists\, k \in \omega)\,(\forall\, n \geq k)\ \ x_n \in N$. A compatible metric ρ is *complete* if any Cauchy sequence converges.

The space is *first-countable* if for all $y \in X$ there is some countable neighborhood basis at y, and it is *second-countable* if there is some countable basis for the space. The space is *zero-dimensional* if it has a basis consisting of clopen sets. The space is *sequential* if for all $F \subseteq X$, F is a closed set if and only if for all $\{x_n : n \in \omega\} \subseteq F$ such that $x_n \to x$, the point x belongs to F. The space is *Fréchet–Urysohn* if for all $A \in X$ and all $x \in X$, $x \in \mathrm{cl}(A)$ if and only if there is some $\{a_n : n \in \omega\} \subseteq A$ such that $a_n \to x$. The space is *metrizable* if there is some compatible metric ρ on X. The space is *regular* if for any closed set F and any point $x \in X$ there is a pair of open disjoint sets U, V such that $x \in U$ and $F \subseteq V$. The space is *Hausdorff* if for any pair of points $x, y \in X$ there is a pair of open disjoint sets U, V such that $x \in U$.

A point x is a *limit point of* $Y \subseteq X$ if $(\forall\, N \in \eta(x))\ \ N \cap Y \setminus \{x\} \neq \emptyset$. For $A \subseteq X$, the *derived set* of A is $A' = \{x \in X : x \text{ is limit point of } A\}$. A set $N \subseteq X$ is *nowhere dense* if $\mathrm{int}(\mathrm{cl}(N)) = \emptyset$. A set $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets. A set $C \subseteq X$ is *comeager* if $X \setminus C$ is meager. A set $B \subseteq X$ has the *Baire property* if there is some open set $U \subseteq X$ such that $B \triangle U$ is meager. A set $D \subseteq X$ is *dense* if $(\forall\, U \in \tau)\ \ U \cap D = \emptyset\ \to\ U = \emptyset$. For $Y \subseteq X$, a family $\mathcal{U} \subseteq \tau$ is an *open cover of* Y if $Y \subseteq \bigcup \mathcal{U}$. The space is *Lindelöf* if any open cover of X admits a countable subcover. A set $K \subseteq X$ is *compact* if any open cover of K admits a finite subcover. A set $K \subseteq X$ is *sequentially compact* if any sequence of it has a convergent subsequence converging to a point in K. The space is *σ-compact* if it is a countable union of compacts sets. The space is a *k-space* if for all $U \subseteq X$, U is an open set if and only if $U \cap K$ is open in the space $(K, \tau\restriction_K)$ for all compact set $K \subseteq X$. The *extent* of the space, denoted by $e(X, \tau)$, is the supremum of sizes of closed discrete subspaces of X.

Let (Y, υ) be a topological space. A map $f : X \to Y$ is *continuous* if $f^{-1}[V] \in \tau$ for all $V \in \upsilon$; and it is *sequentially continuous* if $f(x_n) \to f(x)$ for any $\{x_n : n \in \omega\} \subseteq X$ such that $x_n \to x$. Continuous implies sequentially continuous, and the converse holds if (X, τ) is sequential. A bijective map $f : X \to Y$ is an *homeomorphism* if it and its inverse map $f^{-1} : Y \to X$ are continuous. The space is *homogeneous* if for any pair of points $x, y \in X$ there is some homeomorphism φ from X to itself such that $\varphi(x) = y$. The *product topology* is the coarsest topology on $X \times Y$ for which all the projections maps are continuous. We analogously define it for arbitrary products, in particular for 2^ω and ω^ω, where 2 and ω has the discrete topology.

1.3 Descriptive Set Theory

An ordered set $(T, \leq)$ (or simply T when the order is know) is a *tree* if for all $t \in T$ the set of *predecessors of* t, defined as $\operatorname{pred}(t) = \{s \in T : s \leq t\}$, is well-ordered. The elements of a tree are called *nodes*. For a node s of a tree T we use the usual notation for $\langle s \rangle = \{t \in T : s \subseteq t\}$ and $\operatorname{len}(s) = |\operatorname{pred}(s)|$. A tree T is *pruned* if every node has a successor, that is $(\forall\, t \in T)\,(\exists\, s \in T)\ \ t < s$. The two main pruned trees considered in this work are the *Cantor Tree* $2^{<\omega}$ (the set of all finite binary sequences ordered by $\subseteq$) and the *Baire Tree* $\omega^{<\omega}$ (the set of all finite sequences of natural numbers ordered by $\subseteq$). For $A \subseteq 2^{<\omega}$, the set of *branches of* A is $[A] = \{x \in 2^\omega : (\forall\, n \in \omega)\, x\restriction_n \in A\}$ and for $s \in 2^{<\omega}$ the *cone of* s is $[s] = \{x \in 2^\omega : s \subseteq x\}$. For $x \in 2^\omega$, we denote the set of *nodes of* x by $\operatorname{nd}(x) = \{x\restriction_n : n \in \omega\} \subseteq 2^{<\omega}$. For $s \in 2^{<\omega}$ and $b \in \{0, 1\}$ its *concatenation* is denoted by $s^\frown\langle b\rangle$, that is, the node r with $\operatorname{len}(r) = \operatorname{len}(s) + 1$ such that $r\restriction_{\operatorname{len}(s)} = s$ and $r(\operatorname{len}(s)) = b$. More generally, for $s, t \in 2^{<\omega}$ its *concatenation* is denoted by $s^\frown t$, that is, the node r with $\operatorname{len}(r) = \operatorname{len}(s) + \operatorname{len}(t)$ such that $r\restriction_{\operatorname{len}(s)} = s$ and $(\forall\, k \in \operatorname{len}(t))\ \ r(\operatorname{len}(s) + k) = t(k)$. Similarly, we defined the notations for $\omega^{<\omega}$ (or in general, for any set of the form $A^{<\omega}$).

A topological space (X, τ) is *completely metrizable* if it is metrizable by a complete metric, and it is *separable* if there is a countable dense subset of X. A topological space is a *Polish space* if it is completely metrizable and separable. For a set A the map $d : A^\omega \times A^\omega \to \mathbb{R}$ given by $d(f, g) = 2^{-\Delta(f,g)}$ is a metric on A^ω, where $\Delta(f, g) = \min\{n \in \omega : f(n) \neq g(n)\}$ for two distinct functions $f, g \in A^\omega$ (that is, the first natural number where two distinct functions differ). Moreover, if we consider to A as a metric space with the discrete metric, then d is a complete compatible metric for the product topology on A^ω. Thus, $\{[s] : s \in A^{<\omega}\}$ is a base of clopen sets for the space A^ω. The two main Polish spaces considered in this work are the *Cantor Space* 2^ω (the set of all binary sequences) and the *Baire Space* ω^ω (the set of all sequences of natural numbers), both endowed with the product topology. Hence, the set $\mathcal{P}(\omega)$ can be seen as a Polish space homeomorphic to 2^ω.

For a topological space (X, τ), the *Borel σ-algebra of* X (denoted by $Borel(X)$) is the minimal σ-algebra on X which contains τ. The elements of $Borel(X)$ are called the *Borel sets of* X. $\Sigma^0_1(X)$ denotes the family of all open sets from X, $\Pi^0_1(X)$ denotes the family of all closes sets from X and for all $1 < \alpha < \omega_1$ we have that $\Sigma^0_\alpha(X)$ denotes the family of all countable unions of elements which lies in $\bigcup\left\{\Pi^0_\beta(X) : \beta < \alpha\right\}$ and $\Pi^0_\alpha(X)$ denoted the family of all complements of elements in Σ^0_α. Hence, $\Sigma^0_2(X)$ is the family of the *F_σ-subsets of* X, Π^0_2 is the family of the *G_δ-subsets of* X, $\Sigma^0_3(X)$ is the family of the *$G_{\delta\sigma}$-subsets of* X, Π^0_3 is the family of the *$F_{\sigma\delta}$-subsets of* X, and so on. If X is a metrizable space then $Borel(X) = \bigcup\{\Sigma^0_\alpha : 0 < \alpha < \omega_1\}$. For a Polish space

X, a subset $A \subseteq X$ is an *analytic set* if there are a Polish space Y and a continuous map $f : Y \to X$ such that $f[Y] = A$.

An ideal $\mathcal{I}$ on ω is *analytic* (respectively *Borel*, F_σ, *meager*, etc.) if it is an analytic (respectively Borel, F_σ, meager, etc.) subspace of $\mathcal{P}(\omega)$, which is, as mentioned before, a Polish space.

These concepts, and others related, can be consulted in [19] and [30].

1.4 Cardinal Characteristics of the Continuum

An ideal $\mathcal{I}$ on a set X has the following associated cardinals.

- The *additivity of* $\mathcal{I}$ given by

  $$\operatorname{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \wedge \ \bigcup \mathcal{A} \notin \mathcal{I}\}.$$

  Which is the smallest number of elements in the ideal whose union does not lies in the ideal.

- The *covering of* $\mathcal{I}$ given by

  $$\operatorname{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \wedge \ \bigcup \mathcal{A} = X\}.$$

  Which is the smallest number of elements in the ideal that covers the set X.

- The *uniformity of* $\mathcal{I}$ given by

  $$\operatorname{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq X \ \wedge \ Y \notin \mathcal{I}\}.$$

  Which is the smallest size of any positive set of the ideal $\mathcal{I}$.

- The *cofinality of* $\mathcal{I}$ given by

  $$\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \wedge \ (\forall I \in \mathcal{I})\,(\exists A \in \mathcal{A}) \ I \subseteq A\}.$$

  Which is the smallest size of any basis for the ideal $\mathcal{I}$.

They have the following relations (in this and all the following diagrams we use the usual notation $\kappa \to \lambda$ for $\kappa \leq \lambda$).

$$\begin{tikzcd}
 & \mathrm{non}\,(\mathcal{I}) \arrow[dr] & \\
\mathrm{add}\,(\mathcal{I}) \arrow[ur] \arrow[dr] & & \mathrm{cof}\,(\mathcal{I}) \\
 & \mathrm{cov}\,(\mathcal{I}) \arrow[ur] &
\end{tikzcd}$$

Since any ideal $\mathcal{I}$ is closed under finite unions, we have that $\mathrm{add}\,(\mathcal{I}) \geq \omega$. Also, if the ideal $\mathcal{I}$ has a basis consisting of Borel sets, then $\mathrm{cof}\,(\mathcal{I}) \leq \mathfrak{c}$ because there are only $\mathfrak{c}$ Borel sets.

On the space ω^ω we can define the relation $\leq^*$ between functions given by $f \leq^* g$ if and only if $|\{n \in \omega : f(n) > g(n)\}| < \omega$ (simillary we can define $\geq^*$, $<^*$, $>^*$). This relation has the following related cardinals.

- ◦ $\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega \ \wedge \ (\forall\, f \in \omega^\omega)\,(\exists\, g \in \mathcal{B}) \ \ g \not\leq^* f\}$,
- ◦ $\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \subseteq \omega^\omega \ \wedge \ (\forall\, f \in \omega^\omega)\,(\exists\, g \in \mathcal{B}) \ \ f \leq^* g\}$.

The cardinal $\mathfrak{b}$ is know as *the bounding number* and $\mathfrak{d}$ as the *dominating number*. Also, a family which satisfies the condition for $\mathfrak{b}$ (resp. $\mathfrak{d}$) is called *unbounded family of* ω^ω (resp. *dominating family of* ω^ω).

On the space 2^ω we consider a couple of important σ-ideals: the *null ideal*, $\mathcal{N}$, of all null subsets (according to the product measure); and the *meager ideal*, $\mathcal{M}$, of all meager sets (for an extensive study of these ideals see [3, Chapter 2]). An additional related ideal on 2^ω is the *nowhere dense ideal*, $\mathcal{NWD}$, of all nowhere dense subsets, which is no a σ-ideal. The cardinals of $\mathcal{N}$ and $\mathcal{M}$, besides $\mathfrak{b}$ and $\mathfrak{d}$, are related by the *Cichoń's Diagram*:

$$\begin{tikzcd}
\mathrm{cov}\,(\mathcal{N}) \arrow[r] & \mathrm{non}\,(\mathcal{M}) \arrow[r] & \mathrm{cof}\,(\mathcal{M}) \arrow[r] & \mathrm{cof}\,(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}\,(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}\,(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}\,(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}\,(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

Another relevant σ-ideal is the *ideal of σ-compact sets* on ω^ω, denoted by $\mathcal{K}_\sigma$, which is the family of all sets that can be covered by countably many compact sets. It's know that $\mathrm{add}\,(\mathcal{K}_\sigma) = \mathrm{non}\,(\mathcal{K}_\sigma) = \mathfrak{b}$ and $\mathrm{cov}\,(\mathcal{K}_\sigma) = \mathrm{cof}\,(\mathcal{K}_\sigma) = \mathfrak{d}$.

Several main results and other important cardinals can be found in [5].

1.5 Games and Determinacy

For a non-empty set A, a *game on* A, denoted by $G(T, X)$, consists of a pruned subtree $T \subseteq A^{<\omega}$, whose elements are called the *legal positions*, and a set $X \subseteq [T]$ named the *pay set*. Two players (named *Player I* and *Player II*) take turns in the game choosing elements of A such that in each step the finite sequence $\langle a_i : i < n\rangle$ of them choices is a node of T, the final resulting sequence $\langle a_n : n \in \omega\rangle \in [T]$ is called a *play from the game*. *Player I wins* the game if $\langle a_n : n \in \omega\rangle \in X$, otherwise *Player II wins* the game. A map $\sigma : T \to A$ is a *winning strategy for Player I* if it satisfies that $\{x \in [T] : (\forall\, n \in \omega)\ \ x(2n+1) = \sigma(x{\restriction}_{2n})\} = X$. Respectively, a function $\rho : T \to A$ is a *winning strategy for Player II* if it satisfies that $\big\{x \in [T] : (\forall\, n \in \omega)\ \ x(2n+2) = \rho(x{\restriction}_{2n+1})\big\} = [T] \setminus X$.

A game is represented by the following board, which shows the choices of the players at each step of the game (that is, a_{2n} is the n-th movement of Player I and a_{2n+1} is the n-th movement of Player II).

$$
\begin{array}{r|cccccccccc}
\mathrm{I} & a_0 & & a_2 & & \cdots & & a_{2n} & & \cdots & \\
\hline
\mathrm{II} & & a_1 & & a_3 & & \cdots & & a_{2n+1} & & \cdots
\end{array}
$$

A game is *determined* if either player has a winning strategy (observe that at most one of them can have a winning strategy). The Axiom of Choice implies that not every game is determined. However, D. Gale and F. Stewart (see [18]) proved that for open or closed pay sets the game is determined and more generally D. Martin (see [35]) guaranteed the determinacy for Borel pay sets.

Theorem 1.5.1 (Borel Determinacy)**.** Let $G(T, X)$ a game such that X is a Borel subset of $[T]$, then $G(T, X)$ is determined.

Chapter 2

Ideals on ω

Any ideal $\mathcal{I}$ on a countable set can be seen as an ideal on ω via its enumeration, whence henceforward we assume that an *ideal on* ω make reference to an ideal on a countable set. In this case, all the related cardinals defined in the preliminaries, except maybe the cofinality, are equal to ω. However, there are other interesting related cardinals. For a deeper understanding on ideals see [24], [39] and [51].

Definition 2.0.1. Let $\mathcal{I}$ be an ideal on ω. Define the following cardinals.

a) $\mathrm{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})\,(\exists A \in \mathcal{A})\ A \not\subseteq^* I\}$.

b) $\mathrm{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in [\omega]^\omega \cap \mathcal{I})\,(\exists A \in \mathcal{A})\ |A \cap X| = \aleph_0\}$.

c) $\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{X}| : \mathcal{X} \subseteq [\omega]^\omega \cap \mathcal{I} \wedge (\forall I \in \mathcal{I})\,(\exists X \in \mathcal{X})\ |I \cap X| < \aleph_0\}$.

d) $\mathrm{cof}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})\,(\exists A \in \mathcal{A})\ I \subseteq^* A\}$.

An ideal on ω is *tall* if for every $A \in [\omega]^\omega$ there is some $B \in [A]^\omega$ such that $B \in \mathcal{I}$. Originally, the previous cardinals were only defined for tall ideals (in this case, the definition of $\mathrm{cov}^*(\mathcal{I})$ and $\mathrm{non}^*(\mathcal{I})$ is equivalent if we use $[\omega]^\omega$ instead of $[\omega]^\omega \cap \mathcal{I}$) and as mentioned in [24], the choice of names can be attributed to the fact that for any tall ideal $\mathcal{I}$ on ω, there is an ideal $\widehat{\mathcal{I}}$ on $\mathcal{P}(\omega)$ such that $\mathcal{X} \subseteq \mathcal{P}(\omega)$ belongs to $\widehat{\mathcal{I}}$ if and only if there is some $I \in \mathcal{I}$ with $\mathcal{X} \subseteq \{X \subseteq \omega : |X \cap I| = \aleph_0\}$. Then, $\mathrm{add}(\widehat{\mathcal{I}}) = \mathrm{add}^*(\mathcal{I})$ and the same by the other tree cardinals. Even more, $\mathcal{I}$ is a P-ideal if and only if $\widehat{\mathcal{I}}$ is a σ-ideal. These cardinals satisfies the following relations.

$$\begin{tikzcd}
 & & \operatorname{non}^*(\mathcal{J}) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \operatorname{add}^*(\mathcal{J}) \arrow[ur] \arrow[dr] & & \operatorname{cof}^*(\mathcal{J}) \arrow[r] & 2^{\aleph_0} \\
 & & \operatorname{cov}^*(\mathcal{J}) \arrow[ur] & &
\end{tikzcd}$$

For an ideal $\mathcal{J}$ on ω, we have that $\operatorname{cof}^*(\mathcal{J}) = \operatorname{cof}(\mathcal{J})$ since any witness family for $\operatorname{cof}(\mathcal{J})$ is clearly also a witness for $\operatorname{cof}^*(\mathcal{J})$ and for a witness family $\mathcal{B}^* \subseteq \mathcal{J}$ for $\operatorname{cof}^*(\mathcal{J})$, the family $\mathcal{B} = \{B \cup F : B \in \mathcal{B}^* \wedge F \in [\omega]^{<\omega}\}$ satisfies the condition for $\operatorname{cof}(\mathcal{J})$. The ideal $\mathcal{J}$ is *ω-generated* if $\operatorname{cof}(\mathcal{J}) = \omega$ (and therefore, all its previous related cardinals are also equal to ω), that is, if $\mathcal{J}$ has a countable basis. The only countably generated ideals, up to ideal isomorphism, are Fin and $\mathrm{Fin} \times \emptyset$, defined below (see [15, Proposition 1.2.8]).

Recall that we can see an ideal as a subset of the Cantor space, via the characteristic map, hence it has the induced topology from the metric space 2^ω, denoted by τ. The topological concepts that we will mention for subsets of an ideal $\mathcal{J}$ refer to the space $(\mathcal{J}, \tau\restriction_{\mathcal{J}})$.

We will introduce some usual ideals on ω, later we will study more in depth about some. Now we only give their definitions, their Borel complexity and few basic properties. In order to do that, let $\mathrm{IP}(\omega) = \{\Pi \subseteq [\omega]^\omega : \Pi \text{ is a partition of } \omega\}$. For more details on these ideals see [39, Chapter 1].

- $\mathrm{Fin} = [\omega]^{<\omega}$. It is the unique countable ideal on ω. It is an F_σ-ideal but neither a tall nor P-ideal.

- $\mathrm{Fin} \times \emptyset = \{A \subseteq \omega \times \omega : (\forall^\infty n \in \omega)\, A \cap (\{n\} \times \omega) = \emptyset\}$. This ideal can be seen as the ideal on ω generated by some $\Pi \in \mathrm{IP}(\omega)$. It is an F_σ-ideal but neither a tall nor P-ideal.

- $\emptyset \times \mathrm{Fin} = \{A \subseteq \omega \times \omega : (\forall n \in \omega)\, A \cap (\{n\} \times \omega) \in \mathrm{Fin}\}$. Let $\Pi \in \mathrm{IP}(\omega)$, then the elements of the ideal can be seen as the sets whose intersection with any element of Π is finite. It is a F_σ and P-ideal but not a tall ideal.

- $\mathrm{Fin} \times \mathrm{Fin} = \{A \subseteq \omega \times \omega : \{n \in \omega : A \cap (\{n\} \times \omega) \notin \mathrm{Fin}\} \in \mathrm{Fin}\}$. Seen as an ideal on ω, is the generated by the previous pair of ideals. It is a tall and $\mathrm{F}_{\sigma\delta\sigma}$-ideal but not a P-ideal.

- $\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists\, m \in \omega)\, (\forall^\infty\, n \in \omega)\, |A \cap (\{n\} \times \omega)| \leq m\}$, the *eventually different ideal.* In order to see it as an ideal on ω, let $\{P_n : n \in \omega\} \in \mathrm{IP}(\omega)$ and $\mathcal{S} = \{f \in \omega^\omega : (\forall\, n \in \omega)\ \ f(n) \in P_n\}$. Then $\mathcal{ED}$ is the ideal generated by $\{P_n : n \in \omega\} \cup \{f[\omega] : f \in \mathcal{S}\}$. It is a tall and F_σ-ideal but not a P-ideal.
- $nwd = \{N \subseteq \mathbb{Q} : N \text{ is a nowhere dense set}\}$, the *nowhere dense ideal.* It is a tall and $\mathrm{F}_{\sigma\delta}$-ideal but not a P-ideal.
- $\mathcal{I}_{1/n} = \{A \subseteq \omega : \sum \{n^{-1} : n \in A \setminus \{0\}\} < \infty\}$, the *summable ideal.* It is a tall, F_σ and P-ideal.
- $\mathcal{Z} = \left\{A \subseteq \omega : \lim_n \frac{|A \cap n|}{n} = 0\right\}$, the *asymptotic density zero ideal.* It is a tall, $\mathrm{F}_{\sigma\delta}$ and P-ideal.

2.1 Tukey Order

In this section we focused in the Tukey order introduced by J. Tukey in [50] and further studied by J. R. Isbell e.g. in [20] and [26]. Let be $\mathcal{I}$ an ideal, we say that a set $\mathcal{X} \subseteq \mathcal{I}$ is *bounded* if $\bigcup \mathcal{X} \in \mathcal{I}$ (thus, $\mathrm{add}\,(\mathcal{I})$ is the smallest size of any unbounded set of $\mathcal{I}$). A set $I \in \mathcal{I}$ is a *bound of* $\mathcal{X} \subseteq \mathcal{I}$ if $\bigcup \mathcal{X} \subseteq I$ (clearly, $\mathcal{X} \subseteq \mathcal{I}$ is bounded if and only if it has a bound). A set $\mathcal{Y} \subseteq \mathcal{I}$ is *unbounded* if it is not bounded.

Definition 2.1.1 (Tukey Order)**.** Let $\mathcal{I}, \mathcal{J}$ be a pair of ideals on ω. $\mathcal{I}$ is *Tukey-below* $\mathcal{J}$ (or $\mathcal{J}$ is *Tukey-above* $\mathcal{I}$) if there is a map $F : \mathcal{I} \to \mathcal{J}$ such that $F^{-1}[\mathcal{X}] \subseteq \mathcal{I}$ is bounded for all bounded $\mathcal{X} \subseteq \mathcal{J}$. This is denoted by $\mathcal{I} \leq_T \mathcal{J}$ and the map F is called a *Tukey map.* These ideals are *Tukey equivalent* (denoted by $\mathcal{I} \equiv_T \mathcal{J}$) if $\mathcal{I} \leq_T \mathcal{J}$ and $\mathcal{J} \leq_T \mathcal{I}$.

Proposition 2.1.2. Let $\mathcal{I}, \mathcal{J}$ be a pair of ideals on ω. The following conditions are equivalent.

i) $\mathcal{I}$ is Tukey-below to $\mathcal{J}$.

ii) There are maps $G : \mathcal{I} \to \mathcal{J}$ and $G^* : \mathcal{J} \to \mathcal{I}$ such that $G(I) \subseteq J \ \Rightarrow\ I \subseteq G^*(J)$ for all $I \in \mathcal{I}$ and all $J \in \mathcal{J}$.

