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ESPACIOS TOPOLÓGICOS DE RAMSEY E IDEALES BORELIANOS

T E S I S

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Presenta:

SONIA NAVARRO FLORES

Asesor:
Dr. Michael Hrušák
Centro de Ciencias Matemáticas, UNAM

Coasesor:
Dr. Natasha Dobrinen
University of Denver

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A Emiliano, quien transformó el miedo en fuerza

Contents

Resumen

La teoría de Ramsey estudia las condiciones que se deben cumplir para que una estructura dada tenga subestructuras homogéneas. Dentro de esta teoría se han buscado teoremas tipo Ramsey para estructuras muy diversas y también para coloraciones de diferentes dimensiones incluida la dimensión infinita. El teorema de Ellentuck es el enunciado que establece las condiciones óptimas para tener un teorema de Ramsey de dimensión infinita y los espacios de Ramsey son una una generalización del espacio de Ellentuck. Una de las razones por las que estudiamos los espacios de Ramsey es porque muchos órdenes parciales σ-cerrados son equivalentes como forcing a algún espacio topológico de Ramsey. Además, en años recientes la teoria de espacios de Ramsey ha demostrado ser crucial para investigar las propiedades combinatorias de esos órdenes parciales y de los ultrafiltros que fuerzan.

En este trabajo investigamos los grados Ramsey de diferentes espacios topológicos de Ramsey y como consecuencia calculamos los respectivos grados de los ultrafiltros genéricos. Parte de esta investigación consistió en usar la versión abstracta del teorema de Nash-Williams para desarrollar un método general que sirve para calcular grados Ramsey. Algunos de los grados que calculamos ya se conocían pero con nuestro método obtuvimos pruebas más directas y simples que las originales.

Se sabe que existe una relación fuerte entre los espacios de Ramsey y los ultrafiltros selectivos. Esta relación se ha explorado de diferentes formas y en este trabajo probamos que cada espacio de Ramsey que satisface los axiomas definidos por Todorčević fuerza un ultrafiltro selectivo. Otro objetivo de este trabajo es clasificar los ideales Borelianos que tienen la propiedad de que la colección de conjuntos positivos es equivalente como forcing a un espacio topológico de Ramsey. Esto nos permitirá usar herramientas de la teoría de espacios de Ramsey para investigar las propiedades combinatorias de los ideales.

Finalmente, también investigamos los números de pseudointersección y torre de diversas clases de espacios de Ramsey. Descubrimos que estos números dependen de la libertad con la que se pueden extender los segmentos iniciales de los elementos de cada espacio. Para los espacios donde hay mucha libertad de extensión los números de pseudointersección y de torre coinciden con $\mathfrak{p}$. Por otro lado, para los espacios donde hay más restricciones para extender a los segmentos iniciales los números de pseudointersección y de torre son iguales a ω_1.

Palabras Clave: espacios topológicos, teoría de Ramsey, ideales Borel, ultrafiltros, grados Ramsey.

Abstract

Ramsey theory studies the conditions that must be imposed to a structure in order to guarantee the existence of homogeneous substructures. To do that, several structures and several colorings have been studied, including infinite dimensional colorings. Ellentuck's theorem states the optimal conditions to guarantee an infinite dimensional Ramsey theorem and Ramsey spaces are a generalization of the Ellentuck space. We are interested in studying Ramsey spaces because several σ-closed partial orders are forcing equivalent to a topological Ramsey space. In addition, in recent years Ramsey space theory has shown to be crucial to investigate combinatorial properties of those partial orders and the ultrafilters forced by them.

In this work we investigate Ramsey degrees for several classes of topological Ramsey spaces. As result of this we calculate Ramsey degrees for several ultrafilters. By using an abstract version of the Nash-Williams theorem we develop a general method to calculate Ramsey degrees. Some of the degrees we calculated were already known but our method provide streamlined and direct proofs.

It is known that Ramsey spaces and selective ultrafilters are closely related. This connection has been explored in different ways. In this thesis we prove that topological Ramsey space satisfyng Todorčević axioms forces a selective ultrafilter. Another objective of this work is to classify the Borel ideals with the property that the collection of positive sets is forcing equivalent to a Ramsey space. By doing this we can use Ramsey theory techniques to investigate combinatorial properties of those ideals. Finally, we investigate pseudointersection and tower numbers for several classes of topological Ramsey spaces and their relationship with the classical pseudointersection and tower numbers. We discover that these numbers are related to the freedom to extend initial segments in every space. For the spaces for which there are freedom to extend initial segments the pseudointersection and tower numbers are equal to $\mathfrak{p}$. On the other hand, for spaces such that there are several restrictions to extend initial segments the pseudointersection and tower numbers are equal to ω_1.

Keywords: topological spaces, Ramsey theory, Borel ideals, Ultrafilters, Ramsey degrees.

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Introducción

0.1 Teoría de Ramsey y el espacio de Ellentuck

El principio de las casillas infinito dice que cualquier coloración finita sobre un conjunto infinito debe contener un subconjunto infinito monocromático. La teoría de Ramsey se ocupa de estudiar las condiciones necesarias para generalizar el principio de casillas a mayores dimensiones. Uno de los primeros resultados de esta área es es el Teorema de Schur [67]:

Teorema 0.1 (Schur,1916)**.** *Si coloreamos los enteros positivos con una cantidad finita de colores, existen tres enteros distintos x, y, z tales que $x + y = z$.*

Schur intuyó que había una idea más profunda detrás de este teorema y conjeturó que hay un resultado similar para progresiones aritméticas. Eventualmente, Van Der Waerden [73] probó esta conjetura. Una *progresión aritmética* de longitud k es un conjunto finito de la forma:

$$\{n, n + d, n + 2d, \dots, n + (k - 1)d\}$$

donde d es un entero positivo.

Teorema 0.2 (Van Der Waerden,1927)**.** *Para cualquier partición finita de $\mathbb{N}$ existe una pieza que contiene progresiones aritméticas arbitrariamente grandes.*

Este teorema fue el cimiento de muchos resultados que hoy conocemos como como teoremas tipo Ramsey. Para leer una prueba corta del teorema de Van Der Waerden puede ver [38]. Un par de años después Ramsey probó su teorema mientras trabajaba en encontrar un procedimiento regular que determine la veracidad o falsedad de una fórmula lógica de primer orden [65].

Teorema 0.3 (Ramsey,1929)**.** *Para cada entero positivo n y para cada coloración finita de $[\mathbb{N}]^n$, existe un conjunto infinito $M \subseteq \mathbb{N}$ tal que el conjunto $[M]^n$ es monocromático.*

Una consecuencia del Teorema de Ramsey es el Teorema de Ramsey finito, el cual ha sido sumamente prolífico en combinatoria finita.

Teorema 0.4 (Teorema de Ramsey finito)**.** *Para cada par de números naturales $n \geq m$ existe $N \in \omega$ tal que para cada coloración $c : [N]^m \to r$ existe un conjunto $F \in [N]^n$ que es monocromático.*

En seguida presentamos otro teorema tipo Ramsey clásico:

Teorema 0.5 (Hindman,1972)**.** *Si partimos* $\mathbb{N}$ *en una cantidad finita de piezas entonces una de las piezas* A *contiene un conjunto infinito* H *tal que si* $a_1, \ldots, a_n$ *son elementos distintos de* H *entonces* $a_1 + \cdots + a_n \in A$*.*

Generalizando teoremas tipo Ramsey surgió una nueva área combinatoria, la teoría de Ramsey. La teoría de Ramsey finita y la teoría de Ramsey de dimensiones altas han sido muy fructíferas. Buscando un teorema de Ramsey de dimensión infinita, se descubrió que es necesario pedir condiciones extra a las coloraciones si queremos garantizar la existencia de conjuntos monocromáticos infinitos.

La siguiente coloración nos muestra que la generalización intuitiva del Teorema de Ramsey a dimensión infinita es falsa. Tomamos los siguientes ejemplos de [71].

Ejemplo 0.1.1 (Baumgartner)**.** Dado $\mathcal{U}$ un ultrafiltro no principal sobre ω, definimos la coloración c de $[\omega]^\omega$ en $\{-1, 1\}$ como sigue:

$$c(M) = \lim_{n \to \mathcal{U}} (-1)^{|M \cap \{0, \ldots, n-1\}|}.$$

Note que para cada $X \in [\omega]^\omega$, $c(X) \neq c(X \setminus \{\text{mín}\, X\})$. En particular, c no puede ser constante en algún $[M]^\omega$ donde M es infinito. Note que la coloración c no es Lebesgue ni Baire medible. En este punto, podemos preguntarnos si añadiendo alguna propiedad a la coloración como Lebesgue o Baire medible podremos garantizar la existencia de conjuntos monocromáticos.

Ejemplo 0.1.2. Ahora definimos la coloración $b : [\omega]^\omega \to \{-1, 1\}$ como sigue:

$$b(M) = \text{mín}\{c(M), \text{mín}_{m,n \in M} (-1)^{|m-n|}\}.$$

Note que $\mathcal{O} = \{M \in [\omega]^\omega : \text{mín}_{m,n \in M}(-1)^{|m-n|} = -1\}$ es un subconjunto abierto y denso de $[\omega]^\omega$ y que la coloración es Lebesgue y Baire medible. También note que para cada $N \in [\omega]^\omega$ existe un conjunto infinito $M \subseteq N$ tal que $[M]^\omega \cap \mathcal{O} = \emptyset$. Por lo tanto esta coloración no satisface la versión de dimensión infinita del teorema de Ramsey.

El proceso que llevó al teorema de Ramsey infinito fue iniciado por Nash-Williams [63] cuando estaba desarrollando su teoría de better-quasi-ordered sets . Galvin-Prikry [34] generalizó el teorema de Nash-Williams, él probó que las coloraciones Borel satisfacen el Teorema de Ramsey de dimensión infinita.

Teorema 0.6 (Galvin-Prikry,1973)**.** *Sea* $k > 0$ *y* $[\omega]^\omega = \bigcup_{i<k} P_i$ *una partición de donde cada* P_i *es un conjunto Borel. Entonces existen un conjunto infinito* $H \subseteq \omega$ *y un índice* $i < k$ *tal que* $[H]^\omega \subseteq P_i$*.*

En [68], Silver mejoró el resultado de Galvin-Prikry cuando probó que las coloraciones analíticas satisfacen el teorema de Ramsey de dimensión infinita.

Teorema 0.7 (Silver,1970)**.** *Para cada subconjunto analítico* P *de* $[\omega]^\omega$ *existe un conjunto infitito* $H \subseteq \omega$ *tal que* $[H]^\omega \subseteq P$ *o* $[H]^\omega \cap P = \emptyset$*.*

Ellentuck fue el primero en introducir nociones de topología en la teoría de Ramsey de dimensión infinita y fue él quien obtuvo el resultado óptimo. Para cada $a \in [\omega]^{<\omega}$ y $B \in [\omega]^{\omega}$ definimos el conjunto básico:

$$[a, B] = \{A \in [\omega]^{\omega} : a \sqsubset A \text{ y } A \subseteq B\}.$$

Estos conjuntos generan una topología sobre $[\omega]^{\omega}$. El espacio $[\omega]^{\omega}$ con esta topología es llamado el espacio de Ellentuck. Ellentuck [31] estudió este espacio cuando buscaba otra prueba para el Teorema de Silver, esto debido a que Silver usa ideas muy sofisticadas en su prueba. El Teorema de Ellentuck establece que las coloraciones sobre $[\omega]^{\omega}$ para las cuales siempre existen conjuntos monocromáticos son exactamente las coloraciones Baire medibles con la topología de Ellentuck.

Teorema 0.8 (Ellentuck,1974)**.** *Sea $\mathcal{X} \subseteq [\omega]^{\omega}$. $\mathcal{X}$ es Ramsey si y solo si $\mathcal{X}$ tiene la Propiedad de Baire en la topología de Ellentuck.*

0.2 Espacios topológicos de Ramsey

En [18], Carlson y Simpson presentan el teorema de Ramsey dual. Mientras la teoría de Ramsey estudia coloraciones sobre k-tuplas de números naturales, la teoría Ramsey dual estudia coloraciones de particiones de ω de tamaño k. En el artículo [18] ellos prueban que muchos teoremas combinatorios bien conocidos se pueden deducir del teorema de Ramsey dual. Aún más, ellos probaron que muchos teoremas que se satisfacen para el espacio de Ellentuck también son verdaderos para el espacio dual. Inspirado en este trabajo, en [16] Carlson prueba que cierta clase de estructuras satisfacen las propiedades combinatorias del espacio de Ellentuck. En este trabajo [16], Carlson enuncia la primera definición de espacios de Ramsey. Él define unas estructuras que consisten de sucesiones infinitas llamadas “variable words”. El resultado principal del artículo es que las estructuras que satisfacen tres axiomas adicionales cumplen la conclusión del teorema de Ellentuck. Todorčević, trabajando sobre el trabajo previo de Carlson y Simpson, en [71] define cuatro axiomas ($\mathbf{A}$,$\mathbf{1}-\mathbf{A}$,$\mathbf{4}$) que garantizan la versión abstracta del teorema de Ellentuck. Todorčević define sus axiomas para estructuras abstractas mucho más generales que las estudiadas por Carlson en [16].

Debido a la fuerza de la teoría de las estructuras Ramsey en estos espacios, en los últimos años han surgido muchos espacios de Ramsey con diferentes propósitos. El primer espacio de Ramsey contruido después del espacio de Ellentuck fue el espacio de Milliken $\mathrm{FIN}^{[\infty]}$ [61]. Este espacio es famoso debido a las aplicaciones de la teoría de Ramsey en bloques de Gowers. Él desarrolló esta teoría cuando investigaba algunos problemas de la geometría de espacios de Banach (vea [36] y [37]). Esta teoría tiene potencial para ser aplicada a otras áreas. Por ejemplo, sirvió para probar que la esfera del espacio de Banach c_0 es de oscilación estable.

En años recientes, la teoría de espacios topológicos de Ramsey ha proveído de diversas herramientas para la investigación de forcings que generan ultrafiltros, los cuales son difíciles de estudiar desde el forcing original. Esta tendencia inicia con el trabajo [29], donde Dobrinen y Todorčević construyen un espacio de Ramsey que es denso en el forcing que Laflamme define en [50], el cual produce un ultrafiltro denotado $\mathcal{U}_1$ que es débilmente Ramsey pero no Ramsey. Ellos construyen este espacio para investigar cómo es la estructura exacta de los órdenes Rudin-Keisler y Tukey debajo de $\mathcal{U}_1$. Esta idea fue extendida en [30], [26], [22], y [23]. En estos trabajos se construyen espacios topológicos de Ramsey los cuales son densos en forcings, algunos forcings son conocidos como los definidos por Baumgartner-Taylor, Blass, Laflamme, y Szymański-Zhou, y también hay otros forcings que son nuevos y producen ultrafiltros con propiedades combinatorias interesantes. El lector puede ver [24] para una vista general del área.

Una vez construidos, resulta que la estructura de los espacios topológicos de Ramsey contenidos en los forcings es muy útil para investigar qué tan Ramsey son los ultrafiltros genéricos para estos forcings. Otro ejemplo de esta investigación es el trabajo de Dobrinen y Hathaway en [25], donde ellos prueban que los ultrafiltros mencionados arriba tienen propiedades similares a las de los ultrafiltros Ramsey en el sentido de las extensiones de barren de Henle, Mathias y Woodin en [39]. En esta Tesis nosotros estudiamos propiedades de los ultrafiltros estudiados en [25], lo hacemos utilizando técnicas de espacios topológicos de Ramsey.