Proof. i) implies ii). Let $F : \mathcal{I} \to \mathcal{J}$ a witness for $\mathcal{I} \leq_T \mathcal{J}$. Let $G^* : \mathcal{J} \to \mathcal{I}$ be a map such that $G^*(J)$ is a bound of the set $F^{-1}[\{Y \in \mathcal{J} : Y \subseteq J\}]$. We claim that $G = F$ and G^* satisfy the desired condition, indeed, let $I \in \mathcal{J}$ and $J \in \mathcal{J}$ such that $G(I) \subseteq J$, then $G(I) \in \{Y \in \mathcal{J} : Y \subseteq J\}$ and hance $F^{-1}(G(I)) \subseteq G^*(J)$. Since $I \subseteq F^{-1}(G(I))$ because $G = F$, we conclude that $I \subseteq G^*(J)$.

ii) implies i). Let $G : \mathcal{I} \to \mathcal{J}$ and $G^* : \mathcal{J} \to \mathcal{I}$ a pair of maps satisfying the hypothesis. Then $F = G$ is a Tukey map, indeed, let $\mathcal{X} \subseteq \mathcal{J}$ be a bounded set, hence $G(I) \subseteq \bigcup \mathcal{X}$ for all $I \in G^{-1}[\mathcal{X}]$, and therefore $G^*(\bigcup \mathcal{X})$ is a bound of $F^{-1}[\mathcal{X}]$. ■

More generally, these concepts of bounded and Tukey order can be defined on directed orders as follows.

Definition 2.1.3. Let $(P, \preceq)$, $(Q, \trianglelefteq)$ be a pair of directed order. Then:

a) $p \in P$ is a $\preceq$-*bound* of a set $A \subseteq P$ if $(\forall\, a \in A)\;\, a \preceq p$.

b) $A \subseteq P$ is $\preceq$-*bounded* if it has a $\preceq$-*bound*.

c) $A \subseteq P$ is $\preceq$-*unbounded* if it is not $\preceq$-*bounded*.

d) $A \subseteq P$ is $\preceq$-*countably bounded* if there is some countable set $B \subseteq P$ such that $(\forall\, a \in A)\,(\exists\, b \in B)\;\, a \preceq b$.

e) A map $F : P \to Q$ is called a *Tukey map* from $(P, \preceq)$ to $(Q, \trianglelefteq)$ if $F^{-1}[B] \subseteq P$ is $\preceq$-bounded for all $\trianglelefteq$-bounded subset $B \subseteq Q$.

f) $(P, \preceq)$ is *Tukey below* $(Q, \trianglelefteq)$ (or $(Q, \trianglelefteq)$ *is Tukey reducible to* $(P, \preceq)$), denoted by $(P, \preceq) \leq_T (Q, \trianglelefteq)$, if there exists a Tukey map from $(P, \preceq)$ to $(Q, \trianglelefteq)$.

g) $(P, \preceq)$ and $(Q, \trianglelefteq)$ are *Tukey equivalent* if $(P, \preceq) \leq_T (Q, \trianglelefteq)$ and $(Q, \trianglelefteq) \leq_T (P, \preceq)$, which is denoted by $(P, \preceq) \equiv_T (Q, \trianglelefteq)$.

When there is no ambiguity in the corresponding orders, we can simplify the notation of parts f) and g) by $P \leq_T Q$ and $P \equiv_T Q$, respectively. The analogue of Proposition 2.1.2 is also true for the previous general definition.

Note that any ideal ordered by $\subseteq$ is a directed order (since the union of two elements of an ideal belongs to it). We have that corresponding concepts agree in this case. Unless otherwise specified, we assume that an ideal $\mathcal{I}$ is an ordered set by the order $\subseteq$, but it can be ordered by other ones, as we discussed later in Definition 2.1.6.

Also, for a directed order $(P, \preceq)$ we can define the following cardinals.

- $\operatorname{add}(P, \preceq) = \min\{|A| : A \subseteq P \;\wedge\; (\forall\, p \in P)\,(\exists\, c \in C)\;\, c \not\preceq p\}$.
- $\operatorname{cof}(P, \preceq) = \min\{|C| : C \subseteq P \;\wedge\; (\forall\, p \in P)\,(\exists\, c \in C)\;\, p \preceq c\}$.

The Tukey order relates these cardinals as follows.

Proposition 2.1.4. Let $(P, \preceq), (Q, \trianglelefteq)$ be a pair of directed orders. If $P \leq_T Q$ then $\mathrm{add}(Q) \leq \mathrm{add}(P)$ and $\mathrm{cof}(P) \leq \mathrm{cof}(Q)$.

Proof. Let $F : P \to Q$ be a map witnessing $P \leq_T Q$. Let $A \in [P]^\kappa$ with $\kappa < \mathrm{add}(Q)$. $F[A] \subseteq Q$ is $\trianglelefteq$-bounded because $|F[A]| < \kappa$, therefore $F^{-1}[F[A]] \subseteq P$ is $\preceq$-bounded, thus A is $\preceq$-bounded too. We conclude that $\kappa < \mathrm{add}(P)$.

Let $G : P \to Q$ and $G^* : Q \to P$ be maps that witness $P \leq_T Q$. Let $C \subseteq Q$ be a cofinal subset, then $G^*[C] \subseteq P$ is a cofinal subset because for all $p \in P$ there is a $c \in C$ such that $G(p) \trianglelefteq c$, then $p \preceq G^*(c)$. We conclude that $\mathrm{cof}(P) \leq \mathrm{cof}(Q)$. ■

Using the $\preceq$-countably bounded sets we can define the following order.

Definition 2.1.5. Let $(P, \preceq), (Q, \trianglelefteq)$ be a pair of directed orders. We say that $P \leq_\omega Q$ if there is a map $F : P \to Q$ such that for every $q \in Q$ the set $\{p \in P : F(p) \trianglelefteq q\} \subseteq P$ is countably bounded.

As we mentioned before, an ideal can be ordered in a different way. Then we have another order between ideals.

Definition 2.1.6. Let $\mathcal{I}, \mathcal{J}$ be a pair of ideals on ω. We use the notation $\mathcal{I} \leq_T^* \mathcal{J}$ for $(\mathcal{I}, \subseteq^*) \leq_T (\mathcal{J}, \subseteq^*)$, considering each ideal ordered by $\subseteq^*$.

The previous definitions are related as follows.

Proposition 2.1.7. Let $\mathcal{I}, \mathcal{J}$ be a pair of P-ideals on ω. The following holds.

i) $\mathcal{I} \leq_\omega \mathcal{J}$ if and only if $\mathcal{I} \leq_T^* \mathcal{J}$.

ii) If $\mathcal{I} \leq_T \mathcal{J}$ then $\mathcal{I} \leq_T^* \mathcal{J}$.

Proof. For a set $J \in \mathcal{J}$ let $J^\downarrow = \{Y \in \mathcal{J} : Y \subseteq J\}$. Let $F : \mathcal{I} \to \mathcal{J}$ be a Tukey map, then $F^{-1}[J^\downarrow] = \{I \in \mathcal{I} : F(I) \subseteq J\}$. This proves that $\mathcal{I} \leq_T \mathcal{J}$ implies $\mathcal{I} \leq_\omega \mathcal{J}$. Therefore it is enough to prove i).

Let $F : \mathcal{I} \to \mathcal{J}$ be a witness for $\mathcal{I} \leq_\omega \mathcal{J}$, we will prove that F is also a witness for $\mathcal{I} \leq_T^* \mathcal{J}$. Let $\mathcal{X} \subseteq \mathcal{J}$ be a $\subseteq^*$-bounded set, hence $F^{-1}[\mathcal{X}] \subseteq \mathcal{I}$ is countable bounded by some family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$, and since this family has a pseudo-union $I \in \mathcal{I}$, then $F^{-1}[\mathcal{X}]$ is $\subseteq^*$-bounded by I. Therefore F proves that $\mathcal{I} \leq_T^* \mathcal{J}$.

The converse does not need the hypothesis of P-ideal. Let $G : \mathcal{I} \to \mathcal{J}$ be a witness for $\mathcal{I} \leq_T^* \mathcal{J}$. We have that for $J \in \mathcal{J}$, the set $G^{-1}[J^\downarrow] \subseteq \mathcal{I}$ is $\subseteq^*$-bounded by some $I \in \mathcal{I}$, therefore $\{I \in \mathcal{I} : G(I) \subseteq J\}$ is countably bounded by $\{I \cup F : F \in [\omega]^{<\omega}\}$. Therefore G proves that $\mathcal{I} \leq_\omega \mathcal{J}$. ■

The notation $J^{\downarrow}$ which was used in the previous proof will be useful in what follows, hence we introduce a more general related definition.

Definition 2.1.8. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$. The *closure under subsets of* $\mathcal{A}$ is the set

$$\mathcal{A}^{\downarrow} = \{X \subseteq \omega : (\exists\, A \in \mathcal{A}) \;\; X \subseteq A\}.$$

Other related concepts among orders are the following.

Definition 2.1.9. Let $(P, \preceq), (Q, \trianglelefteq)$ be a pair of ordered sets.

a) $(P, \preceq), (Q, \trianglelefteq)$ are *order-isomorphic*, denoted by $P \backsimeq Q$, if there is a bijective map $\varphi : P \to Q$ such that $p_1 \preceq p_2 \Leftrightarrow \varphi(p_1) \trianglelefteq \varphi(p_2)$ for all $p_1, p_2 \in P$.

b) $(P, \preceq), (Q, \trianglelefteq)$ are *cofinally similar* if there exists an ordered set $(R, \leq)$ and a pair of $\leq$-cofinal sets $P', Q' \subseteq R$ such that $P \backsimeq P'$ and $Q \backsimeq Q'$.

J. Tukey noticed the following.

Theorem 2.1.10 (Tukey [50])**.** Let $(P, \preceq), (Q, \trianglelefteq)$ be a pair of directed orders. They are cofinally similar if and only if ther are Tukey-equivalent.

The generality of the Tukey order is so broad that it can be applied to various mathematical topics. An example of this is the one we discuss below. In order to do that, for a topological space X let $\mathcal{K}(X) = \{K \subseteq X : X \text{ is compact}\}$. Note that $(\mathcal{K}(X), \subseteq)$ is a directed order.

Theorem 2.1.11 (Fremlin [16])**.** Let X be a metrizable space. Then:

i) $\mathcal{K}(X) \equiv_T \{0\}$ if and only if X is compact.

ii) $\mathcal{K}(X) \equiv_T \omega$ if and only if X is separable, locally compact and non-compact.

iii) $\mathcal{K}(X) \equiv_T \omega^{\omega}$ if and only if X is Polish and no locally compact.

Regarding to the directed sets of size ω_1, S. Todorčević showed the following.

Theorem 2.1.12 (Todorčević [46])**.** Assuming the Proper Forcing Axiom, any directed order of size ω_1 is cofinally similar to one of the following: $\{0\}$, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$.

For larger sizes, S. Todorčević also proved that there are 2^{ω_1} many distinct cofinal types of directed orders of size $\mathfrak{c}$.

About the analytic P-ideals, A. Louveau and B. Veličković showed that the Tukey order is complex.

Theorem 2.1.13 (Louveau and Veličković [34])**.** There is an embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$ into analytic P-ideals ordered by the Tukey order.

K. Mazur proves in [38] that even restricting the Tukey order to some class of F_σ-ideals (the called *gradually fragmented ideals*, see [24, Definition 6.9, 6.10]) is still complex for the same reason as before. We also have the following result for analytic P-ideals using the Tukey order, due to S. Todorčević.

Theorem 2.1.14 (Todorčević [47, 48])**.** Let $\mathcal{I}$ be an analytic P-ideal on ω, then $\mathcal{I} \leq_T \mathcal{I}_{1/n}$. Furthermore, if $\mathcal{I}$ is not ω-generated then $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$.

Due to D. H. Fremlin ([17]), for some of the ideals presented at the beginning of this chapter we know its connection under the Tukey order. In the following diagram $\mathcal{I} \to \mathcal{J}$ means $\mathcal{I} \leq_T \mathcal{J}$.

$$\begin{tikzcd}
 & & & nwd \arrow[dr] & \\
\{0\} \arrow[r] & \omega \arrow[r] & \emptyset \times \mathrm{Fin} \arrow[ur] \arrow[dr] & & \mathcal{I}_{1/n} \\
 & & & \mathcal{Z} \arrow[ur] &
\end{tikzcd}$$

The left side of this diagram is optimal in the following sense.

Theorem 2.1.15 (Louveau and Veličković [34])**.** Let $\mathcal{I}$ be an ideal. If $\omega \leq_T \mathcal{I}$ and $\mathcal{I} \leq_T \emptyset \times \mathrm{Fin}$, then $\mathcal{I} \equiv_T \omega$ or $\mathcal{I} \equiv_T \emptyset \times \mathrm{Fin}$

2.2 Submeasures and Category

In [41] and [42], S. Solecki works with maps which are close to being measures on the set $\mathcal{P}(\omega)$, these maps are called sumbeasures (Definition 2.2.1) and the difference is that they are subadditive instead of σ-additive (a map is σ-additive if the measure of a the union of a countably many disjoint sets is equal to the sum of the measures from each of them). These maps give us a way to define ideals on ω, say a lot about their complexity and allow us to classify them, as shown the main related results given by S. Solecki and K. Mazur (Theorem 2.2.3 and Theorem 2.2.4).

Definition 2.2.1. A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure* if it is

- [proper] $(\forall\, n \in \omega)\ \ \varphi(\{n\}) < \infty$ and $\varphi(\emptyset) = 0$.
- [monotone] $(\forall\, A, B \subseteq \omega)\ \ A \subseteq B\ \to\ \varphi(A) \leq \varphi(B)$
- [subadditive] $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$

If additionally $(\forall\, A \subseteq \omega)\ \ \varphi(A) = \lim_n \varphi(A \cap n)$, then φ is a *lower semicontinuous submeasure.*

Directly from the definition, the following families are ideals on ω.

Definition 2.2.2. Let φ a lower semicontinuous submeasure on ω.

a) If $\varphi(\omega) = \infty$, then $\mathrm{Fin}(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ is the *finite ideal of* φ.

b) If $\lim_n \varphi(\omega \setminus n) > 0$, then $\mathrm{Exh}(\varphi) = \{A \subseteq \omega : \lim_n \varphi(A \setminus n) = 0\}$ is the *exhaustive ideal of* φ.

Theorem 2.2.3 (Mazur [37])**.** Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$. $\mathcal{I}$ is an F_σ-ideal if and only if there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathrm{Fin}(\varphi)$.

Theorem 2.2.4 (Solecki [41, 42])**.** Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$.

i) $\mathcal{I}$ is an analytic P-ideal if and only if there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$.

ii) $\mathcal{I}$ is an F_σ and P-ideal if and only if there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$.

By the previous results, we have many examples of F_σ-ideals and, by the following, it is the smaller complexity we can find for ideals on ω.

Theorem 2.2.5 (Folklore)**.** There is no closed or G_δ-ideal on ω.

Proof. Let $\mathcal{I}$ be an ideal on ω. Since $[\omega]^{<\omega} \subseteq \mathcal{I}$, $\mathcal{I}$ is dense in 2^ω and therefore it cannot be closed. Let $sw : 2^\omega \to 2^\omega$ the *switch map* given by $sw(x)(n) = 1 - x(n)$. Then sw is an homeomorphism and $\mathcal{I} \cap sw[\mathcal{I}] = \emptyset$, hence $\mathcal{I}$ can not be a G_δ-ideal by the Baire Category Theorem. ■

Any analytic (and hence Borel) ideal on ω is meager; this follows directly from the next result.

Theorem 2.2.6 (0-1 Law of Sierpiński)**.** Let $\mathcal{I}$ be an ideal on ω. If $\mathcal{I}$ has the Baire property, then $\mathcal{I}$ is meager.

Proof. Let $\mathcal{I}$ be an ideal on ω with the Baire property. If $\mathcal{I}$ is a non-meager set, then there is some $s \in 2^{<\omega}$ such that $\mathscr{I} \cap \langle s \rangle$ is comeager in $\langle s \rangle$. Let $sw_s : \langle s \rangle \to \langle s \rangle$ be the map given by $sw_s(x)(n) = s(n)$ if $n < \operatorname{len}(s)$ and $sw_s(x)(n) = 1 - x(n)$ if $n \geq \operatorname{len}(s)$, then sw_s if a homeomorphism. Hence $\mathcal{I} \cap \langle s \rangle$ and $sw_s[\mathcal{I}] \cap \langle s \rangle$ are two disjoint comeager sets on $\langle s \rangle$, which is imposible by the Baire Category Theorem. ■

The following is a really useful characterization for meager ideals.

Theorem 2.2.7 (Jalali-Naini [28] and Talagrand [45])**.** Let $\mathcal{I}$ be an ideal on ω. The following conditions are equivalent.

i) $\mathcal{I}$ is meager.

ii) There is an interval partition $\{P_n : n \in \omega\}$ of ω such that

$$(\forall\, I \in \mathcal{I})\, (\forall^\infty\, n \in \omega)\;\; P_n \nsubseteq I.$$

Proof. i) implies ii). Let $\{F_n : n \in \omega\} \subseteq \mathcal{P}(2^\omega)$ be an increasing family of nowhere dense sets such that $\mathcal{I} = \bigcup \{F_n : n \in \omega\}$. We recursively define an increasing sequence $\{n_k : n \in \omega\} \subseteq \omega$ such that $\{[n_k, n_{k+1}) : k \in \omega\}$ is an interval partition which satisfies the condition in ii). Let $n_0 = 0$. Since F_0 is nowhere dense, there is $s_0 \in 2^{<\omega}$ such that $\langle s_0 \rangle \cap F_0 = \emptyset$, let $n_1 = \operatorname{len}(s_0)$. Let suppose defined n_k. Since F_k is nowhere dense, then there is $s_k \in 2^{<\omega}$ such that $\langle r^\frown s_k \rangle \cap F_k = \emptyset$ for each $r \in 2^{n_k}$. Let $n_{k+1} = n_k + \operatorname{len}(s_k)$. Finally, if for some $X \subseteq 2^\omega$ there is a set $A \in [\omega]^\omega$ such that $(\forall\, k \in A)\;\; [n_k, n_{k+1}) \subseteq X$, then $Y = \{\ell \cdot s_k(\ell - n_k) : k \in A,\; \ell \in [n_k, n_{k+1})\} \subseteq X$ satisfies that $(\exists^\infty\, k \in \omega)\;\; Y \notin F_k$, and therefore $X \notin \mathcal{I}$.

ii) implies i). Let $\{P_n : n \in \omega\}$ be an interval partition satisfying the hypothesis. For $n \in \omega$ let $\mathcal{I}_n = \{I \in \mathcal{I} : (\forall\, k \geq n)\;\; P_k \nsubseteq I\}$. We claim that $\mathcal{I}_n$ is a nowhere dense set for every $n \in \omega$, which is enough to prove the ideal is meager. Fix n. Let $s \in 2^{<\omega}$ and $k \geq n$ such that $k > \operatorname{len}(s)$. Let $t \in 2^{<\omega}$ such that $s \subseteq t$ and $(\forall\, m \in P_k)\;\; t(m) = 1$, then $\langle t \rangle \cap \mathscr{I}_n = \emptyset$, hence $\mathscr{I}_n$ is nowhere dense. ■

Chapter 3

The Bounded Topology

In this chapter we always assume that the involved ideals are ideals on ω. We will focus on the concept of bounded set in an ideal and use it to define a topology on it. Let $\mathcal{I}$ be an ideal, remember that $\mathcal{B} \subseteq \mathcal{I}$ is bounded if its union lies in $\mathcal{I}$, conversely, $\mathcal{B} \subseteq \mathcal{I}$ is unbounded if its union is a positive set of $\mathcal{I}$. Definition 3.1.1 presents a property weaker than bounded (named *weakly bounded* or *web*) and a property stronger than unbounded (named *strongly unbounded* or *sun*). First, we show some properties of web and sun sets, and the definition of an associated cardinal for the ideal $\mathcal{I}$, which is related to the work of A. Louveau and B. Veličković presented in [34] (Theorem 3.1.13). Later we will give a partial solution to a conjecture of them (Conjecture 3.4.1).

The main concept of this work is the *bounded topology* (Definition 3.2.3), we will study its properties and its relation with the combinatorics of the ideal. As an application, we present a classification of F_σ-ideals.

3.1 web and sun Sets

The following main definition was originally conceived by J. R. Isbell (see [27]).

Definition 3.1.1. Let $\mathcal{I}$ be an ideal.

a) $\mathcal{W} \subseteq \mathcal{I}$ is a *weakly bounded set* (or a *web set*) of $\mathcal{I}$ if it satisfies the following.

$$(\forall\, \mathcal{X} \in [\mathcal{W}]^{\omega})\, (\exists\, \mathcal{Y} \in [\mathcal{X}]^{\omega})\, \bigcup \mathcal{Y} \in \mathcal{I}.$$

b) $\mathcal{S} \subseteq \mathcal{I}$ is a *strongly unbounded set* (or a *sun set*) of $\mathcal{I}$ if it satisfies the following.

$$(\forall\, \mathcal{X} \in [\mathcal{S}]^{\omega})\, \bigcup \mathcal{X} \notin \mathcal{I}.$$

The previous definitions can be applied to directed orders rather than just ideals, although for a directed order such that any countable subset is bounded (for example in any σ-ideal) these definitions are trivial, because any subset is weakly bounded and no subset is strong unbounded. Under those circumstances, we can give a general definition, using subsets of size the additivity of the order instead of countable ones.

First, we present some immediate results.

Proposition 3.1.2. Let $\mathcal{I}$ be an ideal.

i) $\mathcal{F} \subseteq \mathcal{I}$ is a web and a sun set if and only if it is finite.

ii) Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set and $\mathcal{W}' \subseteq^* \mathcal{W}$, then $\mathcal{W}'$ is a web set.

iii) Let $\mathcal{S} \subseteq \mathcal{I}$ be a sun set and $\mathcal{S}' \subseteq^* \mathcal{S}$, then $\mathcal{S}'$ is a sun set.

iv) Let $\{\mathcal{W}_k : k \in n\} \in \mathcal{P}(\mathcal{I})$ be a family of web sets, then $\bigcup_{k\in n} \mathcal{W}_k$ is a web set.

v) Let $\{\mathcal{S}_k : k \in n\} \subseteq \mathcal{P}(\mathcal{I})$ be a family of sun sets, then $\bigcup_{k\in n} \mathcal{S}_k$ is a sun set.

vi) Let $\mathcal{A} \subseteq \mathcal{I}$ be a bounded set of $\mathcal{I}$, then $\mathcal{A}$ is a web set.

vii) Let $\mathcal{S} \subseteq \mathcal{I}$ be an infinite sun set of $\mathcal{I}$, then $\mathcal{S}$ is a unbounded set.

viii) $\mathcal{X} \subseteq \mathcal{I}$ is not a web set if and only if there is an infinite sun set $\mathcal{S} \subseteq \mathcal{X}$.

ix) If $\mathcal{I}$ is meager, then the partition given by Theorem 2.2.7 is a sun set.

x) Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, then $\mathcal{W}^{\downarrow}$ is a web set.

xi) Let $\mathcal{S} \subseteq \mathcal{I}$ be a sun set and $\mathcal{B} \subseteq \mathcal{I}$ be a basis for $\mathcal{I}$. For each $S \in \mathcal{S}$ let $B_S \in \mathcal{B}$ such that $S \subseteq B_S$. Then $\mathrm{ebs}(\mathcal{S}) = \{B_S : S \in \mathcal{S}\}$ is a sun set. Furthermore, if $\mathcal{S}$ is infinite then $|\mathcal{S}| = |\mathrm{ebs}(\mathcal{S})|$.

Proof. The first nine statements follows directly from Definition 3.1.1.

For x), suppose that $\mathcal{W}$ is infinite and let $\mathcal{X} \subseteq \mathcal{W}^{\downarrow}$ be an infinite subset. For each $I \in \mathcal{X}$ let $I' \in \mathcal{W}$ such that $I \subseteq I'$. If the set $\{I' : I \in \mathcal{X}\}$ is finite, then $\mathcal{X}$ is bounded. Otherwise, there is a countable family $\{I'_n : n \in \omega\} \subseteq \{I' : I \in \mathcal{X}\}$ which is bounded, then $\{I \in \mathcal{X} : (\exists\, n \in \omega)\, I \subseteq I'_n\}$ is a countable bounded subset of $\mathcal{X}$. Therefore $\mathcal{W}^{\downarrow}$ is a web set.

For xi), since $\mathcal{S}$ is a sun set then the set $\{S \in \mathcal{S} : S \subseteq B_S\}$ is finite for each $S \in \mathcal{S}$. Therefore $|\mathcal{S}| = |\mathrm{ebs}(\mathcal{S})|$ when $\mathcal{S}$ is infinite. Let $\mathcal{X} \subseteq \mathrm{ebs}(\mathcal{S})$ be an infinite subset, then the set $\mathcal{X}' = \{S \in \mathcal{S} : (\exists\, X \in \mathcal{X})\;\, S \subseteq X\}$ is a countable subset of $\mathcal{S}$, since $\mathcal{X}'$ is unbounded then $\mathcal{X}$ is unbounded too. Hence $\mathrm{ebs}(\mathcal{S})$ is a sun set. ■

We can use the web sets of an ideal to define a cardinal as the following.

Definition 3.1.3. Let $\mathcal{I}$ be an ideal. The *weakly bounded number* of $\mathcal{I}$ (or the *web-number* of $\mathcal{I}$), denoted by $\operatorname{web}(\mathcal{I})$, is defined as follows.

$$\operatorname{web}(\mathcal{I}) = \min\left\{|\mathcal{X}| : \mathcal{X} \subseteq \mathcal{P}(\mathcal{I}),\ (\forall\, \mathcal{W} \in \mathcal{X})\, \mathcal{W} \text{ is a web set, and } \bigcup \mathcal{X} = \mathcal{I}\right\}.$$

A family $\mathcal{X} \subseteq \mathcal{P}(\mathcal{I})$ that satisfies the condition for $\operatorname{web}(\mathcal{I})$ is a *web-family for* $\mathcal{I}$.

Note that the web number is well-defined because $\left\{I^{\downarrow} : I \in \mathcal{I}\right\} \subseteq \mathcal{P}(\mathcal{I})$ is a web-family for any ideal $\mathcal{I}$. This proves that $\operatorname{web}(\mathcal{I}) \leq |\mathcal{I}|$. We can improve this and give some better bounds for the web number.

Theorem 3.1.4. Let $\mathcal{I}$ be an ideal, then $\omega \leq \operatorname{web}(\mathcal{I}) \leq \operatorname{cof}(\mathcal{I})$.

Proof. Let $\mathcal{S} = \{[0, n] \subseteq \omega : n \in \omega\} \subseteq \mathcal{I}$ be the family of initial segments of ω, which is a sun set of $\mathcal{I}$. Let $\mathcal{X} = \{\mathcal{W}_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(\mathcal{I})$ be a web-family for $\mathcal{I}$, since $\mathcal{S} \cap \mathcal{W}_\alpha$ must be a finite set for each $\alpha \leq \kappa$ and $\bigcup \mathcal{X} = \mathcal{I}$, then we have that $\kappa \geq \omega$. Therefore $\omega \leq \operatorname{web}(\mathcal{I})$.

On the other hand, let $\mathcal{C} \subseteq \mathcal{I}$ be a basis for the ideal $\mathcal{I}$. For any $C \in \mathcal{C}$ the set $C^{\downarrow}$ is bounded (by C itself) then each of these sets is a web set, and since $\mathcal{C}$ is cofinal then $\left\{C^{\downarrow} : C \in \mathcal{C}\right\}$ is a web-family for $\mathcal{I}$. Therefore $\operatorname{web}(\mathcal{I}) \leq \operatorname{cof}(\mathcal{I})$. ■

The arguments used in the first part of the previous proof can be refined to get the following general result.

Theorem 3.1.5. Let $\mathcal{I}$ be an ideal and $\mathcal{S} \subseteq \mathcal{I}$ be a sun set, then $|\mathcal{S}| \leq \operatorname{web}(\mathcal{I})$.

Proof. Let $\mathcal{S} \subseteq \mathcal{I}$ be a sun set. If $\mathcal{S}$ is finite then the inequality holds. Then we can assume that $\mathcal{S}$ is infinite. For an infinite $\kappa < |\mathcal{S}|$ let $\mathcal{X} = \{\mathcal{W}_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(\mathcal{I})$ be a family that covers the ideal $\mathcal{I}$. There is an index $\beta < \kappa$ such that $|\mathcal{W}_\beta \cap \mathcal{S}| \geq \omega$. Then $\mathcal{W}_\beta \cap \mathcal{S}$ is an infinite sun set, hence $\mathcal{W}_\beta$ is not a web set and therefore $\mathcal{X}$ is not a web-family for $\mathcal{I}$. We conclude that $|\mathcal{S}| \leq \operatorname{web}(\mathcal{I})$. ■

By the previous result, $\operatorname{web}(\mathcal{I})$ is the maximum possible size of any sun set of $\mathcal{I}$. The following result is a characterization of when there is a sun set which reaches this maximum size. To enunciate it, we define a *disjoint almost-selection of X* as a map $f : X \to (\bigcup X \cup \{\emptyset\})$ such that there is a subset $Y \subseteq X$ with $|Y| = |X|$ satisfying that $(\forall\, x \in X \setminus Y)\ f(x) = \emptyset$ and $f{\restriction}_Y$ is an one-to-one selection map (that is, $f(y) \in y$ for all $y \in Y$).

Lemma 3.1.6. For an ideal $\mathscr{I}$ the following conditions are equivalent.

i) There is a sun set $\mathcal{S} \subseteq \mathscr{I}$ such that $|\mathcal{S}| = \mathrm{web}(\mathscr{I})$.

ii) For some web-family $\mathscr{X} \subseteq \mathcal{P}(\mathscr{I})$ there exists f which is a disjoint almost-selection of $\mathscr{X}$ such that $f[\mathscr{X}]$ is a sun set.

iii) For all web-family $\mathscr{X} \subseteq \mathcal{P}(\mathscr{I})$ there exists f which is a disjoint almost-selection of $\mathscr{X}$ such that $f[\mathscr{X}]$ is a sun set.

Proof. It follows directly that iii) implies ii) implies i). In order to prove the missing implication, let $\mathscr{X}$ be a web-family for the ideal $\mathscr{I}$ of minimum size and $\mathcal{S} \subseteq \mathscr{I}$ a sun set of size $\mathrm{web}(\mathscr{I})$. Let $g : \mathcal{S} \to \mathscr{X}$ such that $(\forall\, S \in \mathcal{S})\ \ S \in g(S)$. Since every infinite subset of $\mathcal{S}$ is unbounded, then g is finite-to-one. Then $\mathscr{Y} = g[\mathcal{S}] \subseteq \mathscr{X}$ is such that $|\mathscr{Y}| = |\mathscr{X}|$. Finally, a map $f : \mathscr{Y} \to \bigcup \mathscr{X}$ such that $(\forall\, \mathcal{Y} \in \mathscr{Y})\ \ f(\mathcal{Y}) \in g^{-1}[\mathcal{Y}]$ induces a disjoint almost-selection of $\mathscr{X}$. ■

The Tukey order preserves the web number and the size of sun sets in the following sense.

Theorem 3.1.7. Let $\mathscr{I}, \mathscr{J}$ be a pair of ideals and $F : \mathscr{I} \to \mathscr{J}$ be a Tukey map. The following holds.

i) If $\mathcal{W} \subseteq \mathscr{J}$ is a web set then $F^{-1}[\mathcal{W}] \subseteq \mathscr{I}$ is a web set. In particular, $\mathscr{I} \leq_T \mathscr{J}$ implies $\mathrm{web}(\mathscr{I}) \leq \mathrm{web}(\mathscr{J})$.

ii) If $\mathcal{S} \subseteq \mathscr{I}$ is a sun set then $F[\mathcal{S}] \subseteq \mathscr{J}$ is a sun set. In particular, $\mathscr{I} \leq_T \mathscr{J}$ implies that for a sun set $\mathcal{S} \subseteq \mathscr{I}$ there is a sun set $\mathcal{S}' \subseteq \mathscr{J}$ such that $|\mathcal{S}| = |\mathcal{S}'|$.

Proof. For i). Let $\mathcal{X} \subseteq F^{-1}[\mathcal{W}]$ be an infinite set. Then there is a bounded set $\mathcal{Y} \subseteq F[\mathcal{X}] \subseteq \mathcal{W}$ such that $F^{-1}[\mathcal{Y}]$ is infinite (If $F[\mathcal{X}]$ is finite take $\mathcal{Y} = F[\mathcal{X}]$, otherwise use that $\mathcal{W}$ is a web set). Then, $F^{-1}[\mathcal{Y}] \cap \mathcal{X}$ is an infinite bounded subset of $\mathcal{X}$ and therefore $F^{-1}[\mathcal{W}]$ is a web set. Hence, if $\mathscr{X}$ is a witness for $\mathrm{web}(\mathscr{J})$ then $\{F^{-1}[\mathcal{W}] : \mathcal{W} \in \mathscr{X}\}$ satisfies the condition for $\mathrm{web}(\mathscr{I})$. Hence $\mathrm{web}(\mathscr{I}) \leq \mathrm{web}(\mathscr{J})$.

For ii). Let $\mathcal{X} \subseteq F[\mathcal{S}]$ be an infinite set. Since $\mathcal{S}$ is a sun set we have that $F{\restriction}_{\mathcal{S}}$ is a finite-to-one map, then $F{\restriction}_{\mathcal{S}}^{-1}[\mathcal{X}] \subseteq \mathcal{S}$ is an infinite set. Therefore $F{\restriction}_{\mathcal{S}}^{-1}[\mathcal{X}] \subseteq \mathscr{I}$ is an unbounded set and then $\mathcal{X} \subseteq \mathscr{J}$ is also unbounded. This shows that $F[\mathcal{S}] \subseteq \mathscr{J}$ is a sun set. This shows that if $\mathcal{S} \subseteq \mathscr{I}$ is a sun set then $\mathcal{S}' = F[\mathcal{S}] \subseteq \mathscr{J}$ is a sun set such that $|\mathcal{S}| = |\mathcal{S}'|$. ■

As mentioned before, each bounded set of an ideal is a web set, for the inverse implication we define the following class of ideals.

Definition 3.1.8. Let $\mathcal{I}$ be an ideal. We say that $\mathcal{I}$ is a *web-regular ideal* if each web set of $\mathcal{I}$ is bounded.

Proposition 3.1.9. Let $\mathcal{I}$ be a web-regular ideal. Then $\mathcal{I}$ is not a tall ideal and $\mathrm{web}(\mathcal{I}) = \mathrm{cof}\,(\mathcal{I})$.

Proof. Let $\mathcal{S} = \{\{n\} : n \in \omega\} \subseteq \mathcal{I}$. If $\mathcal{I}$ is a tall ideal, then $\mathcal{S}$ is an infinite unbounded web set, then $\mathcal{I}$ is not a web-regular ideal. Therefore the first part of implication holds.

Now, let $\mathcal{X}$ be a web-family from $\mathcal{I}$. Since $\mathcal{I}$ is a web-regular ideal, then for each $\mathcal{W} \in \mathcal{X}$ there is a $I_{\mathcal{W}} \in \mathcal{I}$ such that $\bigcup \mathcal{W} = I_{\mathcal{W}}$. Since $\bigcup \mathcal{X} = \mathcal{I}$ then $\{I_{\mathcal{W}} : \mathcal{W} \in \mathcal{X}\}$ is a cofinal subset of $\mathcal{I}$, therefore $\mathrm{cof}\,(\mathcal{I}) \leq \mathrm{web}(\mathcal{I})$, and because the inverse inequality always holds, we have the equality. ■

In the previous proof, the set $\mathcal{S}$ seems to be *ad hoc*, since although it is a web set in each tall ideal, any of its elements is a finite set (indeed, it is made by singletons). But, even if we restrict the definition of web-regular only for the web sets forming an almost-disjoint family where each of its elements is infinite, the conclusion is the same. Indeed, let $\mathcal{I}$ be a tall ideal, $I \in \mathcal{I}$ be an infinite set and $\{I_n : n \in \omega\}$ be a infinite partition of I in infinite sets, then the set $\{I_n \cup \{n\} : n \in \omega\}$ satisfies the previous conditions and it is an unbounded web set of $\mathcal{I}$.

From the inequalities shown in Theorem 3.1.4, we have that any ω-generated ideal satisfies that $\mathrm{web}(\mathcal{I}) = \mathrm{cof}\,(\mathcal{I})$. The following result shows that actually something stronger is happening.

Theorem 3.1.10. Let $\mathcal{I}$ be a ω-generated ideal. Then $\mathcal{I}$ is a web-regular ideal.

Proof. Let $\{I_n : n \in \omega\}$ be an $\subseteq$-increasing basis for $\mathcal{I}$. For each $I \in \mathcal{I}$ let $n_I = \min\{n \in \omega : I \subseteq I_n\}$. Let $\mathcal{W} \subseteq \mathcal{I}$ be an infinite web set and let $X = \{n_W : W \in \mathcal{W}\}$. If X is infinite then there is a countable set $\mathcal{S} \subseteq \mathcal{W}$ such that $n_S \neq n_{S'}$ whenever S, S' are distinct elements of $\mathcal{S}$. Then, since $(\forall\, \mathcal{X} \in [\mathcal{S}]^{\omega})\,(\forall\, n \in \omega)\ \bigcup \mathcal{X} \nsubseteq I_n$, $\mathcal{S}$ is a sun set. Therefore X has a maximum element N and $\mathcal{W}$ is bounded by I_N. ■

However, the following shows that not every web-regular ideal is ω-generated.

Proposition 3.1.11. $\emptyset \times \mathrm{Fin}$ is web-regular ideal and $\mathrm{web}(\emptyset \times \mathrm{Fin}) = \mathfrak{d}$.

Proof. Let $\mathcal{B} \subseteq \emptyset \times \mathrm{Fin}$ be an unbounded set, then $\bigcup \mathcal{B} \cap \{m\} \times \omega$ is infinite for some $m \in \omega$. Therefore, for any $n \in \omega$ exists $B_n \in \mathcal{B}$ such that $\max\{k : (m,k) \in B_n\} \geq n$. Hence the set $\{B_n : n \in \omega\} \subseteq \mathcal{B}$ is a sun set, thus $\mathcal{B}$ is not a web set. This shows that $\emptyset \times \mathrm{Fin}$ is web-regular ideal.

Let $F : \omega^\omega \to \emptyset \times \mathrm{Fin}$ given by $F(f) = \bigcup\{\{n\} \times f(n) : n \in \omega\}$. The image of the map F is a cofinal subset of $\emptyset \times \mathrm{Fin}$ and then $(\omega^\omega, \leq) \equiv_T F[\omega^\omega] \equiv_T \emptyset \times \mathrm{Fin}$. Thus $\mathrm{web}\,(\emptyset \times \mathrm{Fin}) = \mathrm{cof}\,(\omega^\omega, \leq) = \mathfrak{d}$. ■

Using Theorem 2.1.14 and Proposition 3.1.11 we can improve the bounds of Theorem 3.1.4 for analytic P-ideals as follows.

Proposition 3.1.12. Let $\mathcal{I}$ be an analytic P-ideal. Then either $\mathfrak{d} \leq \mathrm{web}(\mathcal{I}) \leq \mathrm{cof}\,(\mathcal{N})$ or $\mathrm{web}(\mathcal{I}) = \omega$.

Proof. If $\mathcal{I}$ is ω-generated then it is a web-regular ideal and thus $\mathrm{web}(\mathcal{I}) = \mathrm{cof}\,(\mathcal{I}) = \omega$. Then, we assume that $\mathcal{I}$ is not ω-generated. By Theorem 2.1.14 we have that $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I} \leq_T \mathcal{I}_{1/n}$. Using that $\mathrm{cof}(\mathcal{I}_{1/n}) = \mathrm{cof}\,(\mathcal{N})$ (proved in [17]), Theorem 3.1.7 and Proposition 3.1.11, we conclude that $\mathfrak{d} \leq \mathrm{web}(\mathcal{I}) \leq \mathrm{web}\left(\mathcal{I}_{1/n}\right) \leq \mathrm{cof}\,(\mathcal{N})$. ■

Using our notation, we can write a result due to A. Louveau and B. Veličković in the following form (originally in [34, Theorem 1 and 2]). This is a very relevant result for the rest of the chapter. Later we deal with a conjecture given by them about an improvement of this result (Conjecture 3.4.1).

Theorem 3.1.13 (Louveau and Veličković, [34])**.** Let $\mathcal{I}$ be an analytic ideal.

i) $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ if and only if $\mathrm{web}(\mathcal{I}) \geq \mathfrak{d}$.

ii) If $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$, then $\mathcal{I}$ is an F_σ-ideal.

3.2 Bounded Topology

Recall that an ideal $\mathcal{I}$ can be seen has a topological subspace of 2^ω with the usual topology and the related concepts are about this topology. For example in the following result, $\mathcal{U} \cap \mathcal{X}$ *open in* $\mathcal{X}$ means that it is open in the usual subspace $\mathcal{X}$.

Proposition 3.2.1. Let $\mathcal{I}$ be an ideal and $\mathcal{U} \subseteq \mathcal{I}$. The following are equivalent.

i) For any $\mathcal{W} \subseteq \mathcal{I}$ weakly bounded set, $\mathcal{U} \cap \mathcal{W}$ is open in $\mathcal{W}$.

ii) For any $\mathcal{K} \subseteq \mathcal{I}$ weakly bounded and compact set, $\mathcal{U} \cap \mathcal{K}$ is open in $\mathcal{K}$.

iii) For any $\mathcal{B} \subseteq \mathcal{I}$ bounded set, $\mathcal{U} \cap \mathcal{B}$ is open in $\mathcal{B}$.

iv) For any $I \in \mathcal{I}$, $\mathcal{U} \cap \mathcal{P}\,(I)$ is open in $\mathcal{P}\,(I)$.

Proof. Since for every $I \in \mathcal{I}$ the set $\mathcal{P}(I) \subseteq \mathcal{I}$ is bounded and compact, then directly follows that i) implies ii) implies iv) and that i) implies iii) implies iv). To see the missing implication, let $\mathcal{W} \subseteq \mathcal{I}$ be a web set and let $I \in \mathcal{U} \cap \mathcal{W}$. If for all $n \in \omega$ there exists $I_n \in (\langle I\restriction_n\rangle \cap \mathcal{W}) \setminus \mathcal{U}$, then there is a subsequence $\{I_{n_m} : m \in \omega\}$ that converges to I, which is bounded by some $J \in \mathcal{I}$ and disjoint from $\mathcal{U}$. But this contradicts that $\mathcal{U} \cap \mathcal{P}(I \cup J)$ is open in $\mathcal{P}(I \cup J)$. Hence, there is some $n \in \omega$ such that $\langle I\restriction_n\rangle \cap \mathcal{W} \subseteq \mathcal{U}$, therefore $\mathcal{U} \cap \mathcal{W}$ is open in $\mathcal{W}$. ■

The previous proposition is clearly also true replacing «open» by «closed». Also we have the following way to verify if a set satisfies such conditions.

Lemma 3.2.2. Let $\mathcal{I}$ be an ideal. Let $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{I})$ be a family of web sets which is cofinal among the web sets of $\mathcal{I}$. If a set $\mathcal{U} \subseteq \mathcal{I}$ satisfies that $\mathcal{U} \cap \mathcal{X}$ is open in $\mathcal{X}$ for all $\mathcal{X} \in \mathfrak{X}$, then $\mathcal{U}$ satisfies any condition of Proposition 3.2.1.

Proof. Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, then there is some $\mathcal{X} \in \mathfrak{X}$ such that $\mathcal{W} \subseteq \mathcal{X}$. Since $\mathcal{U}$ is open in $\mathcal{X}$, for all $I \in \mathcal{W}$ there is some open set $\mathcal{V} \subseteq \mathcal{I}$ such that $I \in \mathcal{V} \cap \mathcal{X} \subseteq \mathcal{U} \cap \mathcal{X}$. Therefore $I \in \mathcal{V} \cap \mathcal{W} \subseteq \mathcal{U} \cap \mathcal{W}$, and hence $\mathcal{U} \cap \mathcal{W}$ is open in $\mathcal{W}$. ■

We will use Proposition 3.2.1 to define a new topology on ideals, the bounded topology, which is the main concept for this chapter.

Definition 3.2.3. Let $\mathcal{I}$ be an ideal. Define the *bounded topology on* $\mathcal{I}$, denoted by τ_{bd}, letting $\mathcal{U} \in \tau_{\text{bd}}$ if and only if $\mathcal{U} \subseteq \mathcal{I}$ satisfies any of the conditions in Proposition 3.2.1.

Directly from this definition we have the following properties for τ_{bd}.

Proposition 3.2.4. Let $\mathcal{I}$ be an ideal and τ its usual topology. The following holds.

i) τ_{bd} is finer than τ.

ii) $(\mathcal{I}, \tau_{\text{bd}})$ is a Hausdorff space.

iii) $(\mathcal{W}, \tau_{\text{bd}}\restriction_{\mathcal{W}}) = (\mathcal{W}, \tau\restriction_{\mathcal{W}})$ for any web set $\mathcal{W} \subseteq \mathcal{I}$.

As stated before, the topological concepts of an ideal refers to the usual topology. To differentiate between topologies we will use the prefix «τ_{bd}-» on properties. For example, we will say that $\mathcal{K} \subseteq \mathcal{I}$ is a τ_{bd}*-compact* set if it is compact in the bounded topology, and we will say that it is a *compact* set if it is compact in the usual topology.

3.3 Parallel Results

In this section we will present some results that relate combinatorial properties of an ideal $\mathcal{I}$ with the topological properties of its bounded topology, drawing a parallel between these two branches. The following is a key result of this section, determining compact and closed-discrete sets of the space $(\mathcal{I}, \tau_{\text{bd}})$.

Proposition 3.3.1. Let $\mathcal{I}$ be an ideal. The following holds.

i) $\mathcal{K} \subseteq \mathcal{I}$ is a τ_{bd}-compact set if and only if $\mathcal{K}$ is a compact and web set.

ii) $\mathcal{S} \subseteq \mathcal{I}$ is a τ_{bd}-closed and τ_{bd}-discrete set if and only if $\mathcal{S}$ is a sun set.

Proof. For the «if» part of i). Let $\mathcal{K} \subseteq \mathcal{I}$ be a compact and web set and $\mathfrak{U}$ be an τ_{bd}-open cover of $\mathcal{K}$. By definition of bounded topology, $\mathcal{V} = \{\mathcal{U} \cap \mathcal{K} : \mathcal{U} \in \mathfrak{U}\}$ is an open cover of $\mathcal{K}$. Since $\mathcal{K}$ is compact, $\mathcal{V}$ has a finite subcover. Then $\mathfrak{U}$ has the respective finite subcover, therefore $\mathcal{K}$ is a τ_{bd}-compact set.

For ii). Let $\mathcal{S} \subseteq \mathcal{I}$ be a τ_{bd}-closed and τ_{bd}-discrete set. By the previous part, the set $\mathcal{P}(I) \subseteq \mathcal{I}$ is a τ_{bd}-compact set for all $I \in \mathcal{I}$, then $\mathcal{S} \cap \mathcal{P}(I)$ is finite for all $I \in \mathcal{I}$, this proves that $\mathcal{S}$ is a sun set. On the other hand, let $\mathcal{S} \subseteq \mathcal{I}$ be a sun set, then for all web sets $\mathcal{W} \subseteq \mathcal{I}$ we have that $\mathcal{S} \cap \mathcal{W}$ is finite, and hence closed in $\mathcal{W}$. Therefore $\mathcal{S}$ is a τ_{bd}-closed set which any of its subsets is also a τ_{bd}-closed set, then it is a τ_{bd}-discrete set.

For the missing part of i). Let $\mathcal{K} \subseteq \mathcal{I}$ be a τ_{bd}-compact set. $\mathcal{K}$ is a compact set since $\tau \subseteq \tau_{\text{bd}}$. Also, it does not has any infinite τ_{bd}-closed and τ_{bd}-discrete subset, then by the previous, $\mathcal{K}$ does not contain any infinite sun set and therefore it is a web set. ■

The previous result implies that $(\mathcal{I}, \tau_{\text{bd}})$ is a k-space; later we will show that, actually, it is a sequential space. From the i) part we have that any bounded and compact set is τ_{bd}-compact, such as $\mathcal{P}(I)$ for any $I \in \mathcal{I}$. We also can use it in sequences and obtain the following result.

Corollary 3.3.2. Let $\mathcal{I}$ be an ideal, $\mathcal{X} \subseteq \mathcal{I}$ be a sequence and $I \in \mathcal{I}$. Then $\mathcal{X}$ τ_{bd}-converges to I if and only if $\mathcal{X}$ is a web set and converges to I.