0.3 Espacios de Ramsey, ideales y ultrafiltros

Es bien sabido que el álgebra Booleana $\mathcal{P}(\omega)/\mathrm{Fin}$, la cual es forcing equivalente al espacio de Ellentuck, forza un ultrafiltro de Ramsey. Esta conexión entre espacios de Ramsey, ultrafiltros con propiedades de partición interesantes e ideales, ha sido explorada en diferentes formas.

En [21], Todorčević prueba que la conexión entre el espacio de Ellentuck y los ultrafiltros Ramsey es más fuerte de lo que se pensaba cuando probó que asumiendo hipótesis adicionales de cardinales grandes se cumple que cada ultrafiltro Ramsey es genérico para $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq)$ sobre $L(\mathbb{R})$. En [19], DiPrisco, Mijares y Nieto extienden la noción de ultrafiltros semiselectivos a ultrafiltros sobre espacios de Ramsey. Más aún, en ese trabajo ellos generalizan el resultado de Todorčević. Ellos prueban que para cada espacio topológico de Ramsey $\mathcal{R}$, bajo hipótesis adecuadas de cardinales grandes cada ultrafiltro semiselectivo $\mathcal{U} \subset \mathcal{R}$ es genérico para $(\mathcal{R}, \leq^*)$ sobre $L(\mathbb{R})$. Otro ejemplo de la conexión entre espacios de Ramsey y ultrafiltros con propiedades combinatorias son los ultrafiltros stable ordered union asociados al espacio de Milliken. Estos espacios fueron introducidos por Blass y Hindman en [13]. En [33], Fernandez y Hrušák prueban la existencia de estos ultrafiltros en el modelo de Sacks usando un diamante débil. Existen muchos ultrafiltros que aunque no son Ramsey, para cada natural $n \geq 2$ existe t_n tal que para cualquier coloración de las n-tuplas de ω en una cantidad finita de colores existe un elemento de ultrafiltro cuyas n-tuplas usan a lo más t_n colores. Los t_n mínimos con la propiedad mencionada son lo que llamamos grados Ramsey.

En la literatura hay muchos ejemplos de ultrafiltros con grados Ramsey finitos han sido forzados por órdenes parciales σ-cerrados. Los siguientes ejemplos muestran la diversidad de estos órdenes parciales. En [50], Laflamme fuerza una jerarquía de p-puntos rápidos Rudin-Keisler arriba de un ultrafiltro débilmente Ramsey. En [6], Baumgartner y Taylor forzan para cada $k \geq 2$, un ultrafiltro que es k-arrow, pero no $(k+1)$-arrow. Estos ultrafiltros son p-puntos rápidos. En [8], Blass construyó un p-punto que tiene dos p-puntos Rudin-Keisler debajo de él que son Rudin-Keisler incomparables. El forcing $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$ fue investigado por Szymański y Zhou en [70]. Ellos prueban que este forcing produce un ultrafiltro, al que denotan por $\mathcal{G}_2$, el cual no es un p-punto y tiene grado Ramsey $t(\mathcal{G}_2, 2) = 4$. En [12], Blass, Dobrinen y Raghavan prueban que $\mathcal{G}_2$ es un p-punto débil e investigan su tipo Tukey. Los órdenes parciales $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ para $2 \leq k < \omega$ han sido investigados por Kurilić en [49] y por Dobrinen en [22]. De hecho, cada forcing $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes \alpha}$, donde α es un ordinal numerable, es forcing equivalente a un espacio topológico de Ramsey (vea [23]). Para estos y otros ultrafiltros forzados por espacios topológicos de Ramsey hemos encontrado un método para calcular sus grados Ramsey. Además, en el Teorema 2.1 probamos que cada espacio de Ramsey que satisface los axiomas de Todorčević fuerza un ultrafiltro de Ramsey, lo cual era una conjetura de Dobrinen y también probamos que tal ultrafiltro está Rudin-Keisler debajo del ultrafiltro genérico.

Otro aspecto que queremos investigar de la relación mencionada al inicio de la sección, es la conexión entre ideales y espacios de Ramsey. Nos interesa clasificar los ideales en aquellos que están relacionados con un espacio topológico de Ramsey de la misma forma que el ideal Fin está relacionado con el espacio de Ellentuck. A estos ideales los llamamos ideales TRS. Para leer sobre propiedades combinatorias de ideales el lector puede ver [**?**] y [43]. Para los ideales TRS, la teoría de espacios de Ramsey nos proporciona métodos simples y directos para investigar sus propiedades combinatorias. Esta es la razón por la cual estamos interesados en clasificar los ideales que satisfacen esta propiedad. Para los ideales Fin^α, Dobrinen probó que forzar con la colección de los positivos es equivalente a forzar con los high dimensional Ellentuck spaces [22]. En esta tesis clasificamos los ideales $\mathcal{ED}_{fin}$, $\mathcal{ED}$, conv, los ideales sumables y también los ideales producidos por clases de Fraïssé. En [46], Hrušák estudia el orden de Katětov sobre los ideales Borel y pregunta cuántos ideales altos, F_σ y K-uniformes existen. Usando los grados Ramsey que calculamos en el Capítulo 2 probamos que los ideales generados por la clase de Fraïssé de los órdenes lineales finitos forma una cadena numerable en el orden de Katětov de ideales altos, F_σ y K-uniformes. Cuando empezamos a estudiar estos ideales nos preguntamos si para cada ideal TRS $\mathcal{I}$, el cociente $\mathcal{P}(\omega)/\mathcal{I}$ es siempre σ-cerrado, en la Subsección 3.1.1 mostramos la existencia de un ideal TRS para el cual su cociente no es σ-cerrado. Aunque este ideal no es σ-cerrado, sí es localmente σ-cerrado por lo que no agrega reales. Cuando empezamos a estudiar los grados Ramsey de los ultrafiltros forzados por espacios de Ramsey pensamos que los grados Ramsey para los espacios de Ramsey, los ideales y los ultrafiltros genéricos coinciden. En la Subsección 3.1.3 probamos que eso no se cumple para el ideal conv y el espacio de Ramsey asociado a él.

0.4 La propiedad Halpern-Läuchli y el forcing de Sacks

El Teorema de Halpern-Läuchli es otro teorema tipo Ramsey que estudia coloraciones sobre productos de árboles. Este teorema sirvió para dar un modelo de la teoría de conjuntos en el cual el axioma de elección es falso pero el Teorema del ideal primo Booleano sí se satisface. El siguiente Teorema es la versión para árboles perfectos del Teorema de Halpern-Läuchli. Si T es un árbol, $T(n)$ es el n-ésimo nivel del árbol.

Teorema 0.9 (Halpern-Läuchli)**.** *Para cada $k \in \omega$, para cada sucesión finita de árboles perfectos $\{T_i : i < k\}$ y para cada partición finita del producto $\prod_{i<k} T_i$, existe un conjunto infinito $H \subseteq \omega$ y subárboles perfectos S_i de T_i tales que $\cup_{n\in H}(\prod_{i<k} S_i(n)))$ se queda contenido en una pieza.*

El teorema de Halpern-Läuchli está relacionado con el forcing de Sacks. Sabemos que cuando forzamos algunas propiedades de los objetos del modelo base se pueden perder. Baumgartner y Laver [5] y Laver [51] probaron que el forcing de Sacks, el producto del forcing de Sacks y la iteración del forcing de Sacks preservan ultrafiltros selectivos. Es natural preguntarse si esto ocurre para los ultrafiltros genéricos para los espacios topológicos de Ramsey. Yuan Yuan Zheng abordó esta pregunta para múltiples espacios de Ramsey. En [75], Zheng prueba que los ultrafiltros selectivos para el espacio Milliken $\mathrm{FIN}^{[\infty]}$ coinciden con los ultrafiltros union stable ordered y son preservados por el side-by-side Sacks forcing. En [77], Zheng prueba que los ultrafiltros selectivos para la jerarquía de espacios $\mathcal{R}_\alpha$ ($\alpha < \omega_1$) se preservan bajo side-by-side Sacks forcing. Siguiendo esta línea de trabajo, nosotros estamos interesados en saber si los ultrafiltros genéricos para ideales TRS son preservados por el forcing de Sacks. En el Teorema 3.48 probamos que para una clase de ideales TRS el ultrafiltro genérico se preserva por el forcing de Sacks.

0.5 Números de pseudointersección y torre

Los cardinales invariantes del continuo han sido decisivos para entender la estructura combinatoria de $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq)$. Muchas estructuras tienen sus propias nociones de estos cardinales invariantes y recientemente diferentes investigadores han trabajado en calcular los invariantes de estructuras similares a $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq)$. En [52], Majcher-Iwanow estudian los cardinales invariantes de la lattice de particiones sobre ω. En [14], Brendle estudia el diagrama de van Douwen relacionado a la estructura $\mathrm{Dense}(\mathbb{Q})/\mathsf{nwd}$. En [2], Balcar, Hernández-Hernández y Hrušák investigan los cardinales invariantes de la estructura $(\mathrm{Dense}\mathbb{Q}, \subseteq)$.

Por muchos años estuvo abierta la pregunta de si los cardinales $\mathfrak{p}$ y $\mathfrak{t}$ son el mismo cardinal o son diferentes cardinales. Ahora, gracias al trabajo de Malliaris y Shelah sabemos que $\mathfrak{p} = \mathfrak{t}$ (vea [53], [54]). En este trabajo investigamos los números de pseudointersección y de torre para los espacios topológicos de Ramsey, nos interesa saber si son iguales o diferentes y cuál es su relación con los cardinales originales $\mathfrak{p}$ y $\mathfrak{t}$. Pensamos los elementos de los espacios de Ramsey como sucesiones infinitas. Los segmentos iniciales de las suceciones se llaman aproximaciones finitas. Para estudiar los cardinales, dividimos los espacios de Ramsey en dos grupos de acuerdo a su comportamiento. Un grupo consiste de los espacios que satisfacen la propiedad Independent Sequences of Structures. Estos espacios consisten de sucesiones de bloques crecientes y tienen la propiedad de que el segmento inicial de una sucesión no impone restricciones sobre el comportamiento de los siguientes bloques. Para estos espacios tenemos bastante libertad para extender las aproximaciones finitas. Algunos espacios que pertenecen a este grupo son los espacios $\mathcal{R}_\alpha$ y los espacios generados por clases de Fraïssé. En el Teorema 4.10 probamos que para los espacios que satisfacen la propiedad Independent Sequences of Structures se satisface que los números de pseudointersección y torre coinciden y además también coinciden con los cardinales clásicos, es decir,

$$\mathfrak{p}_{\mathcal{R}} = \mathfrak{t}_{\mathcal{R}} = \mathfrak{p}.$$

El otro grupo consiste de aquellos espacios para los cuales los segmentos iniciales de las sucesiones determinan el comportamiento de los bloques subsecuentes. En este caso debemos ser más cuidadosos para extender aproximaciones finitas. Algunos ejemplos de estos espacios son los high dimensional Ellentuck spaces $\mathcal{E}_\alpha$, el espacio de Carlson-Simpson $\mathcal{E}_\infty$ y el espacio de Milliken de los strong trees $\mathcal{S}_\infty(U)$. Para los espacios pertenecientes a este grupo no tenemos un método general para investigar sus cardinales asociados por lo que tenemos que investigar cada espacio por separado. Para el espacio de los strong trees probamos que la relación de almost reduction $\leq^*$ definida por Mijares no es transitiva, por lo que debemos buscar otro orden separativo para $\mathcal{S}_\infty(U)$. Para el espacio de Ellentuck de 2 dimensiones $\mathcal{E}_2$, se sigue de un resultado de Szymański y Zhou [70] que $\mathfrak{p}_{\mathcal{E}_2} = \mathfrak{t}_{\mathcal{E}_2} = \omega_1$. En el Teorema 4.17, extendemos este resultado a los high dimensional Ellentuck spaces. Para el espacio de Carlson-Simpson $\mathcal{E}_\infty$ se sigue de un resultado de Carlson en [55] que $\mathfrak{p}_{\mathcal{E}_\infty} = \mathfrak{t}_{\mathcal{E}_\infty} = \omega_1$. Con la evidencia anterior conjeturamos que para los espacios pertenecientes al segundo grupo los números de pseudointersección y torre son exactamente ω_1.

0.6 Contenido

Esta Tesis consiste de cuatro capítulos. En el Capítulo 1 presentamos los antecedentes necesarios para leer este trabajo. Introducimos la teoría de los espacios topológicos de Ramsey con múltiples ejemplos, también introducimos la teoría de filtros e ideales, el forcing de Sacks y los cardinales $\mathfrak{p}$ y $\mathfrak{t}$.

En el Capítulo 2 investigamos las propiedades combinatorias de ultrafiltros forzados por espacios topológicos de Ramsey. En la Sección 2.1 probamos que cada espacio topológico de Ramsey que satisface los axiomas de Todorčević fuerza un ultrafiltro Ramsey. En la Sección 2.2 presentamos un método general para calcular los grados Ramsey de espacios topológicos de Ramsey con propiedades adicionales. Además, calculamos los grados Ramsey para diferentes espacios topológicos de Ramsey y como consecuencia obtenemos los grados Ramsey para los ultrafiltros forzados por ellos.

En el Capítulo 3 estudiamos a los ideales TRS. En este capítulo probamos que los ideales $\mathcal{ED}$, $\mathcal{ED}_{fin}$, los ideales generados por clases de Fraïsséy el ideal conv son ideales TRS. También probamos que los ideales sumables no son ideales TRS. En la Sección 3.2 presentamos la propiedad HL de ideales. Usamos la propiedad HL para probar que los ideales TRS que tienen cierta independencia en los segmentos iniciales de sus elementos tienen la propiedad de que el ultrafiltro genérico para el cociente se preserva por el forcing de Sacks.

En el último capítulo calculamos los números de pseudointersección y torre espacios topológicos de Ramsey. Demostramos que para los espacios topológicos de Ramsey cuyos elementos consisten de sucesiones infinitas de bloques crecientes que son independientes entre ellos los números de pseudointersección y de torre son iguales a $\mathfrak{p}$. También probamos que para muchos espacios topológicos de Ramsey para los cuales los segmentos iniciales de sus elementos son dependientes, los números de pseudointersección y de torre son iguales a ω_1. La mayor parte del material de los capítulos 2 y 4 son parte del artículo [27].

Introduction

0.7 Ramsey Theory and the Ellentuck space

The infinite pigeonhole principle states that a finite coloring of an infinite set must contain an infinite monochromatic set. Ramsey theory is concerned with studying conditions to generalize this principle to richer structures. One of the earliest result of this combinatorial area is the the Schur's theorem [67]:

Theorem 0.1 (Schur,1916)**.** *If the positive integers are finitely coloured,then there are three distinct integers x, y, z of the same color such that $x + y = z$.*

Schur noticed that there was a striking idea behind this theorem and conjectured that there is a similar result for arithmetic progressions. Eventually, Van Der Waerden [73] proved that conjecture. An *arithmetic progression* of length k is a finite set of the form

$$\{n, n + d, n + 2d, \dots, n + (k - 1)d\}$$

where d is a positive integer.