Proof. If $\mathcal{X}$ τ_{bd}-converges to I then $\mathcal{X} \cup \{I\}$ is a τ_{bd}-compact set. So $\mathcal{X} \cup \{I\}$, and hence $\mathcal{X}$, is a web set. Also, $\mathcal{X}$ converges to I since τ_{bd} is finer that the usual topology. On the other hand, if $\mathcal{X}$ is a web set and converges to I then $\mathcal{X} \cup \{I\}$ is a τ_{bd}-compact set. Since τ_{bd} is finer that the usual topology, for any $X \in \mathcal{X}$ there is a $\mathcal{U}_X \in \tau_{\text{bd}}$ such that $\mathcal{U}_X \cap \mathcal{X} = \{X\}$. This shows that $\mathcal{X}$ τ_{bd}-converges to I. ■

From the previous corollary we have that, in particular, any τ_{bd}-convergent sequence has a bounded subsequence which converges to the same point.

Let $\mathcal{I}$ be an ideal. By Proposition 3.3.1 we have that the extend $e(\mathcal{I}, \tau_{\mathrm{bd}})$ is the supremum of sizes of the sun sets of $\mathcal{I}$. Then by Theorem 3.1.5 and the proof of Theorem 3.1.4 we have that $\omega \leq e(\mathcal{I}, \tau_{\mathrm{bd}}) \leq \mathrm{web}(\mathcal{I})$.

K. Beros and P. Larson show that $\mathcal{I}_{1/n}$ has no uncountable sun sets (in [4, Proposition 3.4]). Then, using Theorem 2.1.14 and Theorem 3.1.7(ii), we have already proved the following result.

Proposition 3.3.3. Let $\mathcal{I}$ be an analytic P-ideal. Then $e(\mathcal{I}, \tau_{\mathrm{bd}}) = \omega$.

An ideal $\mathcal{I}$ has an uncountable sun set if and only if $e(\mathcal{I}, \tau_{\mathrm{bd}}) > \omega$. If this is the case, $(\mathcal{I}, \tau_{\mathrm{bd}})$ is not Lindelöf. We conjecture that this is the only reason for that.

Conjecture 3.3.4. Let $\mathcal{I}$ be an ideal such that $e(\mathcal{I}, \tau_{\mathrm{bd}}) = \omega$. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is Lindelöf.

As mentioned before, a subset $\mathcal{C} \subseteq \mathcal{I}$ is τ_{bd}-closed if and only if $\mathcal{C} \cap \mathcal{W}$ is closed in $\mathcal{W}$ for any web set $\mathcal{W} \subseteq \mathcal{I}$. The following result give us other characterization for them using its bounded convergent sequences.

Lemma 3.3.5. Let $\mathcal{I}$ be an ideal and $\mathcal{F} \subseteq \mathcal{I}$. The following conditions are equivalent.

i) $\mathcal{F}$ is a τ_{bd}-closed set.

ii) If a bounded sequence in $\mathcal{F}$ converges to some $I \in \mathcal{I}$, then $I \in \mathcal{F}$.

Proof. i) implies ii). Let $\mathcal{X} \subseteq \mathcal{F}$ be a bounded sequence that converges to some $I \in \mathcal{I}$, then $\mathcal{X} \cup \{I\}$ is a web set. By Proposition 3.2.1(i) in its closed set form, $\mathcal{F} \cap (\mathcal{X} \cup \{I\})$ is a closed set in $\mathcal{X} \cup \{I\}$, therefore $I \in \mathcal{F}$.

ii) implies i). Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set and let $I \in \mathcal{W}$ be a τ_{bd}-limit point of $\mathcal{F} \cap \mathcal{W}$. Since there is a bounded sequence $\mathcal{X} \subseteq \mathcal{F}$ which converges to I, then by the hypothesis we have that $I \in \mathcal{F} \cap \mathcal{W}$, and therefore $\mathcal{F} \cap \mathcal{W}$ is a closed set in $\mathcal{W}$. ■

In order to give a deeper understanding we deduce some topological properties of the bounded topology in the following result. We will use the following map: for $I \in \mathcal{P}(\omega)$ the bijective map $\mathrm{trs}_I : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ given by $\mathrm{trs}_I(A) = A \triangle I$ is the *translation by* I. Note that for an ideal $\mathcal{I}$ and $I \in \mathcal{I}$ we have that $\mathrm{trs}_I : \mathcal{I} \to \mathcal{I}$.

Theorem 3.3.6. Let $\mathcal{I}$ be an ideal. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a homogeneous, separable and sequential space.

Proof. The previous lemma and Corollary 3.3.2 imply that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is sequential.

We will prove that $[\omega]^{<\omega} \subseteq \mathcal{I}$ is a τ_{bd}-dense set. Let $\mathcal{U} \subseteq \mathcal{I}$ be an τ_{bd}-open set and $I \in \mathcal{U}$. Then $\mathcal{U} \cap \mathcal{P}(I)$ is an open set in $\mathcal{P}(I)$ and there is some $n \in \omega$ such that $\langle I{\restriction}_n \rangle \cap \mathcal{P}(I) \subseteq \mathcal{U} \cap \mathcal{P}(I)$, hence $\mathcal{U}$ contains a finite subset of I.

Let $I \in \mathcal{I}$, then $\mathrm{trs}_I : \mathcal{I} \to \mathcal{I}$ is bijective and $\mathrm{trs}_I(\emptyset) = I$. Since it is its own inverse, if trs_I is τ_{bd}-continuous then it is a τ_{bd}-homeomorphism. Let $\mathcal{X} \subseteq \mathcal{I}$ be a bounded sequence that converges to some $J \in \mathcal{I}$, then $\mathrm{trs}_I[\mathcal{X}] \subseteq \mathcal{I}$ is a bounded sequence that convergent to $\mathrm{trs}_I(J)$, hence trs_I is a τ_{bd}-sequentially continuous map, and therefore it is a τ_{bd}-continuous map. So, $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homogeneous. ■

For metric or second-countable spaces, the concepts of compact and sequentially compact on subsets agree, but in general they are not the same (indeed, there are compact and non-sequentially compact spaces such as the Stone-Čech compactification of ω; on the other hand, $[0, \omega_1)$ is a sequentially compact and non-compact space). Later we will see that $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ is neither metrizable nor second-countable, however, as we show below, both concepts are always equivalent in the bounded topology.

Proposition 3.3.7. Let $\mathcal{I}$ be an ideal and $\mathcal{K} \subseteq \mathcal{I}$. $\mathcal{K}$ is a τ_{bd}-compact set if and only if $\mathcal{K}$ is a τ_{bd}-sequentially compact set.

Proof. Since in metric spaces the concepts of compact and sequentially compact are the same and $(\mathcal{W}, \tau_{\mathrm{bd}}{\restriction}_{\mathcal{W}}) = (\mathcal{W}, \tau{\restriction}_{\mathcal{W}})$ for a web set $\mathcal{W}$, then it is enough to show that $\mathcal{K} \subseteq \mathcal{I}$ is a web set if it is τ_{bd}-compact or τ_{bd}-sequentally compact. For τ_{bd}-compact sets this holds by Proposition 3.3.1(i). Now let $\mathcal{K} \subseteq \mathcal{I}$ be an infinite τ_{bd}-sequentially compact set, then any sequence of $\mathcal{K}$ has a τ_{bd}-convergent subsequence, in particular it has a bounded infinite subsequence, therefore $\mathcal{K}$ is a web set. ■

We recall the following definition.

Definition 3.3.8. Let $\mathcal{A} = (\omega \times \omega) \cup \omega \cup \{\infty\}$ and $\tau_{\mathcal{A}}$ the topology on it defined as follows. The space $(\mathcal{A}, \tau_{\mathcal{A}})$ is know as the *Arens' space* [1].

a) $\omega \times \omega \subseteq \mathcal{A}$ is a discrete subspace.

b) For each $n, m \in \omega$ let $B_{n,m} = \{n\} \cup \{(n,k) \in \omega \times \omega : k \geq m\}$. Then the family $\{B_{n,m} : m \in \omega\} \subseteq \mathcal{P}(\mathcal{A})$ is a neighborhood basis at $n \in \omega \subseteq \mathcal{A}$.

c) For each $f \in \omega^\omega$ and $n \in \omega$ let $B_n^f = \bigcup \{B_{m,f(m)} : m \geq n\}$. Then the family $\{B_n^f : f \in \omega^\omega,\ n \in \omega\} \subseteq \mathcal{P}(\mathcal{A})$ is a neighborhood basis at $\infty \in \mathcal{A}$.

Arens' space is a canonical example of a space that is sequential but it is not Fréchet–Urysohn, and these kind of spaces are determined by it, since a sequential space is Fréchet–Urysohn if and only if it does not contains a copy of Arens' space (see [49, Proposition 2.2]). Using this result, we can determine whenever the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is Fréchet–Urysohn.

Theorem 3.3.9. Let $\mathcal{I}$ be an ideal. $(\mathcal{I}, \tau_{\mathrm{bd}})$ is Fréchet–Urysohn if and only if $\mathcal{I}$ is a P-ideal.

Proof. We will prove that $\mathcal{I}$ is a non-P-ideal if and only if $(\mathcal{I}, \tau_{\mathrm{bd}})$ contains a subspace homeomorphic to Arens' space.

Let $\mathcal{F} = \{I_n : n \in \omega\} \subseteq \mathcal{I} \cap [\omega]^\omega$ be a pairwise disjoint family witness that $\mathcal{I}$ is a non-P-ideal. For all n let $\{i^n_m : m \in \omega\}$ be the increasing enumeration for the set I_n. For $n, m \in \omega$ let

$$J^n_m = \{i^0_n\} \cup \bigcup_{k=1}^{n+1} I_k \setminus \{i^a_b : 1 \leq a \leq n+1,\ b < m\}.$$

We claim that the set $\mathcal{A} = \{J^n_m : n, m \in \omega\} \cup \{\{i^0_n\} : n \in \omega\} \cup \{\emptyset\} \subseteq \mathcal{I}$ is the Arens' space. The sequence $\{\{i^0_n\} : n \in \omega\}$ is bounded by I_0, then it τ_{bd}-converges to $\emptyset$. Since $\mathcal{F}$ is pairwise disjoint, then we can isolate each point of the form J^n_m of $\mathcal{A}$ with the following open subset of $\mathcal{I}$:

$$\mathcal{I} \cap \left\{A \subseteq \omega \setminus \left\{i^{n+2}_m\right\} : \{i^a_m : 1 \leq a \leq n+1\} \subseteq A \ \wedge\ \left\{i^a_{m-1} : 1 \leq a \leq n+1\right\} \cap A = \emptyset\right\}$$

Also, for a fixed n, the sequence $\{J^n_m : m \in \omega\}$ τ_{bd}-converges to $\{i^0_n\}$ since it is bounded by $\bigcup \{I_k : k \leq n+1\}$. Then it only remains to prove that for every $g \in \omega^\omega$, the diagonal sequence $\mathcal{X}_g = \{J^n_{g(n)} : n \in \omega\}$ is a sun set, because then $\mathcal{X}_g$ does not τ_{bd}-converges to $\emptyset$. Let $\mathcal{X} \subseteq \mathcal{X}_g$ be an infinite set, since for every $k \in \omega$ there is some $n_k \in \{n \in \omega : J^n_{g(n)} \in \mathcal{X}\}$ such that $I_k \subseteq^* J^{n_k}_{g(n_k)}$, then a bound for $\mathcal{X}$ is a pseudo-union for $\mathcal{F}$. Therefore $\mathcal{X}_g$ is a sun set.

On the other hand, let $\mathcal{A} = \{I^n_m : n, m \in \omega\} \cup \{I_n : n \in \omega\} \cup \{I\}$ be an Arens' subspace of $(\mathcal{I}, \tau_{\mathrm{bd}})$. The sequence $\{I_n : n \in \omega\}$ is a web set because it τ_{bd}-converges to I, so we can suppose that it is actually bounded by some $L \in \mathcal{I}$. Analogously, for any $n \in \omega$ we can suppose that the sequence $\{I^n_m : m \in \omega\}$ is bounded by some $J_n \in \mathcal{I}$. If $(\forall\, n \in \omega)\ J_n \subseteq^* J$ for some $J \in \mathcal{I}$, then for a fixed $n \in \omega$ there is $m_n \in \omega$ such that $\bigcup \{I^n_m : \Delta(I^n_m, I_n) > m_n\} \subseteq J \cup I_n$. Hence $(\forall\, n \in \omega)\, (\forall^\infty\, m \in \omega)\ I^n_m \subseteq J \cup L \cup I$. Thus, there exists $X \in [\omega]^\omega$ and $g \in \omega^\omega$ such that the sequence $\{I^n_{g(n)} : n \in X\}$ τ_{bd}-converges to I, but this contradicts the hypothesis on $\mathcal{A}$. Therefore $\{J_n : n \in \omega\}$ witnesses that $\mathcal{I}$ is a non-P-ideal. ■

Since any metrizable space is Fréchet–Urysohn, if the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is metrizable (or even first-countable) then $\mathcal{I}$ must be a P-ideal. Then the previous lemma implies that $\tau = \tau_{\mathrm{bd}}$ is only possible for P-ideals. The following result give equivalent conditions for this equality. The condition on iii) part in the following theorem is a particular case of the definition of *basic order* due to S. Solecki and S. Todorčević (see [44, Section 3]); and the equivalence between iii) and iv) was originally proved by N. Dobrinen and S. Todorčević in [11, Theorem 4].

Theorem 3.3.10. Let $\mathcal{I}$ be an ideal. The following conditions are equivalent.

i) $\tau = \tau_{\mathrm{bd}}$.

ii) Any compact set of $\mathcal{I}$ is a web set.

iii) Any convergent sequence of $\mathcal{I}$ has a bounded subsequence.

iv) $\mathcal{I}$ is a non-meager P-ideal.

Proof. Using Proposition 3.3.1 we have that i) implies ii) since any compact set is τ_{bd}-compact and hence a web set. To prove ii) implies iii), let $\mathcal{X}$ be a squence that converges to some $I \in \mathcal{I}$, then $\mathcal{X} \cup \{I\}$ is compact and hence, again by Proposition 3.3.1, $\mathcal{X}$ is a web set. Thus $\mathcal{X}$ has a bounded subsequence .

iii) implies i). Let $\mathcal{U}$ be a τ_{bd}-open set, we will prove that $\mathcal{U}$ is a open set. Let $I \in \mathcal{U}$. If for all n there exists $I_n \in (\langle I{\restriction}_n\rangle \cap \mathcal{I}) \setminus \mathcal{U}$ then, by the hypothesis and since $I_n \to I$, the set $\mathcal{K} = \{I_n : k \in \omega\} \cup \{I\} \subseteq \mathcal{I}$ is compact and weakly bounded. But this contradicts that $\mathcal{U}$ is τ_{bd}-open since $\mathcal{U} \cap \mathcal{K} = \{I\}$. Hence $\tau_{\mathrm{bd}} = \tau$.

iii) implies iv). Let $\{P_n : n \in \omega\}$ be an interval partition of ω, then it converges to $\emptyset$; using the hypothesis, there exists a sequence $\{n_k : n \in \omega\}$ such that $\{P_{n_k} : k \in \omega\}$ is bounded. Then by Theorem 2.2.7, the ideal $\mathcal{I}$ is non-meager. Besides, let $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ and for $n \in \omega$ let $J_n = \bigcup_{k \leq n} I_k \setminus n$. Since $\{J_n : n \in \omega\} \subseteq \mathcal{I}$ converges to $\emptyset$, there is a subsequence bounded by some $I \in \mathcal{I}$ which is a pseudo-union of $\{I_n : n \in \omega\}$, then $\mathcal{I}$ is a P-ideal.

iv) implies iii). Let $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ be a sequence which converges to I. We can suppose that $\Delta(I_n, I) < \Delta(I_{n+1}, I)$ for all $n \in \omega$. Since $\mathcal{I}$ is a P-ideal, the family $\{I_n \setminus I : n \in \omega\}$ has a pseudo-union $J \in \mathcal{I}$. Hence, for all $n \in \omega$ the set $F_n = (I_n \setminus I) \setminus J$ is finite. Let $\{E_m : m \in \omega\}$ be an interval partition of ω such that for all $m \in \omega$ there is some $n_m \in \omega$ with $F_{n_m} \subseteq E_m$ (such partition exist because $\Delta(I_n, I) < \Delta(I_{n+1}, I)$). By Theorem 2.2.7, since $\mathcal{I}$ is non-meager, there is $A \in [\omega]^\omega$ such that $L = \bigcup\{E_m : m \in A\} \in \mathcal{I}$. Therefore, the subsequence $\{I_{n_m} : m \in A\} \subseteq \{I_n : n \in \omega\}$ is bounded by $I \cup J \cup L$. ■

For an ideal $\mathcal{I}$, any infinite set $\mathcal{X} \subseteq \mathcal{I}$ is unbounded if and only if $\mathcal{I} = \mathrm{Fin}$. Hence Fin is the only ideal which is a sun set of itself and then, by Proposition 3.3.1(ii), $(\mathcal{I}, \tau_{\mathrm{bd}}) \simeq \omega$ if and only if $\mathcal{I} = \mathrm{Fin}$. In this case there exists $A \in \mathcal{I}$ such that $\mathcal{P}(A)$ is an τ_{bd}-open set. As we show below, this property only holds for ideals of the form $\mathrm{Fin} \oplus \mathcal{P}(A) = \{I \subseteq \omega : |I \setminus A| < \omega\}$ for some $A \subseteq \omega$. Note that $\mathrm{Fin} \oplus \mathcal{P}(A) \cong \mathrm{Fin}$.

Lemma 3.3.11. Let $\mathcal{I}$ be an ideal and let $A \in \mathcal{I}$. Then $\mathcal{P}(A)$ is an τ_{bd}-open set if and only if $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$.

Proof. If $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$ then is straightforward to see that $\mathcal{P}(A)$ is an τ_{bd}-open set since any web set of $\mathcal{I}$ has only finite points of its Fin part. On the other hand, let $A \in \mathcal{I}$ such that $\mathcal{P}(A)$ is an τ_{bd}-open set and let $B \in \mathcal{I}$, since $\mathcal{P}(A) \cap \mathcal{P}(B)$ is an open set in $\mathcal{P}(B)$, there is some $n \in \omega$ such that $\langle (A \cap B)\restriction_n \rangle \cap \mathcal{P}(B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, and then $B \setminus n \in \mathcal{P}(A)$, therefore $(\forall B \in \mathcal{I})\, B \subseteq^* A$ and hence $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$. ■

The previous lemma implies that $(\mathrm{Fin} \oplus \mathcal{P}(A), \tau_{\mathrm{bd}})$ is homeomorphic either to ω or $\omega \times 2^{\omega}$ (depending on whether A is finite or infinite). In this case the space is locally compact and, by the following result, this is the only way to have that property. This is analogous to the result given by S. Solecki in [41, Corollary 3.2].

Theorem 3.3.12. Let $\mathcal{I}$ be an ideal. If $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a locally compact space, then there is $A \in \mathcal{I}$ such that $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$.

Proof. Let $\mathcal{I}$ be an ideal such that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a locally compact space. Then any point of the space has a τ_{bd}-compact neighborhood which is a metric τ_{bd}-subspace, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ does not contain a copy of Arens space and therefore $\mathcal{I}$ is a P-ideal by Theorem 3.3.9.

Now, let $\mathcal{U}$ be a τ_{bd}-compact neighborhood of $\emptyset \in \mathcal{I}$. For any $F \in [\omega]^{<\omega}$, let $\mathcal{U}_F$ be the translation of $\mathcal{U}$ by F. Since $[\omega]^{<\omega}$ is a τ_{bd}-dense set and translations are τ_{bd}-homeomorphisms, $\{\mathcal{U}_F : F \in [\omega]^{<\omega}\}$ is a countable family of τ_{bd}-compact sets that covers $\mathcal{I}$. Therefore $\mathcal{I}$ is an F_σ-ideal and it is covered by countable many web sets. Using Theorem 3.1.13(i), we conclude that $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$.

Finally, by Theorem 2.1.14, $\mathcal{I}$ is a P-ideal which is countable generated. Let $\{\mathcal{C}_n : n \in \omega\} \subseteq \mathcal{I}$ be a countable basis for the ideal $\mathcal{I}$ and let $A \in \mathcal{I}$ be a pseudo-union for this basis. Then it is easy to see that $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$. ■

The ideal $\mathrm{Fin} \times \emptyset$ will be important to classify F_σ-ideals, we will delve on this in the final section, for now we have the following property for its bounded topology. Recall that τ is a *group topology* for a group G if the group operation and the inverse map are continuous respect to τ. In this case, (G, τ) is a *topological group*.

Proposition 3.3.13. The space $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ is a zero-dimensional topological group with the $\triangle$ operation.

Proof. For $f \in \omega^\omega$, let $\mathcal{U}_f = \{A \in \mathrm{Fin} \times \emptyset : (\forall n \in \omega)\, A \cap (\{n\} \times f(n)) = \emptyset\}$.

Then $\mathcal{U}_f$ is a τ_{bd}-clopen set since $\mathcal{U}_f \cap \mathcal{P}(I)$ is clopen in $\mathcal{P}(I)$ for all $I \in \mathrm{Fin} \times \emptyset$. We will show that $\{\mathcal{U}_f : f \in \omega^\omega\}$ is a local base at $\emptyset$, this will be enough to prove that $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ is a zero-dimensional topological space, since it is homogeneous. Also, this implies that $(\mathrm{Fin} \times \emptyset, \triangle)$ is a topological group since $\mathcal{U}_f \triangle \mathcal{U}_f \subseteq \mathcal{U}_f$.

Let $\mathcal{U}$ be an τ_{bd}-open neighbourhood of $\emptyset$. We will recursively define a map $f \in \omega^\omega$ such that $\mathcal{U}_f \subseteq \mathcal{U}$. Since $\mathcal{P}(\{0\} \times \omega) \cap \mathcal{U}$ is open in $\mathcal{P}(\{0\} \times \omega)$ and $\emptyset$ belongs to it, there is some n_0 such that $\mathcal{P}(\{0\} \times (\omega \setminus n_0)) \subseteq \mathcal{U}$. Let $f(0) = n_0$. Suppose that f is already defined up to $k-1$ and $\mathcal{P}\left(\bigcup_{i<k} \{i\} \times (\omega \setminus f(i))\right) \subseteq \mathcal{U}$. We claim that there is some n_k such that

$$\mathcal{P}\left(\left(\bigcup_{i<k} \{i\} \times (\omega \setminus f(i))\right) \cup (\{k\} \times (\omega \setminus n_k))\right) \subseteq \mathcal{U} \tag{$*$}$$

If there is no such n_k, then for all $n \in \omega$ there is some $I_n \in \mathcal{P}((k+1) \times \omega) \setminus \mathcal{U}$ such that $I_n \cap (k \times \omega) \subseteq \mathcal{P}\left(\bigcup_{i<k} \{i\} \times (\omega \setminus f(i))\right)$ and $I_n \cap (\{k\} \times n) = \emptyset$. Since $\mathcal{P}((k+1) \times \omega)$ is a τ_{bd}-compact set, there is a subsequence $\left\{I_{n_j} : j \in \omega\right\} \subseteq \mathcal{I} \setminus \mathcal{U}$ which τ_{bd}-converges to some $I \in \mathcal{U}$ because $I_{n_j} \cap (\{k\} \times n_j) = \emptyset$, this contradicts that $\mathcal{U}$ is an τ_{bd}-open set, hence $(*)$ holds and let $f(k) = n_k$. By construction, $\mathcal{U}_f \cap (n \times \omega) \subseteq \mathcal{U}$ for all $n \in \omega$, this implies that $\mathcal{U}_f \subseteq \mathcal{U}$ and we are done. ■

Note that for every ideal $\mathcal{I}$, the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is not compact since $\mathcal{I}$ is an unbounded set, and hence not a web set, of itself. Then, collecting all properties together, the space $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ satisfies the following.

- It is sequential, separable and homogeneous (by Theorem 3.3.6).
- It is zero-dimensional and a topological group (by Proposition 3.3.13).
- It is σ-compact, non-compact and no locally compact (by Theorem 3.3.12).
- It is not Fréchet–Urysohn and hence neither metrizable nor second-countable (by Theorem 3.3.9 and the Urysohn metrization theorem).

Since $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ is a zero-dimensional space, then it is a regular space. About this property, we find the following necessary condition for the regularity of the bounded topology.

Definition 3.3.14. An ideal $\mathcal{I}$ has the *shrinking property* if for any pairwise disjoint family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ which is a strongly unbounded set of $\mathcal{I}$, there is a strongly unbounded set $\{F_n : n \in \omega\} \subseteq [\omega]^{<\omega}$ of $\mathcal{I}$ such that $(\forall\, n \in \omega)\ F_n \subseteq I_n$.