Theorem 0.2 (Van Der Waerden,1927)**.** *If $\mathbb{N}$ is partitioned into finitely many pieces then one of the pieces contains arbitrarily long arithmetic progressions.*

This result was the seed for several theorems that today are known as Ramsey-type theorems. For a short proof of Van Der Waerden's theorem see [38]. A couple of years later Ramsey proved his theorem when he was working on the problem of finding a regular procedure to determine the truth or falsity of a given logical formula in the language of first-order logic [65].

Theorem 0.3 (Ramsey,1929)**.** *For every positive integer n and for every finite coloring of the family $[\mathbb{N}]^n$, there is an infinite set $M \subseteq \mathbb{N}$ such that the set $[M]^n$ is monochromatic.*

A consequence of Ramsey's theorem is its finite version, which has been very prolific in finite combinatorics.

Theorem 0.4 (Finite Ramsey Theorem)**.** *For every pair of natural numbers $n \geq m$ there exists some $N \in \omega$ such that for every coloring $c : [N]^m \to r$ there exists $F \in [N]^n$ which is monochromatic.*

Now we present another classic Ramsey-type theorem:

Theorem 0.5 (Hindman,1972)**.** *If $\mathbb{N}$ is partitioned into finitely many pieces then one of the pieces A contains an infinite set H such that $a_1+\cdots+a_n \in A$ whenever $a_1, \ldots, a_n$ are distinct members of H.*

By generalizing Ramsey-type theorems, a new combinatorial area arose. Finite Ramsey theory and high dimensional Ramsey theory were very prolific. When looking for an infinite dimensional Ramsey theorem, it was found that it is necessary to ask extra conditions on the colorings to guarantee the existence of infinite monochromatic sets.

The following colorings show that the infinite dimensional Ramsey theorem is not always true. These examples are taken from [71].

Example 0.6 (Baumgartner)**.** Given a non-principal ultrafilter $\mathcal{U}$ on ω, define the coloring c from $[\omega]^\omega$ into $\{-1,1\}$ as follows:

$$c(M) = \lim_{n\to\mathcal{U}} (-1)^{|M\cap\{0,\ldots,n-1\}|}.$$

Note that for every $X \in [\omega]^\omega$, $c(X) \neq c(X \setminus \{\min X\})$. In particular, c can not be constant in some set $[M]^\omega$ where M is infinite.
The coloring c is neither Lebesgue measurable nor Baire measurable. We could believe that by asking some desirable property to the coloring, like being Lebesgue or Baire measurable, will guarantee the existence of monochromatic sets.

Example 0.7. Now define the coloring $b : [\omega]^\omega \to \{-1,1\}$ as follows:

$$b(M) = \min\{c(M), \min_{m,n\in M} (-1)^{|m-n|}\}.$$

Note that the set $\mathcal{O} = \{M \in [\omega]^\omega : \min_{m,n\in M}(-1)^{|m-n|} = -1\}$ is an open and dense subset of $[\omega]^\omega$ and the coloring is Lebesgue and Baire measurable. Also note that for every $N \in [\omega]^\omega$ there is some infinite set $M \subseteq N$ such that $[M]^\omega \cap \mathcal{O} = \emptyset$. So $b \restriction [M]^\omega = c \restriction [M]^\omega$. Therefore this coloring does not satisfy the infinite dimensional Ramsey theorem.

The infinite-dimensional Ramsey theory was initiated by Nash-Williams [63] when he developed his theory of better-quasi-ordered sets. Galvin-Prikry [34] generalized the Nash-Williams theorem by proving that Borel colorings satisfy the infinite dimensional Ramsey theorem.

Theorem 0.8 (Galvin-Prikry,1973)**.** *Let $k > 0$ be a natural number and $[\omega]^\omega = \bigcup_{i<k} P_i$ be a partition where each P_i is Borel. Then there is an infinite $H \subseteq \omega$ and $i < k$ such that $[H]^\omega \subseteq P_i$.*

In [68], Silver improved Galvin-Prikry's result by proving that analytic colorings satisfy the infinite dimensional Ramsey theorem.

Theorem 0.9 (Silver,1970)**.** *For every analytic subset $P \subseteq [\omega]^\omega$ there is an infinite set $H \subseteq \omega$ such that $[H]^\omega \subseteq P$ or $[H]^\omega \cap P = \emptyset$.*

Ellentuck was the first to introduce topological notions in infinite dimensional Ramsey theory and he obtained the optimal result. For every $a \in [\omega]^{<\omega}$ and $B \in [\omega]^{\omega}$ define the basic set

$$[a, B] = \{A \in [\omega]^{\omega} : a \sqsubset A \text{ and } A \subseteq B\}.$$

These sets generate a topology on $[\omega]^{\omega}$. The space $[\omega]^{\omega}$ with this topology is called the Ellentuck space. Ellentuck [31] studied this space to re-prove Silver's theorem because Silver's proof uses sophisticated mathematical tools. Ellentuck's theorem establishes that colorings on $[\omega]^{\omega}$ for which there are monochromatic sets are exactly the Baire measurable colorings for the Ellentuck's space.

Theorem 0.10 (Ellentuck,1974)**.** *Let $\mathcal{X} \subseteq [\omega]^{\omega}$. Then $\mathcal{X}$ is Ramsey if and only if $\mathcal{X}$ has the Baire Property in the Ellentuck topology.*

0.8 Ramsey spaces

In [18], Carlson and Simpson present the dual Ramsey theorem. While Ramsey theory cares about colorings on k-tuples of natural numbers, the dual Ramsey theory studies colorings of k-element partitions of ω. In this work they prove that many well known combinatorial theorems can be deduced from the dual Ramsey theorem. Moreover, they prove that several theorems that are true for the Ellentuck space are also true for the dual space. Inspired by this work, in [16], Carlson proves that some specific structures satisfy the combinatorial properties of the Ellentuck's space. In this work, Carlson states the first definition of Ramsey space. He defines some structures consisting of infinite sequences called variable words. The main result is that those structures that satisfy a generalization of Ellentuck's theorem. Todorčević , working on prior work of Carlson and Simpson, in [71] states four axioms ($\mathbf{A.1}-\mathbf{A.4}$) that guarantee an abstract version of Ellentuck's theorem. Todorčević's axioms hold for structures more general than those studied by Carlson in [16].

Because of the strength of the Ramsey structure in those spaces, in recent years several Ramsey spaces have emerged for different purposes. The first space built after Ellentuck's space is the Milliken space $\mathrm{FIN}^{[\infty]}$ [61]. This space is famous because of Gowers's successful applications of the "block Ramsey theory"when he investigated some problems of Banach space geometry (see [36] and [37]). This theory has been applicated to other areas. For example, this theory was used to prove that the sphere of the Banach space c_0 is oscillation stable.

In recent years, the use of topological Ramsey spaces to investigate forcings that generate interesting ultrafilters has provided methods for obtaining results that were difficult to find when simply using the original forcings. This began in [29], where Dobrinen and Todorcevic constructed a Ramsey space dense inside of the forcing of Laflamme in [50] which produces a weakly Ramsey, not Ramsey ultrafilter, denoted $\mathcal{U}_1$, in order to calculate the exact Rudin-Keisler and Tukey structures below this ultrafilter. This idea was extended in [30], [26], [22], and [23], providing new collections of topological Ramsey spaces dense in known forcings, such as those of Baumgartner-Taylor, Blass, Laflamme, and Szymański-Zhou mentioned above, as well as creating new forcings which produce ultrafilters with interesting partition relations. Such Ramsey spaces were used to find exact Rudin-Keisler and Tukey structures below those ultrafilters. An overview of this area can be found in [24].

Once constructed, it turned out that the topological Ramsey space structure of these forcings can be used to investigate the resemblance between these ultrafilters and Ramsey ultrafilters. Dobrinen and Hathaway show in [25] that each of the aforementioned ultrafilters has properties similar to those of a Ramsey ultrafilter in the sense of the barren extensions of Henle, Mathias, and Woodin in [39]. In this Thesis, we investigate properties of the ultrafilters investigated in [25] by using topological Ramsey space techniques to better handle the properties of the forcings.

0.9 Ramsey spaces, ideals and ultrafilters

It is known that the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$, which is forcing equivalent to the Ellentuck space, forces a Ramsey ultrafilter. This connection between Ramsey spaces, ultrafilters with interesting partition properties and ideals has been explored in several ways.

In [21], Todorčević shows that the connection between the Ellentuck space and Ramsey ultrafilters is stronger than expected by proving under large cardinal assumptions than every Ramsey ultrafilter is generic for $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq)$ over $L(\mathbb{R})$. In [19], DiPrisco, Mijares and Nieto extend the notion of semiselective ultrafilter to ultrafilters on Ramsey spaces. Moreover, they generalize Todorčević's result. They prove that for every topological Ramsey space $\mathcal{R}$, under suitable large cardinal hypotheses every semiselective ultrafilter $\mathcal{U} \subset \mathcal{R}$ is generic for $(\mathcal{R}, \leq^*)$ over $L(\mathbb{R})$.

Another example of the connection between Ramsey spaces and interesting ultrafilters is given by the stable ordered union ultrafilters associated to Milliken's space. These ultrafilters were introduced by Blass and Hindman in [13]. In [33], Fernandez y Hrušák prove that there exist these ultrafilters in the Sack's model by using a weak diamond.

There are several ultrafilters $\mathcal{U}$ satisfying that they are not Ramsey but for every natural number $n \geq 2$ there is some t_n such that for every coloring from the n-sized sets of ω into finitely many colors there exists a member of the ultrafilter such that every n-sized subset touches at most t_n colors. When such t_n are minimal with this property we call them Ramsey degrees. There are several examples of ultrafilters with finite Ramsey degrees forced by various σ-closed posets. We mention some examples. Laflamme in [50] forced a hierarchy of rapid p-points above a weakly Ramsey ultrafilter. Baumgartner and Taylor in [6] forced k-arrow, not $(k+1)$-arrow ultrafilters, for each $k \geq 2$. These are rapid p-points. In [8], Blass constructed a p-point which has two Rudin-Keisler incomparable p-points below it. The forcing $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$ was investigated by Szymański and Zhou in [70] and shown to produce an ultrafilter, denoted $\mathcal{G}_2$, which is not a p-point but has Ramsey degree $t(\mathcal{G}_2, 2) = 4$. This ultrafilter was shown to be a weak p-point in [12] and investigations of its Tukey type are included in that paper. Further extensions of $\mathcal{P}(\omega)/\mathrm{Fin}$ to finite dimensions $(\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ for $2 \leq k < \omega)$

were investigated by Kurilić in [49] and by Dobrinen in [22]. In fact, the natural hierarchy of forcings $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, for all countable ordinals α, was shown to be forcing equivalent to certain topological Ramsey spaces in [23]. These and other ultrafilters are forced by Ramsey spaces and we find a general method for calculating Ramsey degrees of ultrafilters from these classes. Furthermore, in Theorem 2.1 we prove that every topological Ramsey space satisfying Todorčević axioms forces an ultrafilter which is Rudin-Keisler below the generic ultrafilter. This fact was conjectured by Dobrinen previously.

This leads us to the other connection that will be covered in this study. We are interested in finding which ideals are related to a Ramsey space in the same sense that the Ellentuck space is related with the ideal Fin. We call these ideals TRS ideals. Combinatorial properties for ideals are currently being investigated. The reader interested in this topic is referred to [**?**] and [43]. For those ideals related to Ramsey spaces, the Ramsey space theory provides us with direct and streamlined methods to investigate their partition properties. This is why we are interested in classifying ideals according to this relation. For ideals Fin^α, Dobrinen proved that forcing with the positive sets is equivalent to forcing with the high dimensional Ellentuck spaces [22]. In this work we classify ideals $\mathcal{ED}_{fin}$, $\mathcal{ED}$, conv, summable ideals and all the ideals related to Fraïssé classes. In [46], Hrušák studies the Katětov order on Borel ideals and asks how many F_σ, K-uniform, tall ideals exist. By using the Ramsey degrees we prove that the ideals generated for the Fraïssé class of finite linear orders forms a countable chain of F_σ, K-uniform, tall ideals in the Katětov order. At the beginning we ponder whether for every TRS ideal the quotient is σ-closed. Later we prove that there is a TRS ideal for which the quotient is not σ-closed. Although this space is not σ-closed, it is locally σ-closed. Hence it does not add reals. At the beginning we also believe that Ramsey degrees for ideals are the same as Ramsey degrees for the generic ultrafilters of their quotients. We prove that this is not true for the ideal conv and the Ramsey space related to it.

0.10 The Halpern-Läuchli property and Sacks forcing

The Halpern-Läuchli theorem is a Ramsey type theorem. It was used to give a model of set theory in which the axiom of choice is false but the Boolean prime ideal is true. The following theorem is the perfect tree version of the Halpern-Läuchli theorem. If T is a tree, $T(n)$ is the n-th level of the tree.

Theorem 0.11. *Halpern-Läuchli For every $k \in \omega$ and for every finite sequence $\{T_i : i < k\}$ of perfect trees and every finite partition of their product $\prod_{i<k} T_i$, there are an infinite set $H \subseteq \omega$ and perfect subtrees S_i of T_i such that $\cup_{n\in H}(\prod_{i<k} S_i(n)))$ is contained in one piece.*

The Halpern-Läuchli theorem is closely related to the Sacks forcing. It is known that after forcing with a poset the properties of some objects in the ground model may change. Baumgartner and Laver [5] and Laver [51] proved that selective ultrafilters are preserved by Sacks forcing, product Sacks forcing and iterated Sacks forcing. It is natural to ask whether this occurs for ultrafilters which are generic for topological Ramsey spaces. Yuan Yuan Zheng explored this question for several Ramsey spaces. In [75], Zheng proves that selective ultrafilters on the Milliken space $\mathrm{FIN}^{[\infty]}$ are preserved by side-by-side Sacks forcing. In [77], Zheng proves that selective ultrafilters for a hierarchy of spaces $\mathcal{R}_\alpha$ ($\alpha < \omega_1$) are preserved under countable support side-by-side Sacks forcing. Following this line of work, we are interested in knowing if generic ultrafilters for TRS ideals are preserved by Sacks forcing. In Theorem 3.48 we prove that for a class of TRS ideals the generic ultrafilter is Sacks indestructible.

0.11 Pseudointersection and tower numbers

Cardinal invariants of the continuum are crucial for understanding the combinatorial structure of $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq)$. Different structures have their own notions of cardinal invariants. In [52], Majcher-Iwanow studies cardinal invariants of the lattice of partitions. In [14], Brendle studies van Douwen's diagram related to the structure $\mathrm{Dense}(\mathbb{Q})/\mathsf{nwd}$. In [2], Balcar, Hernández-Hernández and Hrušák investigate cardinal invariants of the structure $(\mathrm{Dense}\mathbb{Q}, \subseteq)$.

It was a longstanding question whether $\mathfrak{p}$ and $\mathfrak{t}$ are the same cardinal or they are different. Now, thanks to the work of Malliaris and Shelah we know that $\mathfrak{p} = \mathfrak{t}$ (see [53], [54]). In this work we investigate pseudointersection and tower numbers for topological Ramsey spaces. We think of members of the Ramsey spaces as infinite sequences. Initial segments of those sequences are called finite approximations. To investigate cardinal invariants of Ramsey spaces we divide Ramsey spaces into two groups according to their behavior. One group of spaces consists of spaces satisfying Independent Sequences of Structures. This spaces consists of sequences of growing blocks. They have the convenient property that an initial segment of a sequence does not put restrictions on the behavior of subsequent blocks. In this case there is freedom to extend finite approximations. Spaces $\mathcal{R}_\alpha$ and spaces generated by Fraïssé classes belong to this group. In Theorem 4.10 we prove that for Ramsey spaces satisfying the Independent Sequences of Structures it holds that that

$$\mathfrak{p}_{\mathcal{R}} = \mathfrak{t}_{\mathcal{R}} = \mathfrak{p}.$$

The other group consists of spaces for which initial segments of sequences determine the behavior of subsequent blocks. In this case we must be more careful to extend finite approximations. High dimensional Ellentuck spaces $\mathcal{E}_\alpha$, the Carlson-Simpson space $\mathcal{E}_\infty$ and the Milliken space $\mathcal{S}_\infty(U)$ belong to this group. For the spaces that belong to this group we do not have a general setting to investigate pseudointersection and tower numbers. For the strong trees space we prove that the almost reduction relation $\leq^*$ defined by Mijares is not transitive, so we must find another separative order for $\mathcal{S}_\infty(U)$. It follows from a result from Szymański and Zhou [70] that $\mathfrak{p}_{\mathcal{E}_2} = \mathfrak{t}_{\mathcal{E}_2} = \omega_1$.

In theorem 4.17 we extend this result to high dimensional Ellentuck spaces. For the Carlson-Simpson space it holds that $\mathfrak{p}_{\mathcal{E}_\infty} = \mathfrak{t}_{\mathcal{E}_\infty} = \omega_1$. We conjecture that for every space in the second group pseudointersection and tower numbers are exactly ω_1.

0.12 Outline of results

This thesis consists of four chapters. In Chapter 1 we present the background needed to read this work. We introduce topological Ramsey space theory with several examples, theory of filters and ideals, the Sacks forcing and the cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$.

In Chapter 2 we investigate combinatorial properties for ultrafilters forced by topological Ramsey spaces. In section 2.1 we prove that every topological Ramsey space satisfying Todorčević axioms forces a Ramsey ultrafilter. In section 2.2 we present a general method to calculate Ramsey degrees for topological Ramsey spaces with additional properties. Furthermore, we calculate Ramsey degrees for several topological Ramsey spaces and by consequence of this we obtain Ramsey degrees for several ultrafilters.

In Chapter 3 we study those ideals that are related to a topological Ramsey space in the same sense that the ideal Fin is related to the Ellentuck space. In this chapter we prove that ideals $\mathcal{ED}$, $\mathcal{ED}_{fin}$, ideals generated by Fraïssé classes and the ideal conv are TRS ideals. We also prove that the collection of summable ideals are not TRS ideals. In section 3.2 we present the HL property on ideals. We use the HL property to prove that for TRS ideals such that the initial segments of their elements has some independence property, the generic ultrafilter forced by the quotient is Sacks indestructible.

In the last chapter we calculate pseudointersection and tower numbers for Ramsey spaces. We prove that for several topological Ramsey spaces that behaves like increasing sequences of independent blocks their pseudointersection and tower numbers are equal to $\mathfrak{p}$. We also prove that for several topological Ramsey spaces satisfying that the initial segments of their elements are dependent, their pseudointersection and tower numbers are equal to ω_1.

Most of the materials in Chapter 2 and Chapter 4 are included in [27].

Chapter 1

Preliminaries

First we fix some notation.

- If $k \in \omega$, we denote the collection of all k-sized sets of natural numbers by

$$[\omega]^k = \{s \subset \omega : |s| = k\},$$

- we denote the collection of all finite sets of natural numbers by

$$[\omega]^{<\omega} = \{s \subset \omega : |s| < \omega\}$$

- and we denote the collection of all infinite sets of natural numbers by

$$[\omega]^\omega = \{s \subset \omega : |s| = \omega\}.$$

For a given family $\mathcal{F}$ of subsets of ω and for a subset M of ω, we consider the following restriction of $\mathcal{F}$ to M.

$$\mathcal{F} \restriction M = \{s \in \mathcal{F} : s \subset M\}.$$

If T is a set ordered by the relation $<$, we say that T is a *tree* if for every $t \in T$ the set $\{s \in T : s < t\}$ is well-ordered. Given a tree $(T, <)$ and $t \in T$ define:

- $pred_T(t) = \{s \in T : s < t\}$,
- we say that s is a successor of t if $t < s$,
- $succ_T(t)$ consist of the immediate successors of t.
- For every $n \in \omega$, let $T_n = \{t \in T : |pred_T(t)| = n\}$.

We will use $T(n)$ instead of T_n to denote the n-th level of T when we write a statement about indexed trees.

We use standard set theoretic conventions and notation.

1.1 Topological Ramsey spaces

Now we introduce the definition of the spaces we are interested in. We saw in the last section that Ellentuck's theorem establishes that colorings on $[\omega]^\omega$ for which there are monochromatic sets are exactly the Baire measurable colorings. Topological Ramsey spaces are a generalization of the Ellentuck space. In his book [71], Todorčević extends the previous work Carlson-Simpson [17], and distills the combinatorial properties of the Ellentuck space.

The following definition is taken from [71]. The axioms **A.1-A.4** are defined for triples $(\mathcal{R}, \leq, r)$ of objects with the following properties. $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \longrightarrow \mathcal{AR}$ is a mapping giving us the sequence $(r_n(\cdot) = r(\cdot, n))$ of approximation mappings. For this work $\mathcal{AR}$ is the collection of all finite approximations to members of $\mathcal{R}$. For $a \in \mathcal{AR}$ and $A, B \in \mathcal{R}$,

$$[a, B] = \{A \in \mathcal{R} : A \leq B \text{ and } \exists n \in \omega (r_n(A) = a)\}.$$

For $a \in \mathcal{AR}$, let $|a|$ be the integer k for which $a = r_k(A)$, for some $A \in \mathcal{R}$. If $m < n$, $a = r_m(A)$ and $b = r_n(A)$ then we will write $a = r_m(b)$. For $a, b \in \mathcal{AR}$, $a \sqsubseteq b$ if and only if $a = r_m(b)$ for some $m \leq |b|$; if $m < n$ we write $a \sqsubset b$. For each $n < \omega$, $\mathcal{AR}_n = \{r_n(A) : A \in \mathcal{R}\}$.

A.1 (a) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.
(b) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.
(c) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

A.2 There is a quasi-ordering $\leq_{\mathsf{fin}}$ on $\mathcal{AR}$ such that

(a) $\{a \in \mathcal{AR} : a \leq_{\mathsf{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,
(b) $A \leq B$ if and only if for each $n \in \omega$ there exists $m \in \omega$ such that $r_n(A) \leq_{\mathsf{fin}} r_m(B)$,
(c) For every $a, b, c \in \mathcal{AR}$, if $a \sqsubset b$ and $b \leq_{\mathsf{fin}} c$ then there exists $d \in \mathcal{AR}$ such that $d \sqsubset c$ and $a \leq_{\mathsf{fin}} d$.

We define $\text{depth}_B(a)$ as the least n, if it exists, such that $a \leq_{\mathsf{fin}} r_n(B)$. If such an n does not exist, then write $\text{depth}_B(a) = \infty$. If $\text{depth}_B(a) = n < \infty$, then $[\text{depth}_B(a), B]$ denotes $[r_n(B), B]$.

A.3 (a) If $\text{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\text{depth}_B(a), B]$.
(b) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

If $n > |a|$, then $r_n[a, A]$ is the collection of all $b \in \mathcal{AR}_n$ such that $a \sqsubset b$ and $b \leq_{\mathsf{fin}} A$.

A.4 If $\mathrm{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, then there is $A \in [\mathrm{depth}_B(a), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$.

The topology on $\mathcal{R}$ is given by the basic open sets $[a, B]$. This topology is called the *Ellentuck topology* on $\mathcal{R}$. Note that the Ellentuck topology is finer than the usual metrizable topology on $\mathcal{R}$ obtained by considering $\mathcal{R}$ as a subspace of the Tychonoff cube $\mathcal{AR}^\omega$. Given the Ellentuck topology on $\mathcal{R}$, the notions of nowhere dense, and hence of meager, are defined in the natural way. Thus we may say that a subset $\mathcal{X}$ of $\mathcal{R}$ has the Baire property if $\mathcal{X} = \mathcal{O} \cap \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 1.1. A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

Definition 1.2. A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if every subset of $\mathcal{R}$ with the Baire property is Ramsey and if every meager subset of $\mathcal{R}$ is Ramsey null.

The following result appears as Theorem 5.4 in [71].

Theorem 1.3 (Abstract Ellentuck Theorem)**.** *If $(\mathcal{R}, \leq, r)$ is closed (as a subspace of $\mathcal{AR}^\omega$) and satisfies axioms* **A.1***,* **A.2***,* **A.3** *and* **A.4***, then the triple $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.*

So every triple that which is closed and satisfies Todorčević's axioms is a topological Ramsey space. So far it is an open question whether every topological Ramsey space satisfies **A.1** − **A.4** or whether there is a weaker version of Todorčević's axioms which guarantee that a given a triple is a topological Ramsey space.

Let $\mathcal{F} \subseteq \mathcal{AR}$ and $A \in \mathcal{R}$ be given, and let

$$\mathcal{F} \upharpoonright A = \{s \in \mathcal{F} : s = r_n(B) \text{ for some } n \in \omega \text{ and } B \leq A\}.$$

Definition 1.4. A family $\mathcal{F} \subseteq \mathcal{AR}$ of finite approximations is

- *Nash-Williams* if for all $a, b \in \mathcal{F}$, $a \sqsubseteq b$ implies $a = b$.
- *Ramsey* if for every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ and every $A \in \mathcal{R}$, there are $B \leq A$ and $i \in \{0, 1\}$ such that $\mathcal{F}_i \upharpoonright B = \emptyset$.

The next theorem appears as Theorem 5.17 in [71].

Theorem 1.5 (Abstract Nash-Williams Theorem)**.** *Suppose $(\mathcal{R}, \leq, r)$ is a closed triple that satisfies* **A.1** − **A.4***. Then every Nash-Williams family of finite approximations is Ramsey.*

This theorem is a very strong result of the Ramsey space theory and it was fundamental to calculating Ramsey degrees.

In [58] Mijares introduced the following generalization of the relation of almost inclusion on $[\omega]^\omega$.

Definition 1.6. For $A, B \in \mathcal{R}$, write $A \leq^* B$ if there exists $a \in \mathcal{AR} \restriction A$ such that $[a, A] \subseteq [a, B]$. In this case we say that A is an *almost reduction* of B.

Note that for each $a \in \mathcal{AR} \restriction A$, there exists $C \in \mathcal{R}$ such that $a \sqsubseteq C$ and $C \leq A$, so $\emptyset \neq [a, A] \subseteq [a, B]$.

Even though for most of the spaces we will study $\leq^*$ is a σ-closed partial order, it is not clear which condition on Ramsey spaces implies this property. In Section 4.4 we will see that there is a topological Ramsey space $\mathcal{R}$ for which $\leq^*$ is not a transitive relation on $\mathcal{R}$.

1.2 Some topological Ramsey spaces and their ultrafilters

In this section we introduce some topological Ramsey spaces and their associated ultrafilters whose Ramsey degrees and tower and pseudointersection numbers we will investigate.

For this work, a *filter* on a topological Ramsey space $\mathcal{R}$ is a collection $\mathcal{G} \subset \mathcal{R}$ such that:

- if $A \in \mathcal{G}$ and $A \leq B$ then $B \in \mathcal{G}$;
- if $A, B \in \mathcal{G}$ then there exists some $C \in \mathcal{G}$ such that $C \leq A, B$.

And we say that $\mathcal{G}$ is an *ultrafilter* is it is a maximal filter.

In [58], Mijares states a definition of ultrafilters on topological Ramsey spaces with additional properties. In this work Mijares defines the notions of Ramsey and selective ideals (semiselective in Definition 1.7) in order to investigate general properties for ultrafilters forced by topological Ramsey spaces. Trujillo [72] proved that under that definition of selective ideals there is an ultrafilter on $\mathcal{R}_1$, a topological Ramsey space built by Dobrinen and Todorčević, such that it is selective but not Ramsey. In [20], Mijares and Nieto define a new notion of selective ultrafilters on topological Ramsey spaces and they prove that under suitable large cardinal hypothesis every semiselective ultrafilter $\mathcal{U} \subset \mathcal{R}$ is generic over $L(\mathbb{R})$. This generalizes the corresponding result for the Ellentuck space (see [21]).

Definition 1.7. Let $\mathcal{R}$ be a topological Ramsey space and $\mathcal{U}$ an ultrafilter on $\mathcal{R}$. We say

- $\mathcal{U}$ is *Nash-Williams* if for every Nash-Williams family $\mathcal{F} \subset \mathcal{AR}$ and every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ there exists $X \in \mathcal{U}$ and $i \in 2$ such that $\mathcal{F}_i \cap \mathcal{AR} \restriction X = \emptyset$.
- $\mathcal{U}$ is *Ramsey* if for all $A \in \mathcal{U}$, $a \in \mathcal{AR} \restriction A$ and $n \in \omega$, and for every $f : \mathcal{AR}_{|a|+n} \to 2$ there exists $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{U}$ such that f is constant on $r_{|a|+n}[a, B]$.
- $\mathcal{U}$ is *selective* if for every $A \in \mathcal{U}$ and every $\{A_a\}_{a \in \mathcal{AR} \restriction A} \subset \mathcal{U} \restriction A$ with $[a, A_a] \neq \emptyset$ for each $a \in \mathcal{AR} \restriction A$ there exists $B \in \mathcal{U} \restriction A$ such that $[a, B] \subset [a, A_a]$ for every $a \in \mathcal{AR} \restriction B$.

- $\mathcal{U}$ is *weakly selective* if for every $A \in \mathcal{U}$ and every $\{A_b\}_{b \in \mathcal{AR}_1} \subset \mathcal{U} \restriction A$ with $[b, A_b] \neq \emptyset$ for each $\mathcal{AR}_1 \restriction A$ there exists $B \in \mathcal{U} \restriction A$ such that $[b, B] \subset [b, A_b]$ for every $b \in \mathcal{AR}_1 \restriction B$.

In her thesis [76], Zheng explores relations between the notions defined above.

Given a topological Ramsey space $(\mathcal{R}, \leq, r)$, the generic filter forced by $(\mathcal{R}, \leq)$ induces an ultrafilter as we now show: In all known examples of topological Ramsey spaces, the collection of first approximations, $\mathcal{AR}_1$, is a countable set. If that is not the case for some particular space $\mathcal{R}$, the restriction $\mathcal{AR}_1 \restriction A$ for any member A of $\mathcal{R}$ is countable by Axiom **A.2**, so one may work below a fixed member of $\mathcal{R}$, if necessary.

Definition 1.8. Given a generic filter $\mathcal{G} \subseteq \mathcal{R}$ for the forcing $(\mathcal{R}, \leq)$, define

$$\mathcal{U}_{\mathcal{R}} = \{X \subseteq \mathcal{AR}_1 : X \supseteq \mathcal{AR}_1 \restriction A \text{ for some } A \in \mathcal{G}\}. \tag{1.2.1}$$

We will be working with σ-closed partial orders $\leq^*$ coarsening $\leq$. In some cases this the partial order $\leq^*$ coincides with the almost reduction relation defined by Mijares in Definition 1.6. For this spaces both definitions of $\mathcal{U}_{\mathcal{R}}$ coincide.