Proposition 3.3.15. Let $\mathcal{I}$ be an ideal. If $(\mathcal{I}, \tau_{\text{bd}})$ is a regular space, then $\mathcal{I}$ has the shrinking property.

Proof. Let $\mathcal{I}$ which does not have the shrinking property and let $\mathcal{S} = \{I_n : n \in \omega\} \subseteq \mathcal{I}$ be a sun set witness of that, we can assume that $\emptyset \notin \mathcal{S}$. Let $\mathcal{U}$ be an τ_{bd}-open set such that $\mathcal{S} \subseteq \mathcal{U}$. Since $\mathcal{U} \cap \mathcal{P}(I_n)$ is open in $\mathcal{P}(I_n)$ for all $n \in \omega$, then there is $F_n \in [\omega]^{<\omega}$ such that $F_n \subseteq I_n$ and $F_n \in \mathcal{U}$. By the hypothesis, and since $F_n \to \emptyset$, there is a bounded sequence $\{F_{n_k} : k \in \omega\}$ which τ_{bd}-converges to $\emptyset$. Therefore $\mathcal{U}$ intersects any τ_{bd}-open neighbourhood of $\emptyset$. Hence, the τ_{bd}-closed set $\mathcal{S}$ and the point $\emptyset$ proves that $(\mathcal{I}, \tau_{\text{bd}})$ is not regular. ■

As it is shown in the following, not every space $(\mathcal{I}, \tau_{\text{bd}})$ is regular.

Corollary 3.3.16. $(\text{Fin} \times \text{Fin}, \tau_{\text{bd}})$ is not a regular space.

Proof. Let $\mathcal{S} = \{\{n\} \times \omega : n \in \omega\} \subseteq \text{Fin} \times \text{Fin}$, then $\mathcal{S}$ is a sun set. On the other hand, any family $\{F_n : n \in \omega\} \subseteq [\omega \times \omega]^{<\omega}$ such that $F_n \subseteq \{n\} \times \omega$ for every $n \in \omega$ is a bounded family of the ideal $\text{Fin} \times \text{Fin}$. Then by the previous proposition we have the conclusion. ■

In order to give a sufficient condition for the regularity of τ_{bd}, we will use a concept give by K. Kawamura, L. Oversteegen and E. Tymchatyn in [29, Definition 1], which is a property weaker than being a zero-dimensional space.

Definition 3.3.17. A topological space (X, τ_1) is an *almost zero-dimensional* space if there is a topology τ_0 coarser than τ_1 such that (X, τ_0) is a zero-dimensional space and there exists a neighborhood basis for τ_1 consisting of τ_0-closed sets.

For $(\mathcal{I}, \tau_{\text{bd}})$, the usual topology is a natural candidate for a witness topology for that property, but this not always holds since any almost zero-dimensional space (X, τ_1) is regular, indeed $\mathcal{B} = \{\text{int}_{\tau_1}(C) : C \subseteq X \text{ is a } \tau_0\text{-closed set}\}$ is a basis for it. Hence the almost zero-dimensionality of $(\mathcal{I}, \tau_{\text{bd}})$ implies its regularity.

Question 3.3.18. If $(\mathcal{I}, \tau_{\text{bd}})$ is an almost zero-dimensional space, then it is regular, which in turn implies that $\mathcal{I}$ has the shrinking property. Which of these implications is reversible?

The last part of this section will be dedicated to analytic P-ideals, and hence it is related to lower semicontinuous submeasures (Definition 2.2.1). The main result about this class of ideals is Theorem 3.3.20, which tells us that the bounded topology is a generalization to the topology given by the work of S. Solecki (in [41]).

Lemma 3.3.19. Let φ be a lower semicontinuous submeasure and $\mathcal{W} \subseteq \operatorname{Exh}(\varphi)$. Then $\mathcal{W}$ is a web set on $\operatorname{Exh}(\varphi)$ if and only if it satisfies the following.

$$(\forall\, \varepsilon > 0)\, (\exists\, N \in \omega)\, (\forall\, I \in \mathcal{W})\ \ \varphi(I \setminus N) < \varepsilon \tag{$*$}$$

Proof. Let $\mathcal{W} \subseteq \operatorname{Exh}(\varphi)$ and $\varepsilon > 0$ with $(\forall\, N \in \omega)\, (\exists\, I \in \mathcal{W})\ \ \varphi(I \setminus N) \geq 2\varepsilon$. Then there is an increasing sequence $\{N_m : m \in \omega\}$ of natural numbers and a family $\mathcal{S} = \{I_m : m \in \omega\} \subseteq \mathcal{W}$ such that $\varphi\,(I_m \cap (N_{m+1} \setminus N_m)) \geq \varepsilon$. Hence for an infinite subset $\mathcal{X} \subseteq \mathcal{S}$ we have $(\forall\, N \in \omega)\ \ \varphi\,(\bigcup \mathcal{X} \setminus N) \geq \varepsilon$, and then $\mathcal{S}$ is a sun set. Therefore any web set must satisfy ($*$).

On the other hand, let $\mathcal{W} \subseteq \operatorname{Exh}(\varphi)$ satisfying ($*$). If $\mathcal{W}$ is finite we are done, then we can suppose that $\mathcal{W}$ is infinite. Let $\mathcal{X} = \{I_n : n \in \omega\} \subseteq \mathcal{W}$ be a countable set. Without loss of generality, we can assume that $\mathcal{X}$ converges to some $I \in \mathcal{P}\,(\omega)$ and $\Delta(I_n, I) < \Delta(I_{n+1}, I)$ for all $n \in \omega$. Since $\mathcal{X}$ satisfies ($*$), then $I \in \operatorname{Exh}(\varphi)$. For all $m \in \omega$ we increasingly choose a natural number N_m as a witness of ($*$) for $\varepsilon_m = \frac{1}{2^m(m+1)}$ such that there is some $I_{n_m} \in \mathcal{X}$ with $\Delta(I_{n_m}, I) \in [N_m, N_{m+1})$. We have that $J = \bigcup \{I_{n_m} : m \geq 1\} \in \operatorname{Exh}(\varphi)$ since $\varphi(J \setminus N_m) \leq \sum_{k \geq m} \frac{1}{2^k}$, then $\mathcal{X}$ has a bounded subsequence and therefore $\mathcal{W}$ is a web set. ■

For a lower semicontinuous submeasure φ, we denote by τ_φ the topology on the ideal $\operatorname{Exh}(\varphi)$ induced by the metric d_φ given by $d_\varphi(I, J) = \varphi(I \triangle J)$.

Theorem 3.3.20. Let $\mathcal{I}$ be an analytic P-ideal and let φ its associate lower semicontinuous submeasure. Then $\tau_{\text{bd}} = \tau_\varphi$.

Proof. For $\varepsilon > 0$ we define $B^\varphi_\varepsilon = \{J \in \mathcal{I} : \varphi(J) < \varepsilon\}$.

To see that $\tau_\varphi \subseteq \tau_{\text{bd}}$ is enough to prove that for all $\varepsilon > 0$, B^φ_ε is an τ_{bd}-open set. Let $\mathcal{K} \subseteq \mathcal{I}$ be a compact and web set, and let $I \in B^\varphi_\varepsilon \cap \mathcal{K}$. If for all $n \in \omega$ there is $I_n \in (\langle I{\restriction}_n \rangle \cap \mathcal{K}) \setminus B^\varphi_\varepsilon$, then there exists $\delta > 0$ such that $\varphi(I_n \setminus I) > \delta$ for all $n \in \omega$. Then the family $\{I_n : n \in \omega\} \subseteq \mathcal{K}$ is an infinite sun set, which contradicts that $\mathcal{K}$ is a web set. Then, there is some $n \in \omega$ such that $I \in \langle I_n \rangle \cap \mathcal{K} \subseteq B^\varphi_\varepsilon$, and therefore B^φ_ε is an τ_{bd}-open set.

On the other hand, let $\mathcal{U}$ be an τ_{bd}-open neighborhood of $\emptyset$. By homogeneity, it is enough to prove that there exists some $B^\varphi_\varepsilon \subseteq \mathcal{U}$. If it is not the case, for all $n \in \omega$ there is some $I_n \in B^\varphi_{1/n} \setminus \mathcal{U}$. By the previous lemma $\mathcal{K} = \{I_n : n \in \omega\} \cup \{\emptyset\}$ is a compact and web set, but this contradicts that $\mathcal{U}$ is a τ_{bd}-open set since $\mathcal{U} \cap \mathcal{K} = \{\emptyset\}$. ■

By Theorem 3.3.10 and Theorem 3.3.20 we know that for a P-ideal, if it is analytic or non-meager, then its bounded topology is metrizable, we conjecture that this holds for all P-ideals.

Conjecture 3.3.21. If $\mathcal{I}$ is a P-ideal, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a metric space.

About the concept of almost zero-dimensionality we have the following.

Proposition 3.3.22. Let $\mathcal{I}$ be an analytic P-ideal. The space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is almost zero-dimensional respect to the usual topology.

Proof. Let φ a lower semicontinuous submeasure such that $\mathcal{I} = \mathrm{Exh}\,(\varphi)$. For $r > 0$, the set $\{A \subseteq \omega : \varphi(A) \leq r\} \subseteq \mathcal{I}$ is a neighborhood of $\emptyset$, which is closed in $\mathcal{I}$ because φ is semicontinuous. By homogeneity, we have the desired conclusion. ■

Lemma 3.3.23. Let $\mathcal{I}$ be an ideal and $\mathcal{W} \subseteq \mathcal{I}$ be a web set. Then the closure of $\mathcal{W}$ in the space 2^ω is the same that the closure of $\mathcal{W}$ in the subspace $\mathcal{I}$.

Proof. Let $I \in \mathrm{cl}_{2^\omega}(\mathcal{W})$ and let $\{I_n : n \in \omega\} \subseteq \mathcal{W}$ be a sequence which converges to I. Since $\mathcal{W}$ is a web set, we can suppose that $\{I_n : n \in \omega\}$ is bounded. Hence $I \subseteq \bigcup\{I_n : n \in \omega\} \in \mathcal{I}$, therefore we have that $\mathrm{cl}_{\mathcal{I}}(\mathcal{W}) = \mathrm{cl}_{2^\omega}(\mathcal{W}) \cap \mathcal{I} = \mathrm{cl}_{2^\omega}(\mathcal{W})$. ■

By the previous result, it does not matter in which suitable space we consider the closure of a web set $\mathcal{W}$. Hence, we can simply use the notation $\mathrm{cl}(\mathcal{W})$.

Proposition 3.3.24. Let $\mathcal{I}$ be an analytic P-ideal on ω. Then the closure of any web set of $\mathcal{I}$ is also a web set of $\mathcal{I}$.

Proof. Let φ a lower semicontinuous submeasure with $\mathcal{I} = \mathrm{Exh}\,(\varphi)$. Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, by the previous lemma $\mathrm{cl}(\mathcal{W}) \subseteq \mathcal{I}$. Let $\mathcal{X} \subseteq \mathrm{cl}(\mathcal{W})$ an infinite set, then there is a sequence $\{I_n : n \in \omega\} \subseteq \mathcal{X}$ which converges to I for some $I \in \mathcal{I}$.

We claim that $\lim_n \varphi(I_n \setminus I) = 0$. If not, there exist $\varepsilon > 0$ and a subsequence $\{I_{n_k} : k \in \omega\}$ such that $(\forall\, k \in \omega)\ \ \varphi((I_{n_k} \setminus I) \cap m_k) > \varepsilon$, where $m_k = \min(I_{n_k+1} \setminus I)$. Since $I_{n_k} \in \mathrm{cl}(\mathcal{W})$, for all $k \in \omega$ there is some $W_k \in \mathcal{W}$ such that $W_k \cap m_k = I_{n_k} \cap m_k$. This implies that $\{W_k : k \in \omega\} \subseteq \mathcal{W}$ is a sun set; thus the claim holds. Finally, the previous lemma implies that the sequence $\{I_n : n \in \omega\}$ is a web set and therefore $\mathrm{cl}(\mathcal{W})$ is a web set too. ■

The property in the conclusion of previous proposition will be relevant in the following section.

3.4 Conjecture of Louveau and Veličković

In [34, Conjecture 1], A. Louveau and B. Veličković conjecture that their result, shown in Theorem 3.1.13, can be improved in the following sense.

Conjecture 3.4.1 (Louveau and Veličković, [34])**.** Let $\mathcal{I}$ be an F_σ-ideal. Then $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ if and only if $\mathrm{web}(\mathcal{I}) > \omega$.

We will prove this holds for the following class of ideals.

Definition 3.4.2. Let $\mathcal{I}$ be an ideal. Then $\mathcal{I}$ has the *weakly bounded closure property* (or the *web-closure property*) if the closure of any web set of $\mathcal{I}$ is a web set of $\mathcal{I}$.

Proposition 3.4.3. Let $\mathcal{I}$ be an ideal. Then $\mathcal{I}$ has the web-closure property if and only if any web set of $\mathcal{I}$ is contained in some τ_{bd}-compact set of $\mathcal{I}$.

Proof. For the «only if» part. Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, by hypothesis $\mathrm{cl}(\mathcal{W}) \subseteq \mathcal{I}$ is a web set, and since it is a closed set of 2^ω, then it is compact. Therefore, $\mathrm{cl}(\mathcal{W})$ is a τ_{bd}-compact which contains to $\mathcal{W}$.

For the «if» part. Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, by hypothesis there is some τ_{bd}-compact set $\mathcal{K}$ such that $\mathcal{W} \subseteq \mathcal{K}$. Using Proposition 3.3.1 we have that $\mathcal{K}$ is a compact and a web set, in particular $\mathcal{K}$ is closed then $\mathrm{cl}(\mathcal{W}) \subseteq \mathcal{K}$. Then $\mathrm{cl}(\mathcal{W})$ is also a web set. Since $\mathrm{cl}(\mathcal{W})$ is compact, then it is a τ_{bd}-compact set. ■

Corollary 3.4.4. Let $\mathcal{I}$ be an ideal with the web-closure property. Then $\mathrm{web}(\mathcal{I})$ is the minimum size of a family of τ_{bd}-compact sets which covers $\mathcal{I}$.

Proof. Let $\mathfrak{K}(\mathcal{I}) = \min\{|\mathscr{K}| : (\forall \mathcal{K} \in \mathscr{K})\ \mathcal{K} \text{ is a } \tau_{\mathrm{bd}}\text{-compact set, and } \bigcup \mathscr{K} = \mathcal{I}\}$.

Since any τ_{bd}-compact set is a web set, we always have that $\mathrm{web}(\mathcal{I}) \leq \mathfrak{K}(\mathcal{I})$. Moreover, since $\mathcal{I}$ has the web-closure property, then any web set is contained in some τ_{bd}-compact set. Hence a witness for $\mathrm{web}(\mathcal{I})$ give us a witness for $\mathfrak{K}(\mathcal{I})$, and therefore these cardinals are equal. ■

By Proposition 3.3.24 we know that any P-ideal has the web-closure property, nevertheless, these classes are not equal. A witness for that is the following ideal introduced by A. Louveau and B. Veličković in [34, Example 1].

Definition 3.4.5. The *polynomial growth ideal* is defined by

$$\mathcal{I}_{\mathcal{P}} = \left\{A \subseteq \omega : (\exists\, k \in \omega)\, (\forall\, n \in \omega)\ \ |A \cap 2^n| \leq n^k\right\}.$$

Proposition 3.4.6. $\mathcal{I}_{\mathcal{P}}$ is a tall, not countably generated, F_σ and not P-ideal. Moreover, $\mathrm{web}(\mathcal{I}_{\mathcal{P}}) = \omega$ and it has the web-closure property.

Proof. For an infinite set $X \subseteq \omega$, let $Y \subseteq X$ such that $|Y \cap 2^n| \leq n$ by picking at most one element of X in each interval $[2^n, 2^{n+1})$, then $Y \in \mathcal{I}_{\mathcal{P}}$ and hence $\mathcal{I}_{\mathcal{P}}$ is a tall ideal. In particular it is not countably generated. Also, for all $k \in \omega$ let $I_k \in \mathcal{I}_{\mathcal{P}}$ such that $(\forall^\infty n \in \omega)\ |I_k \cap [2^n, 2^{n+1})| = n^k$, then any pseudo-union of $\{I_k : k \in \omega\}$ does not belong $\mathcal{I}$, therefore $\mathcal{I}_{\mathcal{P}}$ is not a P-ideal.

Besides, for any $k \in \omega$ the set $\mathcal{W}_k = \left\{A \subseteq \omega : (\forall n \in \omega)\ |A \cap 2^n| \leq n^k\right\} \subseteq \mathcal{I}_{\mathcal{P}}$ is τ_{bd}-compact. Indeed, fix k and let $\mathcal{X} = \{I_n : n \in \omega\} \subseteq \mathcal{W}_k$. Then $\mathcal{X}$ converges to some $X \in 2^\omega$ and we can assume that $I_n \cap 2^n = X \cap 2^n$ for all $n \in \omega$. Thus $X \in \mathcal{W}_k$ and $\mathcal{W}_k$ is compact. Even more, $\bigcup \{I_n : n \in \omega\} \in \mathcal{W}_{k+1}$. This proves that $\mathcal{W}_k$ is a web set. Hence $\{\mathcal{W}_k : k \in \omega\}$ testifies that $\mathrm{web}(\mathcal{I}_{\mathcal{P}}) = \omega$, and in particular $\mathcal{I}$ is a F_σ-ideal. Finally, using Theorem 3.5.2, $\mathcal{I}_{\mathcal{P}}$ has the web-closure property. ∎

In particular $\mathrm{cof}\,(\mathcal{I}_{\mathcal{P}}) > \omega = \mathrm{web}(\mathcal{I}_{\mathcal{P}})$, and we have an example in which this cardinals are distinct. In order to prove that no every ideal has the web-closure property we introduce the following ideal.

Definition 3.4.7. Let $T \subseteq \omega \times \omega$.

a) T is *infinite-triangular* if there is an increasing sequence $\{n_k : k \in \omega\}$ such that $T = \bigcup_{k \in \omega} \{n_k\} \times (n_{k+1} - n_k)$.

b) T is *finite-triangular* if there is an increasing finite sequence $\{n_0, \ldots, n_m\}$ such that $T = \bigcup_{k<m} \{n_k\} \times (n_{k+1} - n_k) \cup \{n_m\} \times \omega$.

c) T is *triangular* if it is either finite-triangular or infinite-triangular.

The *triangular ideal* $\mathcal{I}_{\mathcal{T}}$ is the ideal on $\omega \times \omega$ which is generated by all triangular sets.

Note that $\emptyset$ is finite-triangular by the empty sequence and $\mathcal{I}_{\mathcal{T}} \subseteq \mathrm{Fin} \times \mathrm{Fin}$.

Proposition 3.4.8. $\mathcal{I}_{\mathcal{T}}$ is a tall F_σ-ideal which does not have neither the shrinking property nor the web-closure property and it has a sun set of size $\mathfrak{c}$.

Proof. For $A \subseteq \omega$ and $n \in \omega$ let $A(n)$ be the n-th element of A, if exists. Then $\mathcal{P}(\omega)$ is in correspondence with the family of all triangular sets via the function $\mathbb{T} : \mathcal{P}(\omega) \to \omega \times \omega$ given by $\mathbb{T}(A) = \bigcup_{n \in \omega} \{A(n)\} \times (A(n+1) - A(n))$ if A is infinite, $\mathbb{T}(A) = \left(\bigcup_{n=0}^{k} \{A(n)\} \times (A(n+1) - A(n))\right) \cup \{A(k+1)\} \times \omega$ if $|A| = k+1$ and

$\mathbb{T}(\emptyset) = \emptyset$. Since $\mathbb{T}$ is a continuous map, the set of triangular sets is compact, and therefore $\mathcal{I}_{\mathcal{T}}$ is an F_σ-ideal.

The definition of a triangular set directly implies that $\mathcal{I}_{\mathcal{T}}$ is a tall ideal. Hence, the set $\mathcal{W} = \{\{n\} \times m : n, m \in \omega\} \subseteq \mathcal{I}_{\mathcal{T}}$ is a web set. Now, for all $n \in \omega$, the set $\{n\} \times \omega$ belongs to the closure of $\mathcal{W}$, and since $\{\{n\} \times \omega : n \in \omega\} \subseteq \mathcal{I}_{\mathcal{T}}$ is a sun set, the ideal $\mathcal{I}_{\mathcal{T}}$ does not has the web-closure property.

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family of size $\mathfrak{c}$ (see [5, Proposition 8.1]). We will prove that $\mathcal{S} = \{\mathbb{T}(A) : A \in \mathcal{A}\} \subseteq \mathcal{I}_{\mathcal{T}}$ is a sun set, which clearly satisfies $|\mathcal{S}| = \mathfrak{c}$. Let $A_0, \ldots, A_n$ be $n \geq 1$ distinct elements of $\mathcal{A}$, then there is some $N \in \omega$ such that the family $\{A_i \setminus N : i \leq n\}$ is pairwise disjoint, therefore exists $m_0, \ldots, m_n \geq N$ such that $(\forall\, k < n)\ A_k(m_k) < A_n(m_n) < A_k(m_k + 1)$ and, reindexing if necessary, also $(\forall\, i < j < n)\ A_i(m_i) < A_j(m_j)$. Let $B_k = \{A_k(m_k)\} \times (A_k(m_k + 1) - A_k(m_k))$ for $k \leq n$, by previous, if a triangular set covers B_k, then it cannot cover any B_l for $k < l \leq n$, this implies that $\bigcup_{k \leq n} \mathbb{T}(A_k)$ cannot be covered by n triangular sets. Hence any infinite subset of $\mathcal{S}$ is unbounded, therefore $\mathcal{S}$ is a sun set.

Finally, let $\{A_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ be a countable pairwise disjoint family such that $(\forall\, n \in \omega)\, (\forall\, k < n)\ A_n(0) > A_k(n - k)$ (that is, the first element of A_n is greater that the 1-th element of A_{n-1}, the 2-th element of A_{n-2} and so on). By the previous paragraph, we have that $\{\mathbb{T}(A_n) : n \in \omega\}$ is a sun set. Let $F_n \in [\mathbb{T}(A_n)]^{<\omega}$ for all $n \in \omega$. Recursively define sequences $\{n_k : k \in \omega\}$, $\{m_k : k \in \omega\} \subseteq \omega$ as follows. Start with $n_0 = 0$. Suppose n_k is already defined, since F_{n_k} is finite, let m_k such that $F_{n_k} \subseteq \mathbb{T}(A_{n_k}) \cap A_{n_k}(m_k) \times \omega$ and choose n_{k+1} such that $A_{n_{k+1}}(0) > A_{n_k}(m_k)$. Then $I = \bigcup \{\mathbb{T}(A_{n_k}) \cap A_{n_k}(m_k) \times \omega : k \in \omega\} \in \mathcal{I}_{\mathcal{T}}$ and hence $\{F_n : n \in \omega\}$ has a bounded infinite subset, therefore $\mathcal{I}$ has no the shrinking property. ■

To prove that any ideal with the web-closure property satisfies Conjecture 3.4.1, we need some previous lemmas.

Lemma 3.4.9. Let $\mathcal{I}$ be a meager ideal and let $\mathcal{W} \subseteq \mathcal{I}$ be a weakly bounded set. Then $\mathcal{W}$ is nowhere dense in $\mathcal{I}$.

Proof. Let $\{P_n : n \in \omega\}$ be the interval partition of ω given by Theorem 2.2.7. Suppose that $\mathcal{W}$ is dense in $\mathcal{I}$ above some $s \in 2^{<\omega}$. Then, there is an increasing sequence $\{s_k : k \in \omega\} \subseteq 2^{<\omega}$ such that $s_0 = s$ and for every $k \geq 1$ exists $n_k \in \omega$ satisfying $(\forall\, m \in P_{n_k})\, s_k(m) = 1$. Hence we can choose a subset $\mathcal{S} = \{S_k : k \in \omega\} \subseteq \mathcal{W}$ satisfying $(\forall\, k \in \omega)\ P_{n_k} \subseteq S_k$. Thus, due the property of the partition, $\mathcal{S}$ is a sun set of $\mathcal{I}$, a contradiction. Therefore, $\mathcal{W}$ is a nowhere dense set on the ideal $\mathcal{I}$. ■

In what follows, for finite sets $a, b \in [\omega]^{<\omega}$ we use the notation $a \sqsubseteq b$ if $a \subseteq b$ and $(\forall\, n \in a)\,(\forall\, m \in b)\;\; m \leq n \;\rightarrow\; m \in a$.