Lemma 1.9. *Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space, $\leq^*$ be a σ-closed quasi-order coarsening $\leq$, and $\mathcal{G} \subseteq \mathcal{R}$ be a generic filter for $(\mathcal{R}, \leq^*)$. Let $\mathcal{U}_{\mathcal{R}}$ be the filter on base set $\mathcal{AR}_1$ generated by $\mathcal{G}$. Then $\mathcal{U}_{\mathcal{R}}$ is an ultrafilter on the base set $\mathcal{AR}_1$.*

Proof. This follows by the Abstract Nash-Williams Theorem and genericity of $\mathcal{G}$. ∎

This section introduces some topological Ramsey spaces and their associated ultrafilters whose Ramsey degrees, pseudointersection and tower numbers will be investigated in subsequent sections.

1.2.1 The topological Ramsey spaces $\mathcal{R}_\alpha$, $1 \leq \alpha < \omega_1$

In [50], Laflamme constructed a forcing, denoted $\mathbb{P}_1$, which generates a weakly Ramsey ultrafilter, denoted $\mathcal{U}_1$, which is not Ramsey. Although Blass had already shown such ultrafilters exist consistently (see [9]), the point of $\mathbb{P}_1$ was to construct a weakly Ramsey ultrafilter with complete combinatorics, analogous to the result that any Ramsey ultrafilter in the model $V[G]$ obtained by Lévy collapsing a Mahlo cardinal to $\aleph_1$ is $([\omega]^\omega, \subseteq^*)$-generic over $\mathrm{HOD}(\mathbb{R})^{V[G]}$ (see [11] and [56]). One advantage of forcing with topological Ramsey spaces is that the associated ultrafilter automatically has complete combinatorics in the presence of large cardinals (see [19] for the result and [24] for an overview of this area). In [29], a topological Ramsey space denoted $\mathcal{R}_1$ was constructed which forms a dense subset of Laflamme's forcing $\mathbb{P}_1$. This space generates the same weakly Ramsey ultrafilter. It was built to find the exact Tukey structure below $\mathcal{U}_1$ and the precise structure of the Rudin-Keisler classes within these Tukey types, which were indeed found in [29]. Here, we reproduce a few definitions and facts relevant to this paper.

Definition 1.10 (($\mathcal{R}_1, \leq, r$), [29])**.** Let $\mathbb{T}_1$ denote the following infinite tree of height 2.

$$\mathbb{T}_1 = \{\langle\rangle\} \cup \{\langle n\rangle : n < \omega\} \cup \bigcup_{n<\omega} \{\langle n, i\rangle : i \leq n\}.$$

$\mathbb{T}_1$ can be thought of as an infinite sequence of finite trees of height 2, where the n-th *subtree* of $\mathbb{T}_1$ is

$$\mathbb{T}_1(n) = \{\langle\rangle, \langle n\rangle, \langle n, i\rangle : i \leq n\}.$$

The members of $\mathcal{R}_1$ are infinite subtrees of $\mathbb{T}_1$ which have the same structure as $\mathbb{T}_1$. That is, a tree $X \subseteq \mathbb{T}_1$ is in $\mathcal{R}_1$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega}$ such that

1. $X \cap \mathbb{T}_1(k_n) \cong \mathbb{T}_1(n)$ for each $n < \omega$; and
2. whenever $X \cap \mathbb{T}_1(j) \neq \emptyset$, then $j = k_n$ for some $n < \omega$.

When this holds, we let $X(n)$ denote $X \cap \mathbb{T}_1(k_n)$, and call $X(n)$ the *n-th subtree* of X. For $n < \omega$, $r_n(X)$ denotes $\bigcup_{i<n} X(i)$.

For $X, Y \in \mathcal{R}_1$, define $Y \leq X$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega}$ such that for each n, $Y(n)$ is a subtree of $X(k_n)$. Notice that by the structure of the members of $\mathcal{R}_1$, $Y \leq X$ exactly when $Y \subseteq X$. Given $a, b \in \mathcal{AR}$, define $b \leq_{\mathsf{fin}} a$ if and only if $b \subseteq a$.

The following figure presents the first five "blocks" of the maximal member of $\mathcal{R}_1$.

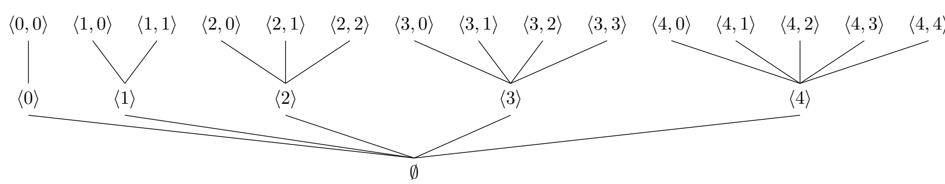


Figure 1.1: $r_5(\mathbb{T}_1)$

The members of $\mathcal{R}_1$ are subtrees of $\mathbb{T}_1$ which are isomorphic to $\mathbb{T}_1$. As the first step toward the main theorem of [29], the following was proved.

Theorem 1.11 (Dobrinen and Todorčević, [29])**.** *($\mathcal{R}_1, \leq, r$) is a topological Ramsey space.*

Notice that by the structure of the members of $\mathcal{R}_1$, given $X, Y \in \mathcal{R}_1$, $Y \leq^* X$ (recall Definition 1.6) holds if and only if there is an $i < \omega$ and a strictly increasing sequence $(k_n)_{n\geq i}$ such that for each $n \geq i$, $Y(n) \subseteq X(k_n)$. Thus, the quasi-order $\leq^*$ turns out to be equivalent to $\subseteq^*$, since $Y \leq^* X$ if and only if $Y \subseteq^* X$. By an *ultrafilter* $\mathcal{U}_{\mathcal{R}_1}$ *associated with the forcing* ($\mathcal{R}_1, \leq^*$) we mean the ultrafilter on base set $\mathcal{AR}_1$ generated by the sets $\mathcal{AR}_1 \restriction X$, $X \in G$, where G is some generic filter for $(\mathcal{R}_1, \leq^*)$. By the density of this topological Ramsey space in Laflamme's forcing, this ultrafilter $\mathcal{U}_{\mathcal{R}_1}$ is isomorphic to the ultrafilter $\mathcal{U}_1$ generic for Laflamme's forcing $\mathbb{P}_1$. Hence, it is weakly Ramsey but not Ramsey.

Continuing in this vein, Laflamme constructed a hierarchy of forcings $\mathbb{P}_\alpha$, $1 \leq \alpha < \omega_1$, in order to produce rapid p-points $\mathcal{U}_\alpha$ satisfying partition relations with decreasing strength as α increases, and such that for $\beta < \alpha$, $\mathcal{U}_\beta$ is Rudin-Keisler below $\mathcal{U}_\alpha$. In [50], Laflamme proved that each $\mathcal{U}_\alpha$ has complete combinatorics, and that below $\mathcal{U}_\alpha$, there is a decreasing chain of length $\alpha+1$ of Rudin-Keisler types, the least one being that of a Ramsey ultrafilter. This left open, though, whether or not this chain is the only Rudin-Keisler structure below $\mathcal{U}_\alpha$.

Topological Ramsey spaces $\mathcal{R}_\alpha$ were constructed in [30] to produce dense subsets of Laflamme's forcings $\mathbb{P}_\alpha$, hence generating the same generic ultrafilters. The reader is referred to [30] for the definition of these spaces. The Ramsey space techniques provided valuable methods for proving in [30] that indeed the Rudin-Keisler, and moreover, the Tukey structure below $\mathcal{U}_\alpha$ is exactly a chain of length $\alpha+1$. Here, we reproduce $\mathcal{R}_2$, with a minor modification not affecting its forcing properties which will make it easier to understand. The reader can then infer the structure of $\mathcal{R}_k$ for each $1 \leq k < \omega$. In Section 2.2, we will only work with $\mathcal{R}_k$ for $1 \leq k < \omega$, since the Ramsey degree $t(\mathcal{U}_\omega, 2) = \omega$. However, Chapter 4 will consider pseudointersection and tower numbers of $\mathcal{R}_\alpha$, for all $1 \leq \alpha < \omega_1$.

1.2.2 Ramsey spaces from Fraïssé classes

This subsection introduces topological Ramsey spaces constructed in [26]. The motivation for these spaces was to find dense subsets of some forcings of Blass in [8] and of Baumgartner and Taylor in [6] in order to better study properties of their forced ultrafilters (more details provided below). The reader can see [48] for more background about Fraïssé theory.

This subsection is taken from [26]. We shall call $L = \{<\} \cup \{R_i\}_{i \in I}$ an *ordered relational signature* if it consists of the order relation symbol $<$ and a countable collection of *relation symbols* R_i, where for each $i \in I$, we let denote by $n(i)$ the *arity* of R_i. A structure for L is of the form $\mathbf{A} = \langle |\mathbf{A}, <^{\mathbf{A}} |, \{R_i^{\mathbf{A}}\}_{i \in I}\rangle$, where $|\mathbf{A}| \neq \emptyset$ is the universe of $\mathbf{A}$, $<^{\mathbf{A}}$ is a linear ordering of $|\mathbf{A}|$, and for each $i \in I$, $R_i^{\mathbf{A}} \subseteq |\mathbf{A}|^{n(i)}$. An *embedding* between structures $\mathbf{A}, \mathbf{B}$ for L is an injection $\iota : |\mathbf{A}| \to |\mathbf{B}|$ such that for any two $a, a' \in |\mathbf{A}|$, $a <^{\mathbf{A}} a' \leftrightarrow \iota(a) <^{\mathbf{B}} \iota(a')$, and for all $i \in I$, $R_i^{\mathbf{A}}(a_1, \cdot, a_{n(i)}) \leftrightarrow R_i^{\mathbf{B}}(\iota(a_1), \cdot, \iota(a_{n(i)}))$. If ι is the identity map, then we say that $\mathbf{A}$ is a substructure of $\mathbf{B}$. We say that ι is an *isomorphism* if ι is an onto embedding. We write $\mathbf{A} \leq \mathbf{B}$ to denote that $\mathbf{A}$ can be embedded into $\mathbf{B}$; we write $\mathbf{A} \cong \mathbf{B}$ to denote that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.

A class $\mathcal{K}$ of finite structures for an ordered relational signature L is called *hereditary* if whenever $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \leq \mathbf{B}$, then also $\mathbf{A} \in \mathcal{K}$. $\mathcal{K}$ satisfies the *joint embedding property* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is a $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{B} \leq \mathbf{C}$. We say that $\mathcal{K}$ satisfies the *amalgamation property* if for any embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and there are embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$. A class of finite structures $\mathcal{K}$ is called a *Fraïssé class of ordered relational structures* for an ordered relational signature L if it is hereditary, satisfies the joint embedding and amalgamation properties, contains (up to isomorphism) only countably many structures, and contains structures of arbitrarily large finite cardinality.

Let $\mathcal{K}$ be a hereditary class of finite structures for an ordered relational signature L. For $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ with $\mathbf{A} \leq \mathbf{B}$, we use $\binom{\mathbf{B}}{\mathbf{A}}$ to denote the set of all substructures of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. Given structures $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$ in $\mathcal{K}$, we write

$$\mathbf{C} \to (\mathbf{B})_k^{\mathbf{A}}$$

to denote that for each coloring of $\binom{\mathbf{C}}{\mathbf{A}}$ into k colors, there is a $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}'}{\mathbf{A}}$ is homogeneous, meaning that every member of $\binom{\mathbf{B}'}{\mathbf{A}}$ has the same color. We say that $\mathcal{K}$ has the *Ramsey property* if and only if for any two structures $\mathbf{A} \leq \mathbf{B}$ in $\mathcal{K}$ and any natural number $k \geq 2$, there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{B} \leq \mathbf{C}$ such that $\mathbf{C} \to (\mathbf{B})_k^{\mathbf{A}}$.

For finitely many Fraïssé classes $\mathcal{K}_j$, $j \in J$ for some $J < \omega$, we write $\binom{(\mathbf{B}_j)_{j\in J}}{(\mathbf{A}_j)_{j\in J}}$ to denote the set of all sequences $(\mathbf{D}_j)_{j\in J}$ such that for each $j \in J$, $\mathbf{D}_j \in \binom{\mathbf{B}_j}{\mathbf{A}_j}$. For structures $\mathbf{A}_j \leq \mathbf{B}_j \leq \mathbf{C}_j \in \mathcal{K}_j$, $j \in J$, we write

$$(\mathbf{C}_j)_{j\in J} \to ((\mathbf{B}_j)_{j\in J})_k^{(\mathbf{A}_j)_{j\in J}}$$

to denote that for each coloring of $\binom{(\mathbf{C}_j)_{j\in J}}{(\mathbf{A}_j)_{j\in J}}$ into k colors, there is a $(\mathbf{B}'_j)_{j\in J} \in \binom{(\mathbf{C}_j)_{j\in J}}{(\mathbf{B}_j)_{j\in J}}$ such that $\binom{(\mathbf{B}'_j)_{j\in J}}{(\mathbf{A}_j)_{j\in J}}$ is homogeneous, meaning that every member of $\binom{(\mathbf{B}'_j)_{j\in J}}{(\mathbf{A}_j)_{j\in J}}$ has the same color. When no k appears in the expression, it is assumed that $k = 2$.

By Theorem A of Nešetřil, Jaroslav and Rödl in [64], there is a large class of Fraïssé classes of finite ordered relational structures with the Ramsey property. In particular, the collection of all finite linear orderings, the collection of all finite ordered n-clique free graphs, and the collection of all finite ordered complete graphs are examples of Fraïssé classes fulfilling the requirements. Moreover, finite products of members of such classes preserve the Ramsey property. The following theorem for products of Ramsey classes of finite objects is due to Sokiśc in his PhD thesis [69].

Theorem 1.12 (Product Ramsey Theorem, Sokiśc)**.** *Let s and k be fixed natural numbers and let $\mathcal{K}_j$, $j \in s$, be a sequence of Ramsey classes of finite objects. Fix two sequences $(\mathbf{A}_j)_{j\in s}$ and $(\mathbf{B}_j)_{j\in s}$ such that for each $j \in s$, we have $\mathbf{A}_j, \mathbf{B}_j \in \mathcal{K}_j$ and $\mathbf{A}_j \leq \mathbf{B}_j$. Then there is a sequence $(\mathbf{C}_j)_{j\in s}$ such that $\mathbf{C}_j \in \mathcal{K}_j$ for each $j \in s$, and*

$$(\mathbf{C}_j)_{j\in s} \to ((\mathbf{B}_j)_{j\in s})_k^{(\mathbf{A}_j)_{j\in s}}.$$

Definition 1.13 (The space $\mathcal{R}(\mathbb{A})$, [26])**.** Fix some natural number $J \geq 1$, and for each $j < J$, let $\mathcal{K}_j$ be a Fraïssé class of finite linearly ordered relational structures with the Ramsey property. We say that $\mathbb{A} = \langle (\mathbf{A}_{k,j})_{k<\omega} : j < J\rangle$ is a *generating sequence* if for each $j < J$, the following hold:

1. For each $k < \omega$, $\mathbf{A}_{k,j}$ is a member of $\mathcal{K}_j$, and $\mathbf{A}_{0,j}$ has universe of cardinality 1.

2. Each $\mathbf{A}_{k,j}$ is a substructure of $\mathbf{A}_{k+1,j}$.

3. For each structure $\mathbf{B} \in \mathcal{K}_j$, there is a k such that $\mathbf{B}$ embeds into $\mathbf{A}_{k,j}$.

4. For each pair $k < m < \omega$, there is an $n > m$ large enough that the following Ramsey property holds:
$$\mathbf{A}_{n,j} \to (\mathbf{A}_{m,j})^{\mathbf{A}_{k,j}}.$$

Let $\mathbf{A}_k$ denote the n-tuple of structures $(\mathbf{A}_{k,j})_{j<J}$. It can be convenient to think of this as the product $\prod_{j<J} \mathbf{A}_{k,j}$ with no additional relations. Let $\mathbb{A} = \langle\langle k, \mathbf{A}_k\rangle : k < \omega\rangle$. This infinite sequence $\mathbb{A}$ of J-tuples of finite structures is the maximal member of the space $\mathcal{R}(\mathbb{A})$. We define B to be a member of $\mathcal{R}(\mathbb{A})$ if and only if $B = \langle\langle n_k, \mathbf{B}_k\rangle : k < \omega\rangle$, where

1. $(n_k)_{k<\omega}$ is some strictly increasing sequence of natural numbers; and
2. for each $k < \omega$, $\mathbf{B}_k$ is an J-tuple $(\mathbf{B}_{k,j})_{j<J}$, where each $\mathbf{B}_{k,j}$ is a substructure of $\mathbf{A}_{n_k,j}$ isomorphic to $\mathbf{A}_{k,j}$.

We use $B(k)$ to denote $\langle n_k, \mathbf{B}_k\rangle$, the k-th block of B. The m-th approximation of B is $r_m(B) = \langle B(0), ..., B(m-1)\rangle$.

Define the partial order $\le$ as follows: For $B = \langle\langle m_k, \mathbf{B}_k\rangle : k < \omega\rangle$ and $C = \langle\langle n_k, \mathbf{C}_k\rangle : k < \omega\rangle$, define $C \le B$ if and only if for each k there is an l_k such that $n_k = m_{l_k}$ and for all $j < J$, $\mathbf{C}_{k,j}$ is a substructure of $\mathbf{B}_{l_k,j}$. The partial order $\le_{\mathsf{fin}}$ on the collection of finite approximations, $\mathcal{AR}$, is defined as follows: For $b = \langle\langle m_k, \mathbf{B}_k\rangle : k < p\rangle\rangle$ and $c = \langle\langle n_k, \mathbf{C}_k\rangle : k < q\rangle$, where $p, q < \omega$, define $c \le_{\mathsf{fin}} b$ if and only if there are $C \le B$ such that $c = r_q(C)$, $b = r_p(B)$. For these spaces, the naturally associated σ-closed partial order $\le^*$ from Definition 1.6 is simply $\subseteq^*$.

Theorem 1.14 (Dobrinen, Mijares and Trujillo, [26])**.** *Given a generating sequence $\langle(\mathbf{A}_{k,j})_{k<\omega} : j < J\rangle$, the triple $(\mathcal{R}(\mathbb{A}), \le r)$ forms a topological Ramsey space.*

Letting $\mathcal{R}$ denote $\mathcal{R}(\mathbb{A})$, given a generic filter G for the forcing $(\mathcal{R}, \le^*)$, we let $\mathcal{U}_{\mathcal{R}}$ denote the ultrafilter on base set $\mathcal{AR}_1$ generated by the sets $\mathcal{AR}_1 \restriction X$, $X \in G$. The motivation for these spaces came from studying the Tukey types below ultrafilters constructed in [8] and [6]. The special case where $n = 2$ and both $\mathcal{K}_0$ and $\mathcal{K}_1$ are the classes of finite linear orders produces a Ramsey space which is dense inside the *n-square forcing* of Blass in [8], which he constructed to produce a p-point which has two Rudin-Keisler incomparable selective ultrafilters Rudin-Keisler below it. Given $n \ge 2$, we shall let $\mathcal{H}^n$ denote the Ramsey space produced when each $\mathcal{K}_j$, $j < n$, is the class of finite linear orders; call this space the *n-hypercube space*. The space $\mathcal{H}^2$ is dense in Blass' forcing, and hence the ultrafilter $\mathcal{U}_{\mathcal{H}^2}$ is isomorphic to the one constructed by Blass. The collection of spaces $\mathcal{H}^n$, $n \ge 2$, form a hierarchy of forcings such that each ultrafilter $\mathcal{U}_{\mathcal{H}^n}$ projects to the ultrafilter $\mathcal{U}_{\mathcal{H}^m}$ for $m < n$. It is shown in [26] that the initial Tukey structure below $\mathcal{U}_{\mathcal{H}^n}$ is isomorphic to the Boolean algebra $\mathcal{P}(n)$. In another direction, the special cases where $J = 1$, $k \ge 3$ is fixed, and $\mathcal{K}_0$ is the class of all finite ordered k-clique-free graphs produces Ramsey spaces which are dense inside partial orders constructed by Baumgartner and Taylor in [6] which produce p-points which have asymmetric partition relations, called k-arrow ultrafilters. Results on the initial Rudin-Keisler and Tukey structures of ultrafilters constructed by Ramsey spaces from generating sequences appear in [26], which includes some work of Trujillo from his thesis [72].

Note that for every $n \in \omega$, $(\mathcal{H}^n, \leq^*)$ is a σ-closed partial order. Let $\mathcal{G}_{\mathcal{H}^n}$ denote the $\langle \mathcal{H}^n, \leq^* \rangle$-generic filter. For every $D \in \mathcal{H}^n$, let $[D]$ denote the set of all terminal nodes on $\bigcup_{k\in\omega} \mathbf{A}_k$. Let $\mathcal{U}$ denote the filter on $[\mathbb{A}]$ generated by $\{[D] : D \in \mathcal{G}_{\mathcal{H}^n}\}$. Note that we can identify D with $[D]$ because D and $[D]$ contain the same information. Also note that we can identify $\mathcal{AR}_1$ with $[\mathbb{A}]$. Then we can identify $\mathcal{U}$ with $\mathcal{U}_{\mathcal{H}_n}$, the first-approximation ultrafilter on $\mathcal{AR}_1$. We shall refer to $\mathcal{U}_{\mathcal{H}^n}$ as an ultrafilter on $\mathcal{H}^n$.

1.2.3 High dimensional Ellentuck spaces

The next topological Ramsey spaces that we are going to present are the high dimensional Ellentuck spaces. The construction of and results about high dimensional Ellentuck spaces are developed in [22]. We shall let $\mathcal{E}_1$ denote the Ellentuck space. The first of the high dimensional Ellentuck spaces, $\mathcal{E}_2$, was motivated by the problem of finding the precise structure of the Tukey types of ultrafilters Tukey reducible to the generic ultrafilter forced by $\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$, denoted by $\mathcal{G}_2$. The construction of $\mathcal{E}_2$ was generalized to find topological Ramsey spaces which are forcing equivalent to the Boolean algebras $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, for each $k > 2$, in order to find Tukey exact structures of $\mathcal{G}_k$. We borrow the following construction from [1].

Definition 1.15. For $k \geq 2$, denote by $\omega^{\not\downarrow \leq k}$ the collection of all non-decreasing sequences of members of ω of length less than or equal to k.

Definition 1.16 (The well-order $<_{\mathrm{lex}}$)**.** Let $(s_1, \dots, s_i)$ and $(t_1, \dots, t_j)$, with $i, j \geq 1$, be in $\omega^{\not\downarrow \leq k}$. We say that $(s_1, \dots, s_i)$ is lexicographically below $(t_1, \dots, t_j)$, written $(s_1, \dots, s_i) <_{\mathrm{lex}} (t_1, \dots, t_j)$, if and only if there is a non-negative integer m with the following properties:

(i) $m \leq i$ and $m \leq j$;

(ii) for every positive integer $n \leq m$, $s_n = t_n$; and

(iii) either $s_{m+1} < t_{m+1}$, or $m = i$ and $m < j$.

This is just a generalization of the way the alphabetical order of words is based on the alphabetical order of their component letters.

Definition 1.17 (The well-ordered set $(\omega^{\not\downarrow \leq k}, \prec)$)**.** Set the empty sequence $()$ to be the $\prec$-minimum element of $\omega^{\not\downarrow \leq k}$, so there for all nonempty sequences s in $\omega^{\not\downarrow \leq k}$, we have $() \prec s$. In general, given $(s_1, \dots, s_i)$ and $(t_1, \dots, t_j)$ in $\omega^{\not\downarrow \leq k}$ with $i, j \geq 1$, define $(s_1, \dots, s_i) \prec (t_1, \dots, t_j)$ if and only if either

1. $s_i < t_j$, or

2. $s_i = t_j$ and $(s_1, \dots, s_i) <_{\mathrm{lex}} (t_1, \dots, t_j)$.

Notation. Since $\prec$ well-orders $\omega^{\not\downarrow\le k}$ in order-type ω, we fix the notation of letting $\vec{s}_m$ denote the m-th member of $(\omega^{\not\downarrow\le k}, \prec)$. Let $\omega^{\not\downarrow k}$ denote the collection of all non-decreasing sequences of length k of members of ω. Note that $\prec$ also well-orders $\omega^{\not\downarrow k}$ in order type ω. Fix the notation of letting $\vec{u}_n$ denote the n-th member of $(\omega^{\not\downarrow k}, \prec)$. For $s, t \in \omega^{\not\downarrow\le k}$, we say that s is an *initial segment* of t and write $s \sqsubset t$ if $s = (s_1, \dots, s_i)$, $t = (t_1, \dots, t_j)$, $i < j$, and for all $m \le i$, $s_m = t_m$. Recall the concatenation operation: Given sequences $s = (s_1, \dots, s_i)$ and $t = (t_1, \dots, t_j)$, $s^\frown t$ denotes the *concatenation* of s and t, which is the sequence $(s_1, \dots, s_i, t_1, \dots, t_j)$ of length $i + j$. As is standard, for a natural number n, $s^\frown n$ denotes the sequence $(s_1, \dots, s_i, n)$.

Definition 1.18 (The spaces $(\mathcal{E}_k, \le, r)$, $k \ge 2$)**.** An $\mathcal{E}_k$*-tree* is a function $\widehat{X}$ from $\omega^{\not\downarrow\le k}$ into $\omega^{\not\downarrow\le k}$ that preserves the well-order $\prec$ and initial segments $\sqsubset$. For $\widehat{X}$ an $\mathcal{E}_k$-tree, let X denote the restriction of $\widehat{X}$ to $\omega^{\not\downarrow k}$. The space $\mathcal{E}_k$ is defined to be the collection of all X such that $\widehat{X}$ is an $\mathcal{E}_k$-tree. We identify X with its range and usually will write $X = \{v_1, v_2, \dots\}$, where $v_1 = X(\vec{u}_1) \prec v_2 = X(\vec{u}_2) \prec \cdots$. The partial ordering on $\mathcal{E}_k$ is defined to be simply inclusion; that is, given $X, Y \in \mathcal{E}_k$, $X \le Y$ if and only if (the range of) X is a subset of (the range of) Y. For each $n < \omega$, the n-th restriction function r_n on $\mathcal{E}_k$ is defined by $r_n(X) = \{v_1, v_2, \dots, v_n\}$ that is, the $\prec$-least n members of X. When necessary for clarity, we write $r^k_n(X)$ to highlight that X is a member of $\mathcal{E}_k$. We set

$$\mathcal{AR}^k_n := \{r_n(X) : X \in \mathcal{E}_k\} \qquad \text{and} \qquad \mathcal{AR}^k := \{r_n(X) : n < \omega, X \in \mathcal{E}_k\} \quad (1.2.2)$$

to denote the set of all n-th approximations to members of $\mathcal{E}_k$, and the set of all finite approximations to members of $\mathcal{E}_k$, respectively.

Theorem 1.19 ([22])**.** *For each $2 \le k < \omega$, $(\mathcal{E}_k, \le, r)$ is a topological Ramsey space.*

Example 1.20. (The space $\mathcal{E}_2$) The members of $\mathcal{E}_2$ look like ω copies of the Ellentuck space. The well-order $\langle \omega^{\not\downarrow\le 2}, \prec \rangle$ begins as follows:

$$() \prec (0) \prec (0,0) \prec (0,1) \prec (1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2) \prec (2,2) \prec \dots$$

The tree structure of $\omega^{\not\downarrow\le 2}$ under the lexicographic order, looks like ω copies of ω, and has order type the countable ordinal ω^2. Here we picture an initial segment of $\omega^{\not\downarrow\le 2}$.

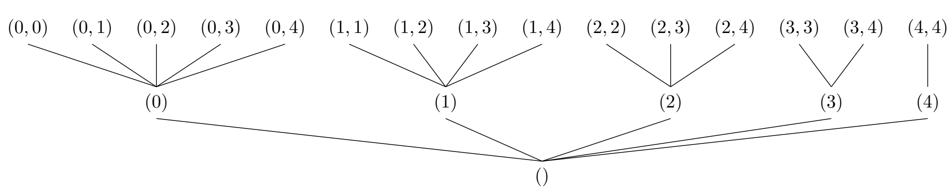


Figure 1.2: The initial structure of $\omega^{\not\downarrow\le 2}$

Let ω^2 denote $\omega \times \omega$ and let $\mathrm{Fin}^{\otimes 2}$ denote the ideal $\mathrm{Fin} \otimes \mathrm{Fin}$, which is the collection of all subsets A of $\omega \times \omega$ such that for all but finitely many $i \in \omega$, the fiber $A(i) := \{j < \omega : (i,j) \in A\}$ is finite. Abusing notation, we also let $\mathrm{Fin}^{\otimes 2}$ denote the ideal on $[\omega]^2$ consisting of sets $A \subseteq [\omega]^2$ such that for all but finitely many $i \in \omega$, the set $\{j > i : \{i,j\} \in A\}$ is finite. Given $X, Y \subseteq [\omega]^2$, we write $X \subseteq^{\mathrm{Fin}^{\otimes 2}} Y$ if and only if $X \backslash Y \in \mathrm{Fin}^{\otimes 2}$.

Proposition 1.21. *$\langle \mathcal{E}_2, \subseteq^{\mathrm{Fin}^{\otimes 2}} \rangle$ is forcing equivalent to $\mathcal{P}(\omega^2) / \mathrm{Fin}^{\otimes 2}$.*

Example 1.22. (The space $\mathcal{E}_3$) The well-order $\langle \omega^{\not\downarrow \le 3}, \prec \rangle$ begins as follows:

$$\begin{aligned} &\emptyset \prec (0) \prec (0,0) \prec (0,0,0) \prec (0,0,1) \prec (0,1) \prec (0,1,1) \prec (1) \\ &\quad \prec (1,1) \prec (1,1,1) \prec (0,0,2) \prec (0,1,2) \prec (0,2) \prec (0,2,2) \\ &\quad \prec (1,1,2) \prec (1,2) \prec (1,2,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (0,0,3) \prec \dots \end{aligned}$$

The set $\omega^{\not\downarrow \le 3}$ is a tree of height three with each non-maximal node branching into ω many nodes. The following figure shows the initial structure of $\omega^{\not\downarrow \le 3}$.