Definition 3.4.10. Let $\mathcal{J}$ be an ideal. A family $\mathcal{A} = \{a_s : s \in \omega^{<\omega}\} \subseteq [\omega]^{<\omega}$ is a *sun-branching tree on* $\mathcal{J}$ if it satisfies the following.

- $a_\emptyset = \emptyset$.
- $(\forall\, s \in \omega^{<\omega})\,(\forall\, n \in \omega)\;\; a_s \sqsubseteq a_{s^\frown\langle n\rangle}$.
- $(\forall\, s \in \omega^{<\omega})\;\; \{a_{s^\frown\langle n\rangle} : n \in \omega\}$ is a sun set of $\mathcal{J}$.
- $(\forall\, x \in \omega^\omega)\;\; \bigcup\{a_{x\restriction_n} : n \in \omega\} \in \mathcal{J}$.

A family $\mathcal{F} \subseteq [\omega]^{<\omega}$ is a *finite-branching tree on* $\mathcal{J}$ if it satisfies all previous conditions but the third one replacing «sun» by «finite»; that is, only finitely many of the sets $a_{s^\frown\langle n\rangle}$ are different.

Lemma 3.4.11. If there is a sun-branching tree on an ideal $\mathcal{J}$, then $\mathrm{web}(\mathcal{J}) \geq \mathfrak{d}$.

Proof. Let $\mathcal{A} = \{a_s : s \in \omega^{<\omega}\} \subseteq [\omega]^{<\omega}$ be a sun-branching tree on $\mathcal{J}$. For $\mathcal{X} \subseteq \mathcal{A}$ let

$$[\mathcal{X}]_\infty = \left\{\bigcup\left\{a_{x\restriction_n} : n \in \omega\right\} : x \in \omega^\omega \text{ and } (\forall\, n \in \omega)\;\; a_{x\restriction_n} \in \mathcal{X}\right\} \subseteq \mathcal{J}.$$

Thus, $[\mathcal{X}]_\infty$ is the collection of elements in the ideal $\mathcal{J}$ which have infinitely many initial segments in a branch of $\mathcal{X}$. We will prove that for any web set $\mathcal{W} \subseteq \mathcal{J}$ exists $\mathcal{F} \subseteq \mathcal{A}$, a finite-branching tree on $\mathcal{J}$, such that $\mathcal{W} \cap [\mathcal{A}]_\infty \subseteq [\mathcal{F}]_\infty$, which shows that $\mathrm{web}(\mathcal{J}) \geq \mathfrak{d}$ holds because $\mathfrak{d}$ many sets of branches from finite-branching subtrees of the tree $\omega^{<\omega}$ are needed to cover the space ω^ω.

We can assume that $\mathcal{W} = \mathcal{W}^\downarrow$. For all $s \in \omega^{<\omega}$ let $F_s = \mathcal{W} \cap \{a_{s^\frown\langle n\rangle} : n \in \omega\}$, then $\mathcal{F} = \bigcup_{s\in\omega} F_s \subseteq \mathcal{A}$ is a finite-branching tree on $\mathcal{J}$. Let $I \in \mathcal{W} \cap [\mathcal{A}]_\infty$, there is $x \in \omega^\omega$ such that $(\forall\, n \in \omega)\;\; a_{x\restriction_{n+1}} \in \mathcal{A} \cap \mathcal{W}$ and $I = \bigcup\left\{a_{x\restriction_n} : n \in \omega\right\}$, then $a_{x\restriction_{n+1}} \in F_{x\restriction_n}$ for all $n \in \omega$, and therefore $I \in [\mathcal{F}]_\infty$. ■

Theorem 3.4.12. Let $\mathcal{J}$ be an F_σ-ideal with the web-closure property. Then either $\mathrm{web}(\mathcal{J}) = \omega$ or $\mathrm{web}(\mathcal{J}) \geq \mathfrak{d}$.

Proof. We define an infinite game with perfect information $\mathfrak{G}_{web}(\mathcal{J})$ as follows.

$$\begin{array}{c|ccccccccc}
\mathrm{I} & \mathcal{W}_0 & & \mathcal{W}_1 & & \cdots & & \mathcal{W}_n & & \cdots \\
\hline
\mathrm{II} & & a_0 & & a_1 & & \cdots & & a_n & & \cdots
\end{array}$$

At the nth move, Player I chooses a web set $\mathcal{W}_n$ of $\mathcal{I}$ such that $\mathcal{W}_n = \overline{\mathcal{W}_n^{\downarrow}}$, and Player II chooses a finite subset a_n such that $a_n \notin \mathcal{W}_n$ and $a_n \sqsubseteq a_{n+1}$ for all n, this is possible by Lemma 3.4.9. Player II wins a run of the game if and only if $\bigcup \{a_n : n \in \omega\} \in \mathcal{I}$.

Since $\mathcal{I}$ is F_σ then $\mathfrak{G}_{web}(\mathcal{I})$ is determined due to Martin's Determinacy Theorem for Borel games. So, we will consider the two cases.

Case 1. Player I has a winning strategy, say σ. Since in every move Player II has countably many options to choose, σ determines a countable family of web sets of $\mathcal{I}$, namely $\mathcal{X}$, which consists of all responses of Player I in σ. Now, if there exists $I \in \mathcal{I} \setminus \bigcup \mathcal{X}$ then there is a run of the game in which Player II choose an initial segment of I in every move since $\mathcal{W} = \overline{\mathcal{W}^{\downarrow}}$ for all $\mathcal{W} \in \mathcal{X}$, thus Player II would wins the run, which is not possible by σ. Then $\bigcup \mathcal{W} = \mathcal{I}$ and $\mathrm{web}(\mathcal{I}) = \omega$.

Case 2. Player II has a winning strategy, say λ. We will prove that exists a sun-branching tree on $\mathcal{I}$. Let $\mathscr{X}_t = \left\{a \in [\omega]^{<\omega} : (\exists\, \mathcal{W} \subseteq \mathcal{I} \text{ web set})\ t^\frown\langle \mathcal{W}, a\rangle \in \lambda\right\}$ for every $t \in \lambda$ of even length, $\mathscr{X}_t$ cannot be a web set of $\mathcal{I}$ because Player II could not do his next move if Player I choose $\overline{\mathscr{X}_t^{\downarrow}}$ as move. Then, for every suitable $t \in \lambda$ let $\mathcal{S}_t \subseteq \mathscr{X}_t$ be a countable sun set of $\mathcal{I}$, and for every $a \in \mathcal{S}_t$ let $\mathcal{W}_a^t$ be a web set such that $t^\frown\langle \mathcal{W}_a^t, a\rangle \in \lambda$. Let $N_t = \left\{t^\frown\langle \mathcal{W}_a^t, a\rangle \in \lambda : a \in \mathcal{S}_t\right\}$. Recursively define the sequence $\{M_n \in \mathcal{P}(\lambda) : n \in \omega\}$ given by $M_0 = \{\emptyset\}$ and $M_{n+1} = \bigcup \{N_t : t \in M_n\}$. Finally, $\mathcal{A} = \{\emptyset\} \cup \bigcup_{n\in\omega} \bigcup_{t \in M_n} \mathcal{S}_t$ is a sun-branching tree on $\mathcal{I}$ since every $\mathcal{S}_t$ is a sun set and λ is a winning strategy for Player II. ■

Corollary 3.4.13. Let $\mathcal{I}$ be an analytic ideal with the web-closure property. Then $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ if and only if $\mathrm{web}(\mathcal{I}) > \omega$.

Proof. Let $\mathcal{I}$ be an analytic ideal. If $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ then, by Theorem 3.1.13, we have that $\mathrm{web}(\mathcal{I}) \geq \mathfrak{d} > \omega$. On the other hand, if $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$ then, again by Theorem 3.1.13, $\mathcal{I}$ is an F_σ-ideal and $\mathrm{web}(\mathcal{I}) < \mathfrak{d}$. Using the previous result, we have that $\mathrm{web}(\mathcal{I}) = \omega$. This prove the desired equivalence. ■

We have the following strengthening of Conjecture 3.4.1.

Conjecture 3.4.14. Let $\mathcal{I}$ be an F_σ-ideal. Then either $\mathcal{I}$ has the web-closure property or $\mathcal{I}$ has a strongly unbounded set of size $\mathfrak{c}$.

3.5 Classification for F_σ-ideals

If $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space, then $\mathcal{I}$ is an F_σ-ideal, since any τ_{bd}-compact set is, in particular, a closed set. The following result give us a useful family witnessing the σ-compactness of the space.

Lemma 3.5.1. Let $\mathcal{I}$ be an ideal. If there is an increasing countable family of τ_{bd}-compact sets $\mathcal{K}$ which covers the ideal and $\mathcal{K} = \mathcal{K}^{\downarrow}$ for all $\mathcal{K} \in \mathcal{K}$, then $\mathcal{K}$ is cofinal among the weakly bounded sets of $\mathcal{I}$. Furthermore, if $(\mathcal{I}, \tau_{\mathrm{bd}})$ is σ-compact such family exists.

Proof. For the first part, let $\mathcal{K} = \{\mathcal{K}_n : n \in \omega\}$ be such a family. We will prove that for any web set $\mathcal{W} \subseteq \mathcal{I}$ there is some $n \in \omega$ such that $\mathcal{W} \subseteq \mathcal{K}_n$. Let $\mathcal{S} \subseteq \mathcal{I}$ such that for all $n \in \omega$ there exists $I_n \in \mathcal{S} \setminus \mathcal{K}_n$. Let $\mathcal{X} \subseteq \{I_n : n \in A\}$ be a bounded set. There is $m \in \omega$ such that $\bigcup \mathcal{X} \in \mathcal{K}_m = \mathcal{K}_m^{\downarrow}$ and hence $\mathcal{X} \subseteq \mathcal{K}_m$, thus $\mathcal{X}$ must be finite. We have that only finite subsets of $\{I_n : n \in A\}$ are bounded, therefore $\mathcal{S}$ contains an infinite sun set and it is not a web set.

We now prove that $\mathcal{K}^{\downarrow}$ is a τ_{bd}-compact set if $\mathcal{K}$ is, which is enough to prove the second part. Indeed, let $\mathcal{K}$ be a τ_{bd}-compact set, then $\mathcal{K}^{\downarrow}$ is a web set since $\mathcal{K}$ is. Let $\{A_n : n \in \omega\} \subseteq \mathcal{K}^{\downarrow}$, then there is a family $\{B_n : n \in \omega\} \subseteq \mathcal{K}$ such that $A_n \subseteq B_n$ for all $n \in \omega$. Since 2^ω and $\mathcal{K}$ are compact sets, then there are a pair of subsequences $\{A_{n_k} : k \in \omega\}$, $\{B_{n_k} : k \in \omega\}$ which respectively converge to $X \in 2^\omega$ and $Y \in \mathcal{K}$. Finally, it is easy to see that $X \subseteq Y$, then $\mathcal{K}^{\downarrow}$ is a compact set and therefore it is a τ_{bd}-compact set. ■

Now we can give a combinatorial characterization for the σ-compactness of the bounded topology.

Theorem 3.5.2. Let $\mathcal{I}$ be an ideal. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space if and only if $\mathcal{I}$ has the web-closure property and $\mathrm{web}(\mathcal{I}) = \omega$.

Proof. If $\mathcal{I}$ has the web-closure property and $\mathrm{web}(\mathcal{I}) = \omega$, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is σ-compact by Proposition 3.4.3. On the other hand, if $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space then clearly $\mathrm{web}(\mathcal{I}) = \omega$. Let $\{\mathcal{K}_n : n \in \omega\}$ the family given by the previous lemma. We have that any web set of $\mathcal{I}$ is contained in $\mathcal{K}_n$ for some $n \in \omega$. Thus, again by Proposition 3.4.3, $\mathcal{I}$ has the web-closure property. ■

Using Corollary 3.4.13, we can write the previous result as follows.

Theorem 3.5.3. Let $\mathcal{I}$ be an ideal. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space if and only if $\mathcal{I}$ has the web-closure property and $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$.

For ideals distinct to Fin, we can improve the family given in Lemma 3.5.1.

Lemma 3.5.4. Let $\mathcal{I} \neq \mathrm{Fin}$ be an ideal such that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space. Then there exist an increasing family of τ_{bd}-compact sets $\{\mathcal{K}_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{I})$ which covers $\mathcal{I}$, such that $\mathcal{K}_n = \mathcal{K}_n^{\downarrow}$ and $(\mathcal{K}_n, \tau\restriction_{K_n})$ is homeomorphic to 2^ω for all $n \in \omega$.

Proof. Let $\{\mathcal{F}_n : n \in \omega\}$ be the family given by Lemma 3.5.1. Since $\mathcal{I} \neq \mathrm{Fin}$ we can suppose that every $\mathcal{F}_n$ is uncountable.

Let $n \in \omega$. By Cantor-Bendixon Theorem, there exists a perfect subset and a countable open subset of $\mathcal{F}_n$, namely $\mathcal{K}_n$ and $\mathcal{C}_n$ respectively, such that $\mathcal{F}_n = \mathcal{K}_n \cup \mathcal{C}_n$. Since $\mathcal{F}_n = \mathcal{F}_n^{\downarrow}$, then $I \in \mathcal{F}_n$ is a condensation point of $\mathcal{F}_n$ if and only if there is some $J \in \mathcal{F}_n \cap [\omega]^\omega$ such that $I \subseteq J$. Therefore $\mathcal{K}_n = \mathcal{K}_n^{\downarrow}$ and $\mathcal{C}_n \subseteq [\omega]^{<\omega}$. Also, for all $n \in \omega$ we have that $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$ because $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$. Finally, since for all $F \in [\omega]^{<\omega}$ there exist $X \in \mathcal{I} \cap [\omega]^\omega$ such that $F \subseteq X$, then $\{\mathcal{K}_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{I})$ covers $\mathcal{I}$ and it is the desired family. ■

Lemma 3.5.5. Let $\mathcal{I}$ be an ideal. If $\mathcal{K} \subseteq \mathcal{I}$ is τ_{bd}-compact and $\mathcal{K}^{\downarrow}$ is τ_{bd}-nowhere dense, then there is a τ_{bd}-compact set $\mathcal{K}' \subseteq \mathcal{I}$ such that $\mathcal{K} \subseteq \mathcal{K}'$ and $\mathcal{K}$ is τ_{bd}-nowhere dense in $\mathcal{K}'$.

Proof. We can assume that $\mathcal{K} = \mathcal{K}^{\downarrow}$. Since $\mathcal{K}$ has empty interior and $(\mathcal{I}, \tau_{\mathrm{bd}})$ is sequential, there is a sequence disjoint to $\mathcal{K}$ which τ_{bd}-converges to $\emptyset$, in particular there is a bounded sequence $\mathcal{X} = \{X_n : n \in \omega\} \subseteq \mathcal{I} \setminus \mathcal{K}$ which converges to $\emptyset$. Let $\mathcal{K}' = \mathcal{K} \cup \{I \cup X_n : I \in \mathcal{K},\ n \in \omega\} \subseteq \mathcal{I}$. Note that for all $I \in \mathcal{K}$ and $n \in \omega$, $I \cup X_n \notin \mathcal{K}$ since $\mathcal{K} = \mathcal{K}^{\downarrow}$ and $X_n \notin \mathcal{K}$. We claim that $\mathcal{K}'$ is the desired set.

$\mathcal{K}'$ is a web set since $\mathcal{K}$ is and $\mathcal{X}$ is bounded. To see $\mathcal{K}'$ is compact let $\mathcal{Y} \subseteq \mathcal{K}'$ be a countable subset, then for all $J \in \mathcal{Y}$ there are $I_J \in \mathcal{K}$ and $F_{n_J} \in \mathcal{X}$ such that $J = I_J \cup X_{n_J}$, since $\mathcal{K}$ is compact we can assume that $\mathcal{Z} = \{I_J : J \in \mathcal{Y}\} \subseteq \mathcal{K}$ converges to some $I \in \mathcal{K}$. We claim that $\mathcal{Y}$ has a convergent subsequence. Indeed, if there is an infinite subset $\mathcal{Y}' \subseteq \mathcal{Y}$ such that $(\forall\, I \in \mathcal{Y}')\ X_{n_J} = X_N$ for some $N \in \omega$, then $\mathcal{Y}'$ converges to $I \cup X_N$. Then we can assume that $X_{n_J} \neq X_{n_L}$ for any distinct $J, L \in \mathcal{Y}$. Let $n \in \omega$, since $\mathcal{Z}$ converges to I and $\mathcal{X}$ converges to $\emptyset$, we have that $(\forall^\infty\, J \in \mathcal{Y})\ \ I_J \in \langle I\restriction_n\rangle$ and $(\forall^\infty\, J \in \mathcal{Y})\ \ X_{n_J} \cap n = \emptyset$. Thus $\mathcal{Y}$ converges to I since $(\forall\, n \in \omega)\,(\forall^\infty\, J \in \mathcal{Y})\ \ J \in \langle I\restriction_n\rangle$. Therefore $\mathcal{K}' \subseteq \mathcal{I}$ is τ_{bd}-compact.

Finally, let $\mathcal{U} \subseteq \mathcal{K}'$ be an τ_{bd}-open set of $\mathcal{K}'$ such that there is some $I \in \mathcal{U} \cap \mathcal{K}$. Since $\{I \cup X_n : n \in \omega\} \subseteq \mathcal{K}'$ τ_{bd}-converges to I, there is some $J \in \mathcal{U} \setminus \mathcal{K}$. Since $\mathcal{K}$ is τ_{bd}-closed, there is some τ_{bd}-open set $\mathcal{V}$ such that $J \in \mathcal{V}$ and $\mathcal{V} \cap \mathcal{K} = \emptyset$. Hence, $\mathcal{V} \cap \mathcal{U} \subseteq \mathcal{U}$ is a non-empty τ_{bd}-open set of $\mathcal{K}'$ disjoint to $\mathcal{K}$. Therefore $\mathcal{K}$ is τ_{bd}-nowhere dense in $\mathcal{K}'$. ■

We will use the previous lemmas to show that $(\mathrm{Fin}\times\emptyset,\tau_{\mathrm{bd}})$ is unique (up to homeomorphisms) among ideals whose bounded topology is σ-compact and no locally compact. To prove it, we also need a result due to B. Knaster and M. Reichbach given in [32, Theorem 2].

Theorem 3.5.6 (Knaster and Reichbach, [32])**.** Let X, Y be a pair of compact, perfect, zero-dimensional and metric spaces, and let $X'\subseteq X$, $Y'\subseteq Y$ be closed nowhere dense subsets of its respective space. If $\varphi' : X'\to Y'$ is a homeomorphism, then there exists a homeomorphism $\varphi : X\to Y$ extending φ'.

Theorem 3.5.7. Let $\mathcal{I}$ be an ideal such that $(\mathcal{I},\tau_{\mathrm{bd}})$ is a σ-compact space. Then either $\mathcal{I}=\mathrm{Fin}\oplus\mathcal{P}(A)$ for some $A\in\mathcal{I}$ or $(\mathcal{I},\tau_{\mathrm{bd}})$ is homeomorphic to $(\mathrm{Fin}\times\emptyset,\tau_{\mathrm{bd}})$.

Proof. Suppose that $\mathcal{I}\neq\mathrm{Fin}\oplus\mathcal{P}(A)$ for any $A\in\mathcal{I}$. Let $\{\mathcal{K}_n : n\in\omega\}$ the family given by Lemma 3.5.4, since $\mathcal{I}$ is no locally compact then every $\mathcal{K}_n$ is τ_{bd}-nowhere dense. Using Lemma 3.5.1 and Lemma 3.5.5 we can assume that $\mathcal{K}_n$ is a τ_{bd}-nowhere dense subspace of $\mathcal{K}_{n+1}$ for all $n\in\omega$. Let $C_n=(n+1)\times\omega\in\mathrm{Fin}\times\emptyset$. Using the previous theorem, we have that for all $n\in\omega$ there is a τ_{bd}-homeomorphism $\varphi_n : \mathcal{K}_n\to\mathcal{P}(C_n)$ such that φ_{n+1} extends φ_n. Let $\varphi=\bigcup_{n\in\omega}\varphi_n$, we claim that the bijective map $\varphi : \mathcal{I}\to\mathrm{Fin}\times\emptyset$ is a τ_{bd}-homeomorphism.

Let $\mathcal{U}\subseteq\mathrm{Fin}\times\emptyset$ be an τ_{bd}-open set. Since φ is injective, $\varphi^{-1}[\mathcal{U}]\cap\mathcal{K}_n=\varphi_n^{-1}[\mathcal{U}\cap\mathcal{C}_n]$ and then $\varphi^{-1}[\mathcal{U}]\cap\mathcal{K}_n$ is open in $\mathcal{K}_n$ for all $n\in\omega$. Therefore $\varphi^{-1}[\mathcal{U}]\subseteq\mathcal{I}$ is a τ_{bd}-open set because the family $\{\mathcal{K}_n : n\in\omega\}$ is cofinal among the web sets of $\mathcal{I}$ by Lemma 3.5.1. Thus φ is a τ_{bd}-continuous map. Analogous arguments for φ^{-1} shows that φ is an τ_{bd}-open map, therefore it is a τ_{bd}-homeomorphism. ■

Note that the homeomorphism given in the previous result does not preserve cofinal subsets, because $\mathrm{cof}(\mathrm{Fin}\times\emptyset)=\omega<\mathrm{cof}(\mathcal{J}_{\mathcal{P}})$ although $(\mathrm{Fin}\times\emptyset,\tau_{\mathrm{bd}})$ and $(\mathcal{I}_{\mathcal{P}},\tau_{\mathrm{bd}})$ are homeomorphic spaces.

We present a Polish space introduced by P. Erdös in [14].

Definition 3.5.8. The *complete Erdös space* $\mathfrak{E}_c$ is the closed subspace of ℓ^2 such that $(x_n)_{n\in\omega}\in\mathfrak{E}_c$ if and only if $(\forall\, n\in\omega)\ x_n\in\{{}^1\!/_m : m\in\omega\}\cup\{0\}$.

$\mathfrak{E}_c$ is a Polish space which is totally disconnected, one-dimensional and almost zero dimensional (for that and more about this space see [10]). Also J. Dijkstra and J. van Mill prove that if $\mathcal{I}$ is an F_σ P-ideal, then $(\mathcal{I},\tau_{\mathrm{bd}})$ is not a σ-compact space if and only if it is homeomorphic to $\mathfrak{E}_c$ (see [9, Theorem 4.15]). Using this result we prove the following.

Theorem 3.5.9. Let $\mathcal{I}$ be an F_σ-ideal. Then the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homeomorphic to $\mathfrak{E}_c$ if and only if $\mathcal{I}$ is a P-ideal and $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$.

Proof. If $\mathcal{I}$ is a P-ideal such that $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is no σ-generated by Theorem 3.5.3, and hence $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homeomorphic to $\mathfrak{E}_c$. On the other hand, if $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homeomorphic to $\mathfrak{E}_c$ then the space is metrizable, hence $\mathcal{I}$ is a P-ideal by Theorem 3.3.9. By Proposition 3.3.24 the ideal $\mathcal{I}$ has the web-closure property. Then, again by Theorem 3.5.3, we conclude that $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ since $(\mathcal{I}, \tau_{\mathrm{bd}})$ is no a σ-compact space. ■

Let $\mathcal{I}$ be an F_σ and P-ideal. By Theorem 2.2.3 there is some submeasure φ such that $\mathcal{I} = \mathrm{Fin}(\varphi)$. For $\varepsilon > 0$, if $\{A \subseteq \omega : \varphi(A) = \varepsilon\} = \emptyset$ then $\{A \subseteq \omega : \varphi(A) < \varepsilon\}$ is a τ_{bd}-clopen set. Thus by the previous result, if $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ then there exists $\varepsilon > 0$ such that $(\forall x \in [0, \varepsilon])\,(\exists A \subseteq \omega)\ \varphi(A) = x$, because otherwise $(\mathcal{I}, \tau_{\mathrm{bd}}) \simeq \mathfrak{E}_c$ would be a zero-dimensional space.

Summarizing some of the previous results, we give a classification of F_σ-ideals through its bounded topologies as follows.

Theorem 3.5.10. Let $\mathcal{I}$ be an F_σ-ideal. Then:

i) $(\mathcal{I}, \tau_{\mathrm{bd}}) \simeq \omega$ if and only if $\mathcal{I} = \mathrm{Fin}$.

ii) $(\mathcal{I}, \tau_{\mathrm{bd}}) \simeq \omega \times 2^\omega$ if and only if $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$ for some infinite $A \in \mathcal{I}$.

iii) $(\mathcal{I}, \tau_{\mathrm{bd}}) \simeq (\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ if and only if $\mathcal{I}$ has the web-closure property, $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$ and $\mathcal{I} \neq \mathrm{Fin} \oplus \mathcal{P}(A)$ for any $A \in \mathcal{I}$.

iv) $(\mathcal{I}, \tau_{\mathrm{bd}}) \simeq \mathfrak{E}_c$ if and only if $\mathcal{I}$ is a P-ideal and $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$.

We conclude with some conjectures related with the bounded topology.

Conjecture 3.5.11. Let $\mathcal{I}$ be an F_σ-ideal whose bounded topology does not satisfy any of the conditions in Theorem 3.5.10. Must $\mathcal{I}$ have a sun set of size $\mathfrak{c}$?

If the previous is true then that would imply Conjecture 3.4.14 also in the positive.

Question 3.5.12. Is there a result analogous to Theorem 3.5.10 for P-ideals?

About the previus question, we conjecture the following.

Conjecture 3.5.13. Let $\mathcal{I}$ be a P-ideal such that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is no σ-compact. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homeomorphic either to ω^ω, $\mathfrak{E}_c$ or $\mathfrak{E}_c^\omega$.

Question 3.5.14. Under what conditions is $(\mathcal{I}, \tau_{\mathrm{bd}})$ a topological group?

Question 3.5.15. Which topological spaces are homeomorphic to $(\mathcal{I}, \tau_{\mathrm{bd}})$ for some (Borel) ideal $\mathcal{I}$? We know they have to be homogeneous, separable and sequential with a weaker homogeneous zero-dimensional metric topology. Is this sufficient?

Question 3.5.16. Are there infinitely (uncountably) many Borel ideals $\mathcal{I}$ such that the spaces $\mathcal{I}$ is $(\mathcal{I}, \tau_{\mathrm{bd}})$ are mutually non-homeomorphic?

Question 3.5.17. For which (Borel) ideals $\mathcal{I}$ is $(\mathcal{I}, \tau_{\mathrm{bd}})$ regular (respectively metrizable and Lindelöf)?

Chapter 4

Trace Ideales on ω

In the final part we will study a way to associate an ideal on 2^ω or ω^ω with an ideal on ω via the *trace ideals* (Definition 4.0.4) introduced by J. Brendle and S. Yatabe in [7]. First we will define some related cardinals (Definition 4.1.2) and show how these are related with their usual cardinals (defined in the introduction). At the end, we present some examples of trace ideals and their relationship with the web and sun sets from the previous chapter. For a detailed study of trace ideal, mainly related to their connection to Forcing, see [24] and [25].

The concept of trace can be applied to both, ideals on 2^ω and ideals on ω^ω. We write the definitions and results for ideals on 2^ω although for the other case it is totally analogous. First we define a strongest notion from branches for a subset of a tree, which will be important for the main definition.

Definition 4.0.1. Let $A \subseteq 2^{<\omega}$. The *set of ideal branches of* A is defined by

$$[A]_\infty = \{f \in 2^\omega : (\exists^\infty n \in \omega)\ \ f{\restriction}_n \in A\}.$$

Directly from the definition we have that for a pair of set $A, B \subseteq 2^{<\omega}$ such that $A \subseteq^* B$, then $[A]_\infty \subseteq [B]_\infty$ holds. Also, the set of ideal branches is empty for antichains, finite sets and finite unions of these kind of sets. We will use the following definition to see the complexity of those sets.

Definition 4.0.2. Let $A \subseteq 2^\omega$. We define the following.

a) For $s \in A$, the *rank of* s *in* A is $\mathrm{rk}_A(s) = |\mathrm{pred}\,(s) \cap A|$.

b) For $n \in \omega$, the *n-th rank level of* A is $A^{\langle n\rangle} = \{s \in A : \mathrm{rk}_A(s) = n\}$.

Proposition 4.0.3. Let $G \subseteq 2^\omega$. G is a G_δ-set if and only if there is some $A \subseteq 2^{<\omega}$ such that $[A]_\infty = G$.

Proof. Since $[A]_\infty = \bigcap_{n\in\omega} \bigcup \left\{[s] \subseteq 2^\omega : s \in A^{\langle n\rangle}\right\}$, then it is a G_δ-set. On the other hand, let $G \subseteq 2^\omega$ be a G_δ-set, then there is a decreasing family $\{G_n \subseteq 2^\omega : n \in \omega\}$ of open sets such that $G = \bigcap_{n\in\omega} G_n$. This family defines a sequence of antichains $\{A_n \subseteq 2^{<\omega} : n \in \omega\} \subseteq \mathcal{P}\left(2^{<\omega}\right)$ such that $(\forall\, n \in \omega)\,(\forall\, t \in A_{n+1})\,(\exists\, s \in A_n)\ s \subsetneq t$ and also $(\forall\, n \in \omega)\ G_n = \bigcup\{[s] : s \in A_n\}$. Therefore $[A]_\infty = G$ for $A = \bigcup_{n\in\omega} A_n$. ■

The following is the key definition of this chapter.

Definition 4.0.4. Let $\mathcal{L}$ be an ideal on 2^ω. The *trace ideal of* $\mathcal{L}$ is defined by

$$\mathrm{tr}\,(\mathcal{L}) = \{A \subseteq 2^{<\omega} : [A]_\infty \in \mathcal{L}\},$$

which is, indeed, an ideal on $2^{<\omega}$.

Directly from the previous definition we have the following facts.

Proposition 4.0.5. Let $\mathcal{L}$ be an ideal on 2^ω and $A \subseteq 2^{<\omega}$.

a) If $[A]_\infty$ is finite, then $A \in \mathrm{tr}\,(\mathcal{L})$.

b) If A is an antichain, then $A \in \mathrm{tr}\,(\mathcal{L})$.

c) $(\forall\, f \in 2^\omega)$ $\mathrm{nd}\,(f) \in \mathrm{tr}\,(\mathcal{L})$ (recall that $\mathrm{nd}\,(f) = \{f{\restriction}_n : n \in \omega\}$).

d) $\mathrm{tr}\,(\mathcal{L})$ is a tall ideal.

Proof. Since $\mathcal{L}$ is an ideal, then $[2^\omega]^{<\omega} \subseteq \mathcal{L}$. Then the a) part holds. Parts b) and c) follow from this because $[A]_\infty$ is empty when A is an antichain, and $|\mathrm{nd}\,(f)\,| = 1$.

For d). Let $X \subseteq 2^\omega$ be an infinite set. Either then exists $f \in [X]_\infty$ and therefore $|\mathrm{nd}\,(f) \cap X| = \omega$; or $[X]_\infty = \emptyset$ and therefore there exists an infinite antichain $A \subseteq X$. In any case, X has a infinite subset which lies in $\mathrm{tr}\,(\mathcal{L})$. ■

We will present a way to relate an ideal with its trace.

Definition 4.0.6. Let $\mathcal{L}$ be an ideal on 2^ω. A map $i : \mathcal{L} \to \mathrm{tr}\,(\mathcal{L})$ is *arboreal for* $\mathcal{L}$ if it satisfies $(\forall\, L \in \mathcal{L})$ $L \subseteq [i(L)]_\infty$. If such function exists, $\mathcal{L}$ is called *arboreal ideal.*

It follows from Proposition 4.0.3 that an ideal is arboreal if and only if it has a basis consisting of G_δ sets. We also have the following.

Proposition 4.0.7. Let $\mathcal{L}, \mathcal{H}$ be a pair of ideals on 2^ω. The following holds.

i) If $\mathcal{L} \subseteq \mathcal{H}$, then $\operatorname{tr}(\mathcal{L}) \subseteq \operatorname{tr}(\mathcal{H})$.

ii) If $\operatorname{tr}(\mathcal{L}) \subseteq \operatorname{tr}(\mathcal{H})$ and $\mathcal{L}$ is arboreal, then $\mathcal{L} \subseteq \mathcal{H}$.

iii) If $\mathcal{L}$ is arboreal, then $(\mathcal{L}, \subseteq) \leq_T (\operatorname{tr}(\mathcal{L}), \subseteq^*)$.

Proof. The i) part follows directly from Definition 4.0.4. For the ii) part, let $L \in \mathcal{L}$. Since $\mathcal{L}$ is arboreal then $L \subseteq [i(L)]_\infty$ and due to $\operatorname{tr}(\mathcal{L}) \subseteq \operatorname{tr}(\mathcal{H})$ then $[i(L)]_\infty \in \mathcal{H}$. Thus $\mathcal{L} \subseteq \mathcal{H}$. For the iii) part, let $\mathcal{A} \subseteq \operatorname{tr}(\mathcal{L})$ be a set $\subseteq^*$-bounded by some $X \in \operatorname{tr}(\mathcal{L})$. Then $i^{-1}[\mathcal{A}]$ is bounded by $[X]_\infty$, and thus the arboreal map is a witness for $(\mathcal{L}, \subseteq) \leq_T (\operatorname{tr}(\mathcal{L}), \subseteq^*)$. ■

4.1 Associated Cardinals

From Proposition 4.0.7 and Proposition 2.1.4 we directly have the following.

Proposition 4.1.1. Let $\mathcal{L}$ be an arboreal ideal on 2^ω. The following holds.

i) $\operatorname{cof}(\mathcal{L}) \leq \operatorname{cof}^*(\operatorname{tr}(\mathcal{L}))$.

ii) $\operatorname{add}^*(\operatorname{tr}(\mathcal{L})) \leq \operatorname{add}(\mathcal{L})$.

For an ideal $\mathcal{L}$ on 2^ω we denote with $\mathcal{L}^\delta$ the ideal generated by the G_δ-sets of $\mathcal{L}$. We define the following cardinals.

Definition 4.1.2. Let $\mathcal{L}$ be an ideal on 2^ω. We define

a) $\operatorname{add}_\delta(\mathcal{L}) = \operatorname{add}\left(\mathcal{L}^\delta\right)$.

b) $\operatorname{cov}_\delta(\mathcal{L}) = \operatorname{cov}\left(\mathcal{L}^\delta\right)$.

c) $\operatorname{non}_\delta(\mathcal{L}) = \operatorname{non}\left(\mathcal{L}^\delta\right)$.

d) $\operatorname{cof}_\delta(\mathcal{L}) = \operatorname{cof}\left(\mathcal{L}^\delta\right)$.

Directly from Definition 4.0.4 and Proposition 4.0.3 we have that $\operatorname{tr}(\mathcal{L}) = \operatorname{tr}\left(\mathcal{L}^\delta\right)$. Also $\mathcal{L}$ is arboreal if and only if $\mathcal{L} = \mathcal{L}^\delta$, in this case trivially $\operatorname{non}_\delta(\mathcal{L}) = \operatorname{non}(\mathcal{L})$ and $\operatorname{cov}_\delta(\mathcal{L}) = \operatorname{cov}(\mathcal{L})$, and in general we have the following inequalities.

Proposition 4.1.3. Let $\mathcal{L}$ be an ideal on 2^ω. The following holds.

i) $\operatorname{cov}(\mathcal{L}) \leq \operatorname{cov}_\delta(\mathcal{L}) \leq \operatorname{cov}^*(\operatorname{tr}(\mathcal{L}))$.

ii) $\operatorname{non}^*(\operatorname{tr}(\mathcal{L})) \leq \operatorname{non}_\delta(\mathcal{L}) \leq \operatorname{non}(\mathcal{L})$.

iii) $\operatorname{add}^*(\operatorname{tr}(\mathcal{L})) \leq \operatorname{add}_\delta(\mathcal{L})$ and $\operatorname{cof}_\delta(\mathcal{L}) \leq \operatorname{cov}^*(\operatorname{tr}(\mathcal{L}))$.

Proof. Since $\mathcal{L}^\delta \subseteq \mathcal{L}$, then $\operatorname{cov}(\mathcal{L}) \leq \operatorname{cov}_\delta(\mathcal{L})$ and $\operatorname{non}_\delta(\mathcal{L}) \leq \operatorname{non}(\mathcal{L})$.

For i). Let $\kappa < \operatorname{cov}_\delta(\mathcal{L})$ be a cardinal, $\mathcal{A} \in [\operatorname{tr}(\mathcal{L})]^\kappa$ and $f \in 2^\omega$ such that $f \notin \bigcup\{[A]_\infty : A \in \mathcal{A}\}$, then $|\operatorname{nd}(f)| = \omega$ and it satisfies that $(\forall\, A \in \mathcal{A})\ |\operatorname{nd}(f) \cap A| < \omega$. Therefore $\kappa < \operatorname{cov}^*(\operatorname{tr}(\mathcal{L}))$ and the inequality holds.

For ii). Let $Z \notin \mathcal{L}^\delta$ and $\mathcal{X} = \{\operatorname{nd}(f) : f \in Z\} \subseteq [2^{<\omega}]^\omega$. If there exists $A \in \operatorname{tr}(\mathcal{L})$ such that $(\forall\, f \in Z)\ |\operatorname{nd}(f) \cap A| = \omega$ then $Z \subseteq [A]_\infty$, contradicting the choice of Z. Thus $\mathcal{X}$ satisfies the definition for $\operatorname{non}^*(\operatorname{tr}(\mathcal{L}))$ and the inequality holds.

The part iii) follows from Proposition 4.1.1 and the fact that $\mathcal{L}^\delta$ is arboreal. ■

Corollary 4.1.4. Let $\mathcal{L}$ be an σ-ideal on 2^ω. If $\operatorname{non}_\delta(\mathcal{L}) = \omega$ then $\mathcal{L}$ is not an arboreal ideal and $\operatorname{tr}(\mathcal{L})$ is not an P-ideal.

Proof. Let $\mathcal{L}$ be an ideal such that it satisfies the hypotheses. Then there exists $A \in \mathcal{L} \setminus \mathcal{L}^\delta$ such that $|A| = \omega$, which is a witnesses for $\mathcal{L}$ is not arboreal. On the other hand, since $\operatorname{add}^*(\operatorname{tr}(\mathcal{L})) \leq \operatorname{non}^*(\operatorname{tr}(\mathcal{L})) \leq \operatorname{non}_\delta(\mathcal{L})$, then $\operatorname{add}^*(\operatorname{tr}(\mathcal{L})) = \omega$ and therefore $\operatorname{tr}(\mathcal{L})$ is not an P-ideal. ■

In summary, for an ideal $\mathcal{L}$ on 2^ω we have the following diagram among the previously defined cardinals.

$$\begin{tikzcd}
 & & \operatorname{cof}^*(\operatorname{tr}(\mathcal{L})) & & \\
 & \operatorname{cov}^*(\operatorname{tr}(\mathcal{L})) \arrow[ur] & \operatorname{cof}_\delta(\mathcal{L}) \arrow[u] & & \\
\operatorname{cov}(\mathcal{L}) \arrow[r] & \operatorname{cov}_\delta(\mathcal{L}) \arrow[u] \arrow[ur] & & \operatorname{non}_\delta(\mathcal{L}) \arrow[ul] \arrow[r] & \operatorname{non}(\mathcal{L}) \\
 & & \operatorname{add}_\delta(\mathcal{L}) \arrow[ul] \arrow[ur] & \operatorname{non}^*(\operatorname{tr}(\mathcal{L})) \arrow[u] & \\
 & & \operatorname{add}^*(\operatorname{tr}(\mathcal{L})) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

For $A \in \operatorname{tr}(\mathcal{L})$ and any $X \in \operatorname{tr}(\emptyset)$ we have that $A \cup X \in \operatorname{tr}(\mathcal{L})$, to restrict this only for sets such that $[X]_\infty \subseteq \mathcal{L}$ we can define another kind of trace ideals. In order to do that, we need consider the following hyperspace. Recall that for a topological space X we have that $\mathcal{K}(X) = \{K \subseteq X : X \text{ is compact}\}$.

Definition 4.1.5. The *Vietoris topology* on $\mathcal{K}(X)$ is the topology such that, for all finite family of open sets $\{U_i : i \leq n\} \subseteq \mathcal{P}(X)$, the following is a basic open set:

$$\left\{K \in \mathcal{K}(X) : K \subseteq \bigcup \{U_i : i \leq n\} \ \wedge \ (\forall\, i \leq n) \ \ K \cap U_i \neq \emptyset\right\}.$$

This topology can be generally defined on the closed subsets of the space in a similar way. To delve into this topology see [36] and [40].

Definition 4.1.6. Let X be a space. Then $\mathcal{I} \subseteq \mathcal{K}(P)$ is an *ideal* (or a *ideal of compacts sets*) if it is closed under finite unions and compact subsets, that is, if $(\forall\, F \in [\mathcal{I}]^{<\omega}) \ \bigcup F \in \mathcal{I}$ and $(\forall\, K \in \mathcal{I})\,(\forall\, L \in \mathcal{K}(P)) \ \ L \subseteq K \ \rightarrow \ L \in \mathcal{I}$ hold. Even more, if an ideal $\mathcal{I} \subseteq \mathcal{K}(P)$ satisfies $(\forall\, C \in [\mathcal{I}]^{\omega}) \ \bigcup C \in \mathcal{K}(P) \ \rightarrow \ \bigcup C \in \mathcal{I}$, then $\mathcal{I}$ is an *σ-ideal* (or a *σ-ideal of compacts sets*).

Let X be a compact metrizable space. R. Dougherty, A. Kechris, A. Louveau and W. Woodin show that $\mathcal{I} \subseteq \mathcal{K}(X)$ is an analytic σ-ideal if and only if $\mathcal{I}$ is an G_δ-ideal (in [30, Theorem 3] and [31, Theorem 11]). More generally, for $A \subseteq X$ let $\mathcal{K}(A) = \{K \in \mathcal{K}(X) : K \subseteq A\}$. S. Solecki gives a stronger property than being an σ-ideal, and proves the following result in [43, Proposition 2.1].

Proposition 4.1.7 (Solecki [43])**.** Let X be a compact metrizable space and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be an analytic or coanalytic ideal. Then $\mathcal{I}$ is an G_δ-ideal if it satisfies the following property.

$$(\forall\, \mathcal{X} \in [\mathcal{I}]^{\omega})\,(\exists\, G \subseteq X) \ \ G \text{ is } \mathrm{G}_\delta \ \wedge \ \bigcup \mathcal{X} \subseteq G \ \wedge \ \mathcal{K}(G) \subseteq \mathcal{I}.$$

As mentioned before, we use the previous to define a new kind of trace ideals.

Definition 4.1.8. Let $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ be an ideal. The *compact-trace ideal of* $\mathcal{I}$ is defined as follows.

$$\operatorname{tr}_{\mathcal{K}}(\mathcal{I}) = \langle \{T \subseteq 2^{<\omega} : T \text{ is a pruned tree and } [T] \in \mathcal{I}\}\rangle.$$

Unlike trace ideals, $\operatorname{tr}_{\mathcal{K}}(\mathcal{I})$ is not necessarily a tall ideal (as we will show in the following section). About the cofinality of a compact-trace ideal, from its definition directly follows that $\operatorname{cof}(\operatorname{tr}_{\mathcal{K}}(\mathcal{I})) = \operatorname{cof}(\mathcal{I})$.

4.2 Examples

In this final section we present some examples of trace ideals, showing some of its properties. At first, we have the following.

Proposition 4.2.1. Let $\mathcal{L}$ be an σ-ideal on 2^ω. If $\mathcal{L} \subseteq \mathcal{M}$, then $\mathcal{L}$ is not arboreal and $\operatorname{tr}(\mathcal{L})$ is not an P-ideal.

Proof. By Baire Category Theorem, any dense and G_δ and dense set is non meager. Then, there is no G_δ-set in $\mathcal{M}$ containing the rational numbers. Hence, the set $\{\{q\} : q \in \mathbb{Q}\} \subseteq \mathcal{M}$ witnesses that $\operatorname{non}_\delta(\mathcal{M}) = \omega$ and therefore $\operatorname{non}_\delta(\mathcal{L}) = \omega$. Using Corollary 4.1.4 we are done. ■

Besides the ideal $\mathcal{N}$ is arboreal, and its trace is P-ideal, as show below.

tr $(\mathcal{N})$

We have the following, originally proved by P. Borodulin-Nadzieja and B. Farkas in [6, Example 2.7],

Theorem 4.2.2. There is a lower semicontinuous submeasure $\varphi : \mathcal{P}(2^{<\omega}) \to [0,1]$ such that $\operatorname{tr}(\mathcal{N}) = \operatorname{Exh}(\varphi)$.

Proof. Recall that $A^{\langle 0\rangle} = \{s \in A : \operatorname{pred}(s) \cap A = \emptyset\}$ for $A \subseteq 2^{<\omega}$. Let the map $\varphi : \mathcal{P}(2^{<\omega}) \to [0,1]$ defined by

$$\varphi(A) = \sum \left\{2^{-|s|} : s \in A^{\langle 0\rangle}\right\}.$$

Clearly $\varphi(\emptyset) = 0$. Let $A, B \subseteq 2^{<\omega}$ such that $A \subseteq B$, then $\varphi(A) \leq \varphi(B)$ since for every $s \in B^{\langle 0\rangle} \setminus A^{\langle 0\rangle}$ and every $A_s \subseteq A^{\langle 0\rangle}$ such that $(\forall\, t \in A_s)\ \ s \subseteq t$ follows that $2^{-|s|} \geq \sum_{t\in A_s} 2^{-|t|}$. Now, since $(A \cup B)^{\langle 0\rangle} \subseteq A^{\langle 0\rangle} \cup B^{\langle 0\rangle}$, then $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$. This shows that φ is a submeasure. Finally, since $\bigcup_{n\in\omega} (A \cap 2^{<n})^{\langle 0\rangle} = A^{\langle 0\rangle}$ and φ is defined as a convergent series, then φ is a lower semicontinuous submeasure.

From the definition of φ, follows that $\operatorname{Exh}(\varphi) \subseteq \operatorname{tr}(\mathcal{N})$. Let $A \notin \operatorname{Exh}(\varphi)$ and let $\varepsilon > 0$ a witness of this, that is, $\varphi(A \setminus 2^{<n}) \geq \varepsilon$ for every $n \in \omega$. For $n \in \omega$ let $U_n = \bigcup\{[s] \subseteq 2^\omega : s \in (A \setminus 2^{<n})^{\langle 0\rangle}\}$, then $\{U_n : n \in \omega\}$ is a decreasing sequence of sets with measure not less than ε, thus $\bigcap_{n\in\omega} U_n \subseteq 2^\omega$ has positive measure. Since $\bigcap_{n\in\omega} U_n = [A]_\infty$ then $A \notin \operatorname{tr}(\mathcal{N})$, and the theorem holds. ■

The ideal $\mathcal{N}$ is arboreal because every null set is subset of an G_δ-null set. Furthermore, F. Hernández-Hernández and M. Hrušák proved in [23, Theorem 3.4] that $\mathcal{I}_{1/n} \hookrightarrow \operatorname{tr}(\mathcal{N}) \hookrightarrow \mathcal{Z}$ in the following sense. Recall that $\mathcal{I} \cong \mathcal{J}$ means that the ideals are isomorphic (see the preliminaries 1.1).

Theorem 4.2.3 (Hernández-Hernández and Hrušák [23])**.** The following holds.

i) $\mathcal{I}_{1/n} \cong \left\{A \subseteq 2^{<\omega} : \sum_{n\in\omega} \dfrac{|A \cap 2^n|}{2^n} < \infty\right\} \subseteq \operatorname{tr}(\mathcal{N})$

ii) $\operatorname{tr}(\mathcal{N}) \subseteq \left\{A \subseteq 2^{<\omega} : \lim_{n\in\omega} \dfrac{|A \cap 2^n|}{2^n} = 0\right\} \cong \mathcal{Z}$

This fact, along with the following characterization of the null sets of 2^ω taken from [3, Lemma 2.5.1], shows that there is an arboreal function for $\mathcal{N}$ whose range is the embedding of $\mathcal{I}_{1/n}$ on $\operatorname{tr}(\mathcal{N})$.

Theorem 4.2.4 (Bartoszyński and Judah [3])**.** Let $X \subseteq 2^\omega$. X is a null set if and only if there is a set $A \subseteq 2^{<\omega}$ such that $X \subseteq [A]_\infty$ and $\sum\limits_{n\in\omega} {}^{|A\cap 2^n|}/_{2^n} < \infty$

Proof. For the «if» part. Let A a set that satisfies the last two conditions. By the previous result and since $\sum\left\{{}^{|A\cap 2^n|}/_{2^n} : n \in \omega\right\} < \infty$, we have that $A \in \operatorname{tr}(\mathcal{N})$. Moreover, since $X \subseteq [A]_\infty$ we have that X is a null set.