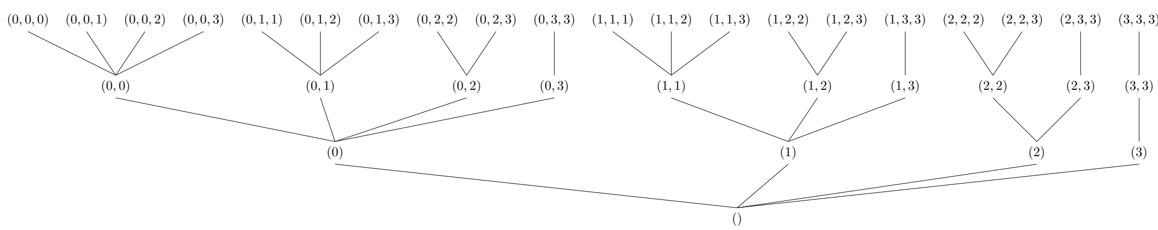


Figure 1.3: $\omega^{\not\downarrow \le 3}$

By $\mathrm{Fin}^{\otimes 3}$, we denote $\mathrm{Fin} \otimes (\mathrm{Fin}^{\otimes 2})$, which consists of all subsets $F \subseteq \omega^3$ such that for finitely many $i \in \omega$, $\{(j,k) : (i,j,k) \in F\}$ is in $(\mathrm{Fin}^{\otimes 2})^+$. Identifying $[\omega]^3$ with $\{(i,j,k) \in \omega^3 : i < j < k\}$, we abuse notation and let $\mathrm{Fin}^{\otimes 3}$ on $[\omega]^3$ denote the collection of all subsets $F \subseteq [\omega]^3$ such that $\{(i,j,k) : \{i,j,k\} \in F\}$ is in $\mathrm{Fin}^{\otimes 3}$ as defined on $[\omega]^3$.

Proposition 1.23. *$\langle \mathcal{E}_3, \subseteq^{\mathrm{Fin}^{\otimes 3}} \rangle$ is forcing equivalent to $\mathcal{P}(\omega^3) / \mathrm{Fin}^{\otimes 3}$.*

Let $k \ge 2$. By $\mathrm{Fin}^{\otimes k}$, we denote $\mathrm{Fin} \otimes \mathrm{Fin}^{\otimes k-1}$, which consists of all subsets $F \subseteq \omega^k$ such that, $\{(j_1, ..., j_{k-1}) : (i, j_1, ..., j_{k-1}) \in F\}$ is in $(\mathrm{Fin}^{\otimes k-1})^+$ for finitely many $i \in \omega$. Identifying $[\omega]^k$ with $\{(j_1, ..., j_k) \in \omega^k : j_i < j_{i+1}, i \in k+1\}$, we abuse notation and let $\mathrm{Fin}^{\otimes k}$ on $[\omega]^k$ denote the collection of all subsets $F \subseteq [\omega]^k$ such that $\{(j_1, ..., j_k) : \{j_1, ..., j_k\} \in F\}$ is in $\mathrm{Fin}^{\otimes k}$ as defined on $[\omega]^k$.

Proposition 1.24. *$\langle \mathcal{E}_k, \subseteq^{\mathrm{Fin}^{\otimes k}} \rangle$ is forcing equivalent to $\mathcal{P}(\omega^k) / \mathrm{Fin}^{\otimes k}$.*

Let $\mathcal{G}_k$ be a generic ultrafilter for $\mathcal{P}(\omega^k) / \mathrm{Fin}^{\otimes k}$. Since $\langle \mathcal{E}_k, \subseteq^{\mathrm{Fin}^{\otimes k}} \rangle$ and $\mathcal{P}(\omega^k) / \mathrm{Fin}^{\otimes k}$ are forcing equivalent, the generic filter of $\langle \mathcal{E}_k, \subseteq^{\mathrm{Fin}^{\otimes k}} \rangle$ is also an ultrafilter. We will abuse notation and call $\mathcal{G}_k$ the generic ultrafilter for $\langle \mathcal{E}_k, \subseteq^{\mathrm{Fin}^{\otimes k}} \rangle$.

1.2.4 The spaces $\mathrm{FIN}_k^{[\infty]}$

Next, we introduce a collection of topological Ramsey spaces that contain infinite sequences of functions. The space $\mathrm{FIN}_1^{[\infty]}$, also denoted simply as $\mathrm{FIN}^{[\infty]}$, is connected with the famous Hindman's Theorem [41]. Milliken later proved that it forms a topological Ramsey space [62]. The general spaces for $k \geq 2$ are based on work of Gowers in [37]. The presentation here comes from [71].

Definition 1.25. For a positive integer k, define

$$\mathrm{FIN}_k = \{f : \mathbb{N} \longrightarrow \{0, 1, ..., k\} : \{n : f(n) \neq 0\} \text{ is finite and } k \in \mathrm{range}(f)\}.$$

We consider FIN_k a partial semigroup under the operation of taking the sum of two disjointly supported elements of FIN_k. For $f \in \mathrm{FIN}_k$, let $\mathrm{supp}(f) = \{n : f(n) \neq 0\}$. A block sequence of members of FIN_k is a (finite or infinite) sequence $F = (f_n)$ such that

$$\max \mathrm{supp}(f_m) < \min \mathrm{supp}(f_n) \text{ whenever } m < n.$$

For $1 \leq d \leq \infty$, let $\mathrm{FIN}_k^{[d]}$ denote the collection of all block sequences of length d. The notion of a partial subsemigroup generated by a given block sequence depends on the operation $T : \mathrm{FIN}_k \longrightarrow \mathrm{FIN}_{k-1}$ defined as follows:

$$T(f)(n) = \max\{f(n) - 1, 0\}.$$

Given a finite or infinite block sequence $F = (f_n)$ of elements of FIN_k and an integer j $(1 \leq j \leq k)$, the partial subsemigroup $[F]_j$ of FIN_j generated by F is the collection of members of FIN_j of the form

$$T^{(i_0)}(f_{n_0}) + ... + T^{(i_l)}(f_{n_l})$$

for some finite sequence $n_0 < ... < n_l$ of non negative integers and some choice $i_0, ..., i_l \in \{0, 1, ..., k\}$. For $F = (f_n)$, $G = (g_n) \in \mathrm{FIN}_k^{[\leq\infty]}$, set $F \leq G$ if $f_n \in [G]_k$ for all n less than the length of the sequence F. Whenever $F \leq G$, we say that F is a *block-subsequence* of G. The partial ordering $\leq$ on $\mathrm{FIN}_k^{[\infty]}$ allows the finitization $\leq_{\mathsf{fin}}$: For $F, G \in \mathrm{FIN}_k^{[<\infty]}$,

$$F \leq_{\mathsf{fin}} G \text{ if and only if } F \leq G \textit{ and } (\forall l < \mathrm{length}(G)) F \not\leq G \restriction l.$$

Theorem 1.26 ([71])**.** *For every positive integer k, the triple $(\mathrm{FIN}_k^{[\infty]}, \leq, r)$ is a topological Ramsey space.*

The space $\mathrm{FIN}_1^{[\infty]}$, also denoted simply as $\mathrm{FIN}^{[\infty]}$, was proved to be a topological Ramsey space by Milliken in [62]. This was the first space that was built on the basis of a substantially different pigeonhole principle, according to Todorcevic in [71]. Its power over the Ellentuck space was not fully realized until Gower's succesful applications of the "Block Ramsey theory" when treating some problems from Banach space geometry.

The ultrafilter $\mathcal{U}_{\mathrm{FIN}^{[\infty]}}$ associated with the space $\mathrm{FIN}^{[\infty]}$ is exactly a stable ordered-union ultrafilter, in the terminology of [10]. Given $f \in \mathrm{FIN}$, let $\min(f)$ denote the minimum of the support of f, and let $\max(f)$ denote the maximum of the support of f. In [10], Blass showed that the min and max projections of the ultrafilter $\mathcal{U}_{\mathrm{FIN}^{[\infty]}}$ are selective ultrafilters which are Rudin-Keisler incomparable. In [28], the analogous result for the Tukey order was shown. More recently, Mildenberger showed in [59] that forcing with $\langle \mathrm{FIN}_k^{[\infty]}, \leq^* \rangle$ produces an ultrafilter with at least $k+1$-near coherence classes of ultrafilters Rudin-Keisler below it.

1.2.5 The Carlson Simpson dual space

The first version of the Abstract Ellentuck Theorem appeared in a paper of Carlson-Simpson, using structures less general than those studied by Todorčević. The dual Ramsey theory was developed by T.J. Carlson and S.G. Simpson in [17]. In this paper they establish a combinatorial theorem which is called the dual of Ramsey's Theorem. The dual form is concerned with colorings of the k-element partitions of a fixed infinite set. Now we will introduce the Carlson-Simpson space. This space is called a dual space because it is dual to the Ellentuck space. While the Ellentuck space is defined by using injective functions from ω to ω, the Carlson Simpson space is defined by using surjectuve functions from ω to ω.

Let $\mathcal{E}_\infty$ be the collection of all equivalence relations E on ω with infinitely many equivalence classes. Each class $[x]_E$ of E has a minimal representative. Let $p(E)$ be the set of all minimal representatives of classes of E. Let $\{p_n(E)\}_{n=0}^{\infty}$ be the increasing enumeration of $p(E)$. Note that $0 \in p(E)$ for all $E \in \mathcal{E}_\infty$, so we have that $p_0(E) = 0$ for all $E \in \mathcal{E}_\infty$.

For $E, F \in \mathcal{E}_\infty$ we say that E is *coarser* than F and write $E \leq F$ if every class of E can be represented as the union of certain set of classes of F. The n-th *approximation* $r_n(E)$ to some $E \in \mathcal{E}_\infty$ is defined as follows:

$$r_n(E) = E \restriction p_n(E).$$

Thus, $r_n(E)$ is simply the restriction of the equivalence relation E to the finite set $\{0, 1, ..., p_n(E) - 1\}$ of integers. Each approximation $a \in \mathcal{AE}_\infty$ has its *length* $|a|$, the integer n such that $a = r_n(E)$ for some $E \in \mathcal{E}_\infty$ (or equivalently, the number of equivalence classes of a) and its *domain*, the integer $p_{|a|}(E) = \{0, 1, ..., p_{|a|}(E) - 1\}$, where E is some member of $\mathcal{E}_\infty$ such that $a = r_{|a|}(E)$. The relation $\leq$ of $\mathcal{E}_\infty$ allows a natural finitization $\leq_{\mathsf{fin}}$ on $\mathcal{AE}_\infty$ satisfying **A.2** and **A.3**: $a \leq_{\mathsf{fin}} b$ if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and a is coarser than b.

We can find the proof of the following result as the Theorem 5.70 in [17] and [71]

Theorem 1.27 (Carlson-Simpson)**.** *The space $(\mathcal{E}_\infty, \leq, r)$ is a topological Ramsey space.*

If $E \in \mathcal{E}_\infty$ we define by recursion a function $f_E : \omega \to \omega$. Let $f_E(0) = 0$. Let $n \in \omega$ be fixed and suppose that for every $i \in n$ we have defined $f_E(i)$, if there is an $i \in n$ such that n and i belong to the same member of the partition, let $f_E(n) = f_E(i)$, otherwise $f_E(n) = \max\{f_E(i) : i \in \omega\} + 1$. Note that we can identify E with the function f_E. Let $h : \omega \to \omega$ be a surjective function such that for every $m \in \omega$, if i is the least natural number such that $h(i) = m$ and j is the least natural number such that $h(j) = m + 1$ then $i < j$. Then the sets $h_i^{-1}(i)$ form a partition of ω; we will denote such a partition as $E_h \in \mathcal{E}_\infty$.

There is a natural correspondence between members of $\mathcal{E}_\infty$ and the collection of all the surjective functions $h : \omega \to \omega$ such that for every $m \in \omega$, if i is the least natural number such that $h(i) = m$ and j is the least natural number such that $h(j) = m + 1$ then $i < j$. Let g, h be functions with properties mentioned above, note that $X_g \leq X_h$ if and only if there exists a function with properties mentioned above f such that $f \circ h = g$ if and only if for each $n, m \in \omega$ such that $h(n) = h(m)$, $g(n) = g(m)$.

In [55], Matet studies the partial order $(\mathcal{E}_\infty, \leq)$ as a lattice and proves the following.

Theorem 1.28. *[55] $(\mathcal{E}_\infty, \leq^*)$ is a σ-closed partial order.*

1.2.6 Ramsey space of strong subtrees

In this subsection, by a *tree* we mean a rooted finitely branching tree of some height $\leq \omega$. Given a tree T and $n \in \omega$, let $T(n)$ denote the n-th level of T. Thus, the height of T is simply the minimal $n \leq \omega$ such that $T(n) = \emptyset$. For a set $A \subseteq \omega$, let $T(A) = \bigcup_{n \in A} T(n)$. Let U be a fixed rooted finitely branching tree of height ω and we study its terminal nodes and we study its subtrees. We say that T is a strong subtree of U if

- $T \subseteq U$ and with the induced ordering T is a rooted tree that in general can be of finite height.
- Every level $T(n)$ of T is a subset of some level $U(m)$ of U.
- There exists $A \subseteq \omega$ such that
    a) for all $n \in A$, $T \subseteq U(A)$ and $T \cap U(n) \neq \emptyset$,
    b) if $m < n$ are two succesive elements of the set A, then for every $s \in T \cap U(m)$, every immediate successor of s in U has exactly one extension in $T \cap U(n)$.

Let $S_\infty(U)$ denote the collection of all strong subtrees of U of infinite height, and for a positive integer k, let $S_k(U)$ denote the collection of all strong subtrees of U of height k. For $T \in S_\infty(U)$ the sequence $(r_n(T))$ of finite approximations is defined as follows:

$$r_n(T) = \bigcup_{m<n} T(m). \tag{1.2.3}$$

Thus, the set of finite approximations to elements of $S_\infty(U)$ is the set

$$S_{<\infty}(U) = \bigcup_{n \in \omega} S_n(U) \tag{1.2.4}$$

of strong subtrees of U of finite heights. Note that $S_\infty(U)$ becomes a closed subset of the Tychonov product of $S_{<\infty}(U)^\omega$.

Theorem 1.29 (Milliken)**.** *For every rooted finitely branching tree U of height ω with no terminal nodes, the triple $(S_\infty(U), \subseteq, r)$ forms a topological Ramsey space.*

1.3 Filters and ideals

Ideals on countable sets have been studied and classified for different purposes. In [46], Hrŭsák classifies ideals by using the Katétov order to characterize definable ideals. The Boolean algebra $\mathcal{P}(\omega)/\text{Fin}$ has been deeply studied. It is known that $\mathcal{P}(\omega)/\text{Fin}$ is forcing equivalent with the Ellentuck space with the almost inclusion order. $\mathcal{P}(\omega)/\text{Fin}$ forces a selective ultrafilter, moreover, in [21] Todorčević proved using large cardinals that all selective ultrafilters are $\mathcal{P}(\omega)/\text{Fin}$-generic over $L(\mathbb{R})$. Motivated by forcing and combinatorial properties of $\mathcal{P}(\omega)/\text{Fin}$, partial orders $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is an ideal are studied from different viewpoints. In works such as [45] and [44] Hrušák and Zapletal, and Hrušák and Verner study which ultrafilters are added by quotients $\mathcal{P}(\omega)/\mathcal{I}$.

An *ideal* on a set X is a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ closed under subsets and finite intersections.