For the «only if» part. Let $\langle G_n \subseteq 2^\omega : n \in \omega\rangle$ a decreasing sequence of open sets such that $X \subseteq G_n$ and $\mu(G_n) < 2^{-n}$. For every $n \in \omega$ let $A_n \subseteq 2^{<\omega}$ satisfying $G_n = \bigcup\{[s] \subseteq 2^\omega : s \in A_n\}$. Let $A = \bigcup\{A_n \subseteq 2^{<\omega} : n \in \omega\}$, we have that

$$\sum_{n\in\omega} {}^{|A\cap 2^n|}/_{2^n} \leq \sum_{n\in\omega} \mu(G_n) \leq 1$$

Finally, we have that $(\forall\, n \in \omega)\, (\exists\, s_n \in A_n)\;\; f \in [s_n]$ for every $f \in X \subseteq \bigcap_{n\in\omega} G_n$, and since $\lim_n \mu(G_n) = 0$ then $\{s_n : n \in \omega\}$ is infinite, therefore $(\exists^\infty\, n \in \omega)\;\; f{\restriction}_n \in A$ and we have that $X \subseteq [A]_\infty$. ■

In [3, Lemma 2.3.3] the authors prove that $\left(\mathcal{I}_{1/n}, \subseteq^*\right) \equiv_T (\mathcal{N}, \subseteq)$ (recall that $\equiv_T$ means that the orders are Tukey-equivalent, see section 2.1). Then, using Theorem 2.1.14, we have that $(\mathcal{I}, \subseteq^*) \leq_T (\mathcal{N}, \subseteq)$ for every $\mathcal{I}$ analytic P-ideal on ω. Thus $\operatorname{add}^*(\mathcal{I}) \geq \operatorname{add}(\mathcal{N})$ and $\operatorname{cof}^*(\mathcal{I}) \leq \operatorname{cof}(\mathcal{N})$ for this kind of ideals. Using also Proposition 4.1.1 we have the following result.

Proposition 4.2.5. Let $\mathcal{L} \subseteq \mathcal{P}(2^\omega)$ be an arboreal ideal whose trace ideal is an analytic P-ideal. The following holds.

i) $\operatorname{add}(\mathcal{L}) \geq \operatorname{add}^*(\operatorname{tr}(\mathcal{L})) \geq \operatorname{add}(\mathcal{N})$

ii) $\operatorname{cof}(\mathcal{L}) \leq \operatorname{cof}^*(\operatorname{tr}(\mathcal{N})) \leq \operatorname{cof}(\mathcal{N})$

Particularly, we have that $\operatorname{add}^*(\operatorname{tr}(\mathcal{N})) = \operatorname{add}(\mathcal{N})$ and $\operatorname{cof}^*(\operatorname{tr}(\mathcal{N})) = \operatorname{cof}(\mathcal{N})$.

For a family of sets $\mathcal{A} \subseteq \operatorname{tr}(\mathcal{N})$ with $|\mathcal{A}| < \operatorname{cov}^*(\operatorname{tr}(\mathcal{N}))$ we know that exists a set $X \in [2^{<\omega}]^\omega$ such that $X \cap A$ is finite for every $A \in \mathcal{A}$, but in general such X does not contain a branch. The following observation gives an equivalency for this.

Proposition 4.2.6. $\operatorname{cov}^*(\operatorname{tr}(\mathcal{N})) = \operatorname{cov}(\mathcal{N})$ if and only if for every $\mathcal{A} \subseteq \operatorname{tr}(\mathcal{N})$ such that $|\mathcal{A}| < \operatorname{cov}^*(\mathcal{N})$ there is a map $f \in 2^\omega$ satisfying $(\forall\, A \in \mathcal{A})\ |\operatorname{nd}(f) \cap A| < \omega$.

Proof. To prove the «if» part. We already have that $\operatorname{cov}(\mathcal{N}) \leq \operatorname{cov}^*(\operatorname{tr}(\mathcal{N}))$. Now let $\kappa < \operatorname{cov}^*(\operatorname{tr}(\mathcal{N}))$ and $\mathcal{X} \subseteq \mathcal{N}$ with $|\mathcal{X}| = \kappa$. Since $\mathcal{N}$ is abroreal, there is a family of sets $\mathcal{A} \subseteq \operatorname{tr}(\mathcal{N})$ of size κ such that $(\forall\, X \in \mathcal{X})\,(\exists\, A \in \mathcal{A})\ X \subseteq [A]_\infty$, then by hypothesis there is exists $f \in 2^\omega \setminus [A]_\infty$ for every $A \in \mathcal{A}$. Thus $f \notin \bigcup \mathcal{X}$ and therefore $\kappa < \operatorname{cov}(\mathcal{N})$.

To prove the «only if» part. Let $\mathcal{A} \subseteq \operatorname{tr}(\mathcal{N})$ with $|\mathcal{A}| < \operatorname{cov}^*(\operatorname{tr}(\mathcal{N})) = \operatorname{cov}(\mathcal{N})$, and let $\mathcal{X} = \{[A]_\infty : A \in \mathcal{A}\} \subseteq \mathcal{P}(\mathcal{N})$. Since $|\mathcal{X}| \leq |\mathcal{A}|$, there is some $f \in 2^\omega \setminus \bigcup \mathcal{X}$. Clearly $|\operatorname{nd}(f) \cap A| < \omega$ for every $A \in \mathcal{A}$. ■

In [23, Questions 2.3] the authors ask if any two analytic, tall and P-ideals are $\leq_T^*$-equivalent. The following result, along with Theorem 2.1.14, shows that $\mathcal{I}_{1/n}$ and $\operatorname{tr}(\mathcal{N})$ are $\leq_T^*$-equivalent.

Proposition 4.2.7. Let $\mathcal{I}$ be an analytic P-ideal on ω. Then $\mathcal{I} \leq_T^* \operatorname{tr}(\mathcal{N})$.

Proof. The prove gives something more general. Let $\mathcal{I}$ be an ideal with the hypothesis and let $\mathcal{L}$ be an arboreal ideal on 2^ω such that $(\mathcal{I}, \subseteq^*) \leq_T (\mathcal{L}, \subseteq)$. Let $F : \mathcal{I} \to \mathcal{L}$ and $F^* : \mathcal{L} \to \mathcal{I}$ be the witness maps of this fact and let $i : \mathcal{L} \to \operatorname{tr}(\mathcal{L})$ an arboreal function for $\mathcal{L}$. We define the functions $G : \mathcal{I} \to \operatorname{tr}(\mathcal{L})$ and $G^* : \operatorname{tr}(\mathcal{L}) \to \mathcal{I}$ by $G(I) = i(F(I))$ and $G^*(A) = F^*([A]_\infty)$. Then, for every $I \in \mathcal{I}$ and every $A \in \operatorname{tr}(\mathcal{L})$ such that $G(I) \subseteq^* A$ we have that $F(I) \subseteq [i(F(I))]_\infty \subseteq [A]_\infty$, therefore $I \subseteq^* G^*(A)$. Thus, $\mathcal{I} \leq_T^* \operatorname{tr}(\mathcal{L})$. ■

tr $(\mathcal{M})$

As mentioned before, $\mathcal{M}$ is not an arboreal ideal, and the following result gives a more detailed way to see that.

Proposition 4.2.8. $\mathcal{M}^\delta = \mathcal{NWD}$.

Proof. Let M be a G_δ-set which is meager, and let U be an open set such that $M \cap U$ is dense in U. Then $M \cap U$ is comeager in U. On the other hand, since U is open, then $M \cap U$ is meager in U, thus $U = \emptyset$ and therefore M is nowhere dense. ■

Due to this result, we can find the Borel complexity of tr $(\mathcal{M})$.

Theorem 4.2.9. tr $(\mathcal{M})$ is an $\mathrm{F}_{\sigma\delta}$-ideal.

Proof. For $t \in 2^{<\omega}$ let $F_t = \{A \subseteq 2^{<\omega} : (\exists\, s \supseteq t)\ A \cap \langle s \rangle = \emptyset\}$, which is an F_σ-set. For every $A \subseteq 2^{<\omega}$ we have that $A \in F_t$ if and only if $[A]_\infty$ is not a dense in the open set $[t]$, then $(\forall\, t \in 2^{<\omega})\ A \in F_t$ if and only if $[A]_\infty \in \mathcal{NWD}$, and by the previous proposition we concluded that tr $(\mathcal{M}) = \bigcap \{F_t : f \in 2^{<\omega}\}$. ■

In [2], an order on $2^{<\omega}$ is defined by the following. Similarly as the case of 2^ω, we use the notation $\Delta(s,t) = \min\{n \in \omega : s(n) \neq t(n)\}$ for two distinct $s, t \in 2^{<\omega}$.

Definition 4.2.10. Let $s, t \in 2^{<\omega}$. Then $s \prec t$ if one of the following conditions holds.

- $t^\frown\langle 0 \rangle \subseteq s$.
- $s^\frown\langle 1 \rangle \subseteq t$.
- $s(\Delta(s,t)) < t(\Delta(s,t))$.

This order is isomorphic to the the order of $\mathbb{Q}$, therefore they are homeomorphic as topological spaces, and the next facts follow from this (see [2, Fact 1.1]).

- $\{\langle s \rangle : s \in 2^{<\omega}\}$ is a π-base for the topology of $\mathbb{Q}$.
- $D \subseteq 2^{<\omega}$ is dense in $\mathbb{Q}$ if and only if $(\forall\, s \in 2^{<\omega})\,(\exists\, t \in D)\ s \subseteq t$.
- $N \subseteq 2^{<\omega}$ is nowhere dense in $\mathbb{Q}$ if and only if $(\forall\, s \in 2^{<\omega})\,(\exists\, t \supseteq s)\ N \cap \langle t \rangle = \emptyset$.

From these and the proof of the previous theorem, we have that tr $(\mathcal{M}) = nwd$. Finally, as mentioned in [24, 3.1] and proved in [2] and [17], $\mathrm{cof}(nwd) = \mathrm{cof}\,(\mathcal{M})$, $\mathrm{cov}^*(nwd) = \mathrm{cov}\,(\mathcal{M})$, and $\mathrm{non}^*(nwd) = \aleph_0$.

$\mathrm{tr}_{\mathcal{K}}(conv_0)$

Let $conv_0 = \{s \in (2^\omega)^\omega : s \to 0\} \cup [2^\omega]^{<\omega}$, that is, the ideal in $\mathcal{K}(2^\omega)$ generated by the sequences in 2^ω which converge to 0, note that this is not a σ-ideal since, for example, any point of a sequence which converges to 1 belongs to the ideal but the sequence itself does not.

Lemma 4.2.11. $\mathrm{cof}\,(conv_0) = \mathfrak{c}$

Proof. Let $\mathcal{A} \subseteq conv_0$ such that $|\mathcal{A}| < \mathfrak{c}$. Since $conv_0$ is generated by sequences, then any element of it is countable and thus $|\bigcup \mathcal{A}| < \mathfrak{c}$. Since $(\forall\, n \in \omega)\ \ |\langle 0\restriction_n\rangle| = \mathfrak{c}$, then for all $n \in \omega$ pick some $x_n \in \langle 0\restriction_n\rangle \setminus \bigcup \mathcal{A}$. Hence we have that the sequence $\{x_n : n \in \omega\} \subseteq 2^\omega$ converges to 0 and does not belongs to $\mathcal{A}$. Therefore $\mathcal{A}$ is not a cofinal set for $conv_0$ and we are done. ■

By the previous and Proposition 4.1.1 we have that $\mathrm{cof}(\mathrm{tr}_{\mathcal{K}}(conv_0)) = \mathfrak{c}$. For $K \in \mathcal{K}(2^\omega)$ we denote by $T_K \subseteq 2^{<\omega}$ to the pruned tree such that $[T_K] = K$. Directly from the definition of $\mathrm{tr}_{\mathcal{K}}(conv_0)$ we have the following.

Proposition 4.2.12. Let $\mathcal{A} \subseteq \mathrm{tr}_{\mathcal{K}}(conv_0)$. $\mathcal{A}$ is bounded if and only if there exists $K \in \mathcal{K}(2^\omega)$ such that $\bigcup A \subseteq T_K$ and $K' \subseteq \{0\}$.

We will prove that $\mathrm{tr}_{\mathcal{K}}(conv_0)$ is a web-regular ideal, and then by Proposition 3.1.9 it is not a tall ideal and $\mathrm{web}\,(\mathrm{tr}_{\mathcal{K}}(conv_0)) = \mathfrak{c}$.

Lemma 4.2.13. Let $X \subseteq 2^{<\omega}$ be an infinite set such that $[X]_\infty = \emptyset$. Then X contains an infinite antichain.

Proof. If $X^{\langle n\rangle}$ is infinite for some $n \in \omega$, then we are done. Hence we can suppose that $|X^{\langle n\rangle}| < \omega$ for all $n \in \omega$. For $t \in 2^\omega$ and $n \in \omega$ let

$$X_t^{\langle n\rangle} = \{x \in X : |\{y \in X : t \subseteq y \subseteq x\}| = n\}.$$

That is, the elements of X with rank relative to t equal to n. Note that $X_t^{\langle n\rangle}$ is finite since all $X^{\langle m\rangle}$ is. We have that for $t \in 2^{<\omega}$ satisfying $|\langle t\rangle \cap X| = \omega$ there exists $n \in \omega$ such that $|X_t^{\langle n\rangle}| \geq 2$, because otherwise $[X]_\infty \neq \emptyset$.

For $\emptyset$ let n such that there is a pair of distinct nodes $t_0, t_0' \in X^{\langle n\rangle}$ satisfying $|\langle t_0'\rangle \cap X| = \omega$. Recursively, if t_k, t_k' are already defined, then for a suitable $n \in \omega$ there exists two distinct nodes $t_{k+1}, t_{k+1}' \in X_{t_k'}^{\langle n\rangle}$ such that $|\langle t_{k+1}'\rangle \cap X| = \omega$. Hence $\{t_k : k \in \omega\} \subseteq X$ is an infinite antichain. ■

Theorem 4.2.14. $\mathrm{tr}_{\mathcal{K}}(conv_0)$ is a web-regular ideal.

Proof. Let $t_0 = \langle 1 \rangle$ and $t_{n+1} = \langle 0 \rangle^\frown t_n$. Then $\{[t_n] : n \in \omega\} \cup \{\{0\}\}$ is a partition of 2^ω. Let $\mathcal{A} \subseteq \mathrm{tr}_{\mathcal{K}}(conv_0)$ be a web set. We claim that $\langle t_n \rangle \cap \bigcup \mathcal{A}$ does not contains an infinite antichain. Indeed, if for some n there exists an infinite antichain $\{a_n : n \in \omega\} \subseteq \langle t_n \rangle \cap \bigcup \mathcal{A}$, then there is some countable set $\mathcal{B} = \{A_n \in \mathcal{A} : n \in \omega\}$ such that $a_n \in A_n$. Hence, for all countable subset $Y \subseteq \mathcal{B}$, if some $K \in \mathcal{K}(2^\omega)$ satisfies that $Y \subseteq T_K$, then necessarily K is infinite and therefore it has a limit point in $[t_n]$. This proves that $\mathcal{B}$ is a sun set, which is not possible because $\mathcal{A}$ is a web set.

Then we have that for all $n \in \omega$, $|\, [\langle t_n \rangle \cap \bigcup \mathcal{A}]_\infty \,| < \omega$ and, using previous lemma, for any $X \subseteq \langle t_n \rangle \cap \bigcup \mathcal{A}$ such that $[X]_\infty = \emptyset$, $|X| < \omega$. Therefore, for all $n \in \omega$ there exists a finite set $F_n \subseteq [t_n]$ such that $\langle t_n \rangle \cap \bigcup \mathcal{A} \subseteq T_{F_n}$. Hence, $\mathcal{A}$ is bounded by T_K where $K = \bigcup \{F_n : n \in \omega\} \cup \{\emptyset\} \in conv_0$. ■

About its sun sets, we have the following.

Proposition 4.2.15. $\mathrm{tr}_{\mathcal{K}}(conv_0)$ has a sun set of size $\mathfrak{c}$.

Proof. Let $\mathcal{S} = \{\mathrm{nd}\,(x) : x \in [\langle 1 \rangle]\} \subseteq \mathrm{tr}_{\mathcal{K}}(conv_0)$, then $|\mathcal{S}| = \mathfrak{c}$. Let $\mathcal{X} \subseteq \mathcal{S}$ be an infinite set. If $\bigcup \mathcal{X} \subseteq T_K$ for some $K \subseteq 2^\omega$, then $K' \nsubseteq \{0\}$ because K has a limit point in $[\langle 1 \rangle]$. Thus $\mathcal{X}$ is unbounded and therefore $\mathcal{S}$ is a sun set. ■

We have the following question about trace ideals.

Question 4.2.16. Is there an ideal $\mathcal{L}$ on 2^ω such that $\mathrm{tr}\,(\mathcal{L})$ is Borel but not a $\mathrm{F}_{\sigma\delta}$-ideal?

Bibliography

[1] R. Arens, *Note on convergence in topology*, Mathematics Magazine, 23 (1950), pp. 229–234.

[2] B. Balcar, F. Hernández-Hernández, and M. Hrušák, *Combinatorics of dense subsets of the rationals*, Fundamenta Mathematicae, 183 (2004), pp. 59–80.

[3] T. Bartoszyński and H. Judah, *Set Theory. On the Structure of the real line*, A K Peters, 1995.

[4] K. A. Beros and P. B. Larson, *Maximal Tukey types, P-ideals and the weak Rudin-Keisler order*, Archive for Mathematical Logic, (2023).

[5] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in Handbook of Set Theory, M. Foreman and A. Kanamori, eds., Springer, 2010, pp. 395–489.

[6] P. Borodulin-Nadzieja and B. Farkas, *Analytic P-ideals and Banach spaces*, Journal of Functional Analysis, 279 (2020).

[7] J. Brendle and S. Yatabe, *Forcing indestructibility of MAD families*, Annals of Pure and Applied Logic, 132 (2005), pp. 271–312.

[8] K. Ciesielski, *Set Theory for the Working Mathematician*, London Mathematical Society Student Texts, Cambridge University Press, 1997.

[9] J. Dijkstra and J. van Mill, *Characterizing complete Erdös space*, Canad. J. Math., 61 (2009), pp. 124–140.

[10] ——, *Erdös space and homeomorphism groups of manifolds*, Memoirs of The American Mathematical Society - MEM AMER MATH SOC, 208 (2010).

[11] N. Dobrinen and S. Todorčević, *Tukey types of ultrafilters*, Illinois Journal of Mathematics, 55 (2011).

[12] J. Dugundji, *Topology*, Allyn and Bacon series in advanced mathematics, Allyn and Bacon, 1966.

[13] R. Engelking, *General Topology*, vol. 6 of Sigma series in pure mathematics, Heldermann, Berlin, 1989.

[14] P. Erdös, *The dimension of the rational points in Hilbert space*, Ann. of Math. (2), 41 (1940), pp. 734–736.

[15] I. Farah, *Analytic quotients - Theory of liftings for quotients over analytic ideals on the integers*, Memoirs of the American Mathematical Society, 48 (2000), p. 177pp.

[16] D. H. Fremlin, *Families of compact sets and Tukey's ordering*, Atti Sem. Mat. Fis. Univ. Mdena, 39 (1991), pp. 29–50.

[17] ——, *The partially ordered sets of measure theory and Tukey's ordering*, in Note di Matematica, vol. 11, 1991, pp. 177–214.

[18] D. Gale and F. M. Stewart, *Infinite games with perfect information*, vol. 2 of Annals of Mathematics Studies, Princeton University Press, 1953.

[19] A. Galicki, *Aspects of classical descriptive set theory*, arXiv: Logic, (2013).

[20] S. Ginsburg and J. R. Isbell, *The category of cofinal types. i*, Transactions of the American Mathematical Society, 116 (1965), pp. 386–393.

[21] F. Hernández-Hernández, *Teoría de conjuntos. Una introducción*, Aportaciones Matemáticas, 2014.

[22] ——, *Curso de topología. Un enfoque conjuntista*, Aportaciones Matemáticas, 2021.

[23] F. Hernández-Hernández and M. Hrušák, *Cardinal invariants of analytic P-ideals*, Canadian Journal of Mathematics, 59 (2007), pp. 575–595.

[24] M. Hrušák, *Combinatorics of filers and ideals*, Contemporary Mathematics, 533 (2011), pp. 29–69.

[25] M. Hrušák and J. Zapletal, *Forcing with quotiens*, Archive for Mathematical Logic, 47 (2004), pp. 719–739.

[26] J. R. Isbell, *The category of cofinal types. ii*, Transactions of the American Mathematical Society, 116 (1965), pp. 394–416.

[27] ——, *Seven cofinal types*, Journal of the London Mathematical Society, s2-4 (1972), pp. 651–654.

[28] S. A. Jalali-Naini, *The monotone subsets of Cantor Space, filters and descriptive set theory*, PhD thesis, University of Oxford, 1976.

[29] K. Kawamura, L. Oversteegen, and E. D. Tymchatyn, *On homogeneous totally disconnected* 1*-dimensional spaces*, Fund. Math., 150 (1996), pp. 97–112.

[30] A. S. Kechris, *Hereditary properties of the class of closed sets of uniqueness for trigonometric series*, Israel Journal of Mathematics, 73 (1991), pp. 189–198.

[31] A. S. Kechris, A. Louveau, and W. H. Woodin, *The structure of σ-ideals of compact sets*, Transactions of the American Mathematical Society, 301 (1987), pp. 263–288.

[32] B. Knaster and M. Reichbach, *Notion d'homogénéité et prolongements des homéomorphies*, Fundamenta Mathematicae, 40 (1953), pp. 180–193.

[33] K. Kunen, *Set theory: an introduction to independence proofs*, North-Holland, 1980.

[34] A. Louveau and B. Veličković, *Analytic ideals and cofinal types*, Annals of Pure and Applied Logic, (1999), pp. 171–195.

[35] D. A. Martin, *Borel determinacy*, Annals of Mathematics, 102 (1975), pp. 363–371.

[36] I. Martínez-Ruiz, *Hiperespacios de espacios topológicos y pseudocompacidad*, master's thesis, Universidad Michoacana de San Nicolás de Hidalgo, 2006.

[37] K. Mazur, F_σ*-ideals and* $\omega_1\omega_1^*$*-gaps in the boolean algebras* $\mathcal{P}(\omega)/\mathrm{I}$., Fundamenta Mathematicae, 138 (1991), pp. 103–111.

[38] ——, *A modification of Louveau and Velickovic's construction for* F_σ*-ideals*, Proceedings of the American Mathematical Society, 128 (2000), pp. 1475–1479.

[39] D. Meza-Alcántara, *Ideals and Filters on countable sets*, PhD thesis, Universidad Nacional Autónoma de México, 2009.

[40] E. Michael, *Topologies on spaces of subsets*, Transactions of the American Mathematical Society, 71 (1951), pp. 152–182.

[41] S. Solecki, *Analytic ideals*, Bulletin of Symbolic Logic, 2 (1996), pp. 339–348.

[42] ——, *Analytic ideals and their applications*, Annals of Pure and Applied Logic, 99 (1999), pp. 51–72.

[43] ——, G_δ *ideals of compact sets*, Zb. Rad. (Beogr.), 4 (2011).

[44] S. Solecki and S. Todorčević, *Cofinal types of topological directed orders*, Annales de l'institut Fourier, 54 (2004), pp. 1877–1911.

[45] M. Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Mathematica, 67 (1980), pp. 13–43.

[46] S. Todorčević, *Directed sets and cofinal types*, Transactions of the American Mathematical Society, 290 (1985), pp. 711–723.

[47] ——, *Analytic gaps*, Fundamenta Mathematicae, 150 (1996).

[48] ——, *Definable ideals and gaps in their quotients*, in Set theory (Curaçao, 1995; Barcelona, 1996), Kluwer Acad. Publ., Dordrecht, 1998, pp. 213–226.

[49] S. Todorčević and C. Uzcátegui-Aylwin, *Analytic k-spaces*, Topology and its Applications, 146-147 (2005), pp. 511–526.

[50] J. W. Tukey, *Convergence and Uniformity in Topology*, no. 2 in Annals of Mathematics Studies, Princeton University Press, Princeton, N. J., 1940.

[51] C. Uzcátegui-Aylwin, *Ideals on countable sets: a survey with questions*, Revista Integración, 37 (2019), pp. 167 – 198.