For this work we are interested in ideals on countable sets; hence the ideals we work with we will be assumed to be on ω. If $\mathcal{I}$ is an ideal, we will denote by $\mathcal{I}^+$ the collection of sets that don't belong to $\mathcal{I}$. We will identify $\mathcal{P}(\omega)$ with 2^ω, so we can view ideals as subsets of 2^ω with the Tychonoff topology.

We write $\omega \to (\mathcal{I}^+)^m_n$ to mean that for every $c : [\omega]^m \to k$ there is an $\mathcal{I}$-positive set Y such that $|c''[Y]^m| \leq n$. Similarly, $\mathcal{I}^+ \to (\mathcal{I}^+)^m_n$ denotes that for every $\mathcal{I}$-positive set X and every coloring $c : [X]^m \to k$ there is an $\mathcal{I}$-positive set $Y \subseteq X$ such that $|c''[Y]^m| \leq n$.

The following ideals are Borel ideals on countable sets.

- The ideal Fin is the collection of finite subsets of ω. Fin is an F_σ ideal.

- The ideal $\text{Fin} \times \text{Fin}$ contains sets $A \subseteq \omega \times \omega$ such that

  $$|\{m \in \omega : |\{n \in \omega : (m, n) \in A\}| = \omega\}| < \omega.$$

  The ideal $\text{Fin} \times \text{Fin}$ is an $F_{\sigma\delta\sigma}$ ideal.

- If $n \in \omega$, given Fin^n define the ideal

  $$\text{Fin}^{n+1} = \{A \subseteq \omega^n : \{m \in \omega : A_n \notin \text{Fin}^n\} \in \text{Fin}\}$$

  where $A_n = (\{n\} \times \omega^n) \cap A$.

- The *eventually different ideal* $\mathcal{ED}$ consists of sets $A \subseteq \omega \times \omega$ such that

  $$(\exists m, n \in \omega)(\forall k > m)(|\{l \in \omega : (k, l) \in A\}| < n).$$

- The ideal $\mathcal{ED}_{\mathsf{fin}} = \{A \cap \Delta : A \in \mathcal{ED}\}$ where $\Delta = \{(m, n) \in \omega \times \omega : m \leq n\}$. $\mathcal{ED}_{\mathsf{fin}}$ is an F_σ ideal.

- Given a function $f : \omega \to \mathbb{R}^+$ tending to zero such that $\sum_{n<\omega} f(n) = \infty$, we define $\mathcal{I}_f = \{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\}$ and we call an ideal *summable* if $\mathcal{I} = \mathcal{I}_f$ for some such function. Ideals $\mathcal{I}_f$ are F_σ P-ideals.

- The ideal $\mathcal{I}_{\frac{1}{n}}$ is the *summable ideal*. Note that in this case $f(n) = \frac{1}{n}$.

- Let $\Omega = \{A \in Clop(2^\omega) : \lambda(A) = \frac{1}{2}\}$. Solecki's ideal $\mathcal{S}$ is generated by sets $I_x = \{A \in \Omega : x \in A\}$, with $x \in 2^\omega$. λ denotes the Haar measure on 2^ω. $\mathcal{S}$ is an F_σ ideal.

- Let n be a natural number, we will define the n-th hypercube ideal $\mathcal{I}_n$. Fix a collection $\{a_{k,j} : k \in \omega, j < n\} \subseteq [\omega]^{<\omega}$ such that for every $k \in \omega$, $a_{k,j} \in [\omega]^{k+1}$ and $\max\{a_{k,j} : j < n\} < \min\{a_{k+1,j} : j < n\}$. For every $k \in \omega$ let $\mathbf{a}_k = \prod_{j<n} a_{k,j}$ and $\mathbb{A} = \bigcup_{k\in\omega} \mathbf{a}_k \subseteq \omega^n$. The n-th hypercube ideal $\mathcal{I}_n$ is such that if $A \subseteq \mathbb{A}$, then $A \notin \mathcal{I}_n$ if $(\forall k \in \omega)(\exists s_0, s_1, ..., s_{n-1} \in [\omega]^k)(\prod_{i\in n} s_i \subseteq A)$.

- The ideal **conv** is generated by sequences on $\mathbb{Q} \cap [0,1]$ which are convergent in $[0,1]$. **conv** is an $F_{\sigma\delta\sigma}$ ideal.

- The *nowhere dense ideal* nwd is the ideal on the set of rational numbers $\mathbb{Q}$ whose elements are the nowhere dense subsets of $\mathbb{Q}$.

- The ideal $\mathcal{Z}$ of subsets of ω of *asymptotic density zero* is the ideal

$$\mathcal{Z} = \{A \subseteq \omega : lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\}.$$

- The *random graph* on ω is the graph on ω that satisfies the following property: Given F and G disjoint finite subsets on ω there is $k < \omega$ such that $\{\{k,l\} : l \in F\} \subseteq E$ and $\{\{k,l\} : l \in F\} \cap E = \emptyset$ where E is the collection of edges of the graph. We denote by $\mathcal{R}$ the ideal on ω generated by the homogeneous sets (cliques and free sets) in Rado's random graph.

- Let $\mathcal{G}_c$ be the ideal such that $I \notin \mathcal{G}_c$ if there exists some $A \in [\omega]^\omega$ such that $[A]^2 \subseteq I$. This ideal is called the *complete graph ideal*.

We say that an ideal $\mathcal{I}$ is *tall* if for every $X \in [\omega]^\omega$, there exists some $I \in \mathcal{I}$ such that $|X \cap I| = \omega$. Note that Fin is not a tall ideal.

A *submeasure* on ω is a function $\varphi : \mathcal{P}(\omega) \to [0,\infty)$ such that

1. $\varphi(\emptyset) = 0$,

2. if $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$,

3. $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

We will assume that for every $a \in [\omega]^{<\omega}$, $\varphi(a) < \infty$. We say that φ is a *lower semicontinuous submeasure* (lssm) if for every $A \subseteq \omega$,

$$\varphi(A) = lim_{n\to\infty}\varphi(A \cap n).$$

To each lscsm φ, there naturally corresponds the following ideal

$$Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}.$$

$Fin(\varphi)$ is an F_σ ideal, and by Mazur's Theorem we know that for every F_σ ideal $\mathcal{I}$ there is some lscsm φ such that $\mathcal{I} = Fin(\varphi)$.

We say that an ideal is *locally* F_σ if for every $X \in \mathcal{I}^+$ there exists some $Y \subseteq X$ such that $Y \in \mathcal{I}^+$ and $\mathcal{I} \restriction Y$ is F_σ.

Let $\mathcal{I}$ be an F_σ ideal. $\mathcal{I}$ is *fragmented* if there is a lscm φ with $\mathcal{I} = Fin(\varphi)$ and a sequence $\{t_i : i \in \omega\} \subseteq [\omega]^{<\omega}$ such that $\sup \varphi(t_n) = \infty$ and for every $X \subseteq \bigcup_{i\in\omega} t_i$, $\varphi(t) = \sup \varphi(X \cap t_n)$.

If $\mathcal{I}$ is an ideal on ω, we say that $\mathcal{I}$ is a

- P-*ideal* if for any sequence $\langle X_n : n \in \omega\rangle$ of members of $\mathcal{I}$, there is an $X \in \mathcal{I}$ such that for all $n \in \omega$, $X_n \subseteq^* X$.
- P^+-*ideal* if for every decreasing sequence $\langle X_n : n \in \omega\rangle$ of $\mathcal{I}$-positive sets there is an $\mathcal{I}$-positive set X such that for all $n \in \omega$, $X \subseteq^* X_n$.

The following fact was first observed in [47]

Lemma 1.30 (Just, Krawczyk)**.** *Every F_σ ideal is P^+.*

An *ultrafilter* on ω is a collection $\mathcal{U} \subseteq \mathcal{P}(\omega) \setminus \{\emptyset\}$ which is closed under supersets and finite unions, and for every $X \in \mathcal{P}(\omega)$, $X \in \mathcal{U}$ holds or $\omega \setminus X \in \mathcal{U}$ holds.

If $\mathcal{U}$ is an ultrafilter on ω we say that:

- $\mathcal{U}$ is a *P-point* if for any sequence $\langle X_n : n \in \omega\rangle \subseteq \mathcal{U}$ there is an $X \in \mathcal{U}$ such that for every $n \in \omega$, $X \subseteq^* X_n$.
- $\mathcal{U}$ is a *Q-point* if for every partition of ω into finite pieces $\{I_n : n \in \omega\}$, there exists an $X \in \mathcal{U}$ such that for each $n \in \omega$, $X \cap I_n$ has at most one element.
- $\mathcal{U}$ is *selective* if for every function $f : \omega \to \omega$ there exists an $X \in \mathcal{U}$ such that $f \restriction X$ is constant or one-to-one.
- $\mathcal{U}$ is *Ramsey* if for every $m \in \omega$ and $c : [\omega]^2 \to m$ coloring, there exists $X \in \mathcal{U}$ such that $|c \restriction [X]^2| = 1$.
- $\mathcal{U}$ is *weakly Ramsey* if for every $m \in \omega$ and $c : [\omega]^2 \to m$ coloring, there exists $X \in \mathcal{U}$ such that $|c \restriction [X]^2| \leq 2$.

Weakly Ramsey properties are interesting because some ultrafilters are not Ramsey, which means that for some colorings it is impossible to find homogeneous sets in the ultrafilter, but for every coloring it is possible to find some set in the ultrafilter that uses only a fixed number of colors.

Proposition 1.31. *Let $\mathcal{U}$ be an ultrafilter.*

- *$\mathcal{U}$ is a Ramsey ultrafilter if and only if $\mathcal{U}$ is a selective.*
- *$\mathcal{U}$ is a P-point if and only if for every function $f : \omega \to \omega$ there exists an $X \in \mathcal{U}$ such that $f \restriction X$ is constant or finite to one.*
- *$\mathcal{U}$ is a Q-point if and only if for every finite-to-one function $f : \omega \to \omega$ there exists an $X \in \mathcal{U}$ such that $f \to X$ is constant or one-to-one.*

For a proof of the last Proposition the reader can see [3]. The following Theorem is proved in [44].

Theorem 1.32 (Hrušák, Verner)**.** *Suppose $\mathcal{I}$ is analytic and $\mathcal{P}(\omega)/\mathcal{I}$ adds no new reals. Then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point if and only if $\mathcal{I}$ is locally F_σ.*

The following Theorem is proved in [43].

Theorem 1.33 (Hrušák, Meza-Alcántara, Thümmel and Uzcátegui)**.** *An analytic ideal $\mathcal{I}$ is P^+ if and only if $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed and locally F_σ.*

1.4 The Katětov order on ideals

In this section we introduce the Katětov order on ideals and some basic properties. Definitions and properties are taken from [46]. The Katětov order has been crucial for classifying ideals. In particular, it has been possible to classify undefinable ideals by using definable ideals and the Katětov order.

Definition 1.34. Let $\mathcal{I}, \mathcal{J}$ be Borel ideals. We say that $\mathcal{I}$ is *Katětov below* $\mathcal{J}$ if there is a function $f : \omega \to \omega$ such that for every $I \in \mathcal{I}$, $f^{-1}[I] \in \mathcal{J}$; in this case we write $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$.

If f is finite to one we write $\mathcal{I} \leq_{KB} \mathcal{J}$ and we will say that $\mathcal{I}$ is *Katětov-Blass below* $\mathcal{J}$. Whenever $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$ we will say that $\mathcal{I}$ and $\mathcal{J}$ are *Katětov equivalent* and we will write $\mathcal{I} \approx_K \mathcal{J}$.

The following are basic properties of the Katětov order. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω.

1. $\mathcal{I} \approx_K \mathsf{fin}$ if and only if $\mathcal{I}$ is not tall.
2. If $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \leq_K \mathcal{J}$.
3. If $X \in \mathcal{I}^+$ then $\mathcal{I} \leq_K \mathcal{I} \restriction X$.

Definition 1.35. An ideal $\mathcal{I}$ on ω is *K-uniform* if for every $X \in \mathcal{I}^+$, $\mathcal{I} \approx_K \mathcal{I} \restriction X$.

The ideal $\mathcal{ED}_{\mathsf{fin}}$ satisfies that it is F_σ, tall and K-uniform. In [46], Hrušák asked if $\mathcal{ED}_{\mathsf{fin}}$ is the only tall F_σ ideal which is K-uniform. In [35], Gomez proved that there exists an ideal which is F_σ and K-uniform but it is not Katětov equivalent to $\mathcal{ED}_{\mathsf{fin}}$. Hrušák reformulated his question by asking if there is an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$ into F_σ, tall and K-uniform ideals ordered by the Katětov order. In Chapter 3 we prove that there is a $\leq_K$-chain of F_σ, tall and K-uniform ideals.

In [46], Hrušák proved the following Katětov relations:

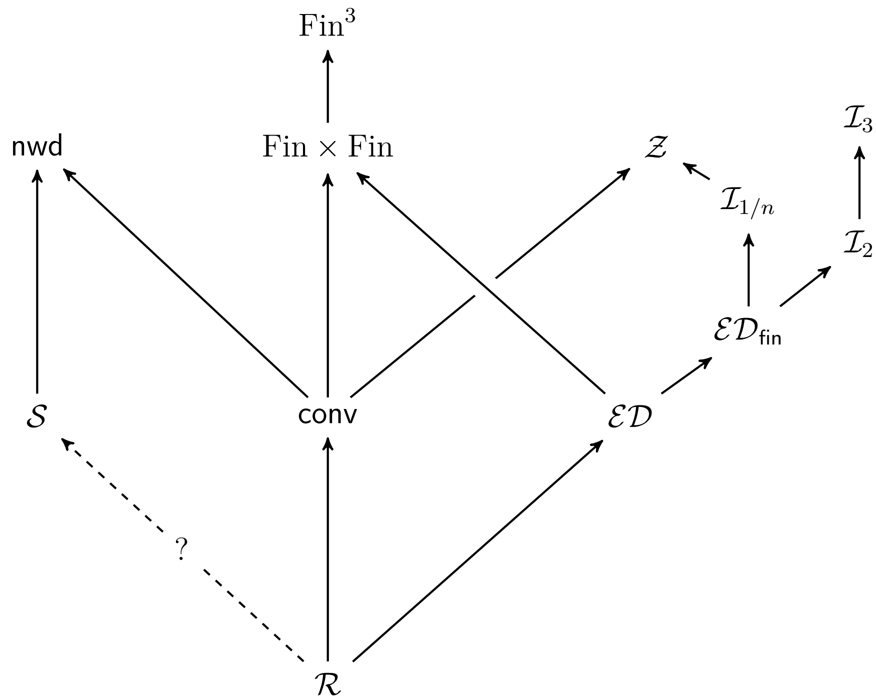
Proposition 1.36. *The following Katětov relations hold:*

1. $\mathcal{R} \leq_K \mathcal{ED}, \mathsf{conv}$.
2. $\mathsf{conv} \leq_K \mathrm{Fin} \times \mathrm{Fin}$ *and* $\mathcal{ED} \leq_K \mathcal{ED}_{\mathsf{fin}}$.
3. $\mathsf{conv} \leq_K \mathsf{nwd}$, $\mathcal{ED} \leq_K \mathrm{Fin} \times \mathrm{Fin}$ *and* $\mathcal{I}_{\frac{1}{n}} \leq_k \mathcal{Z}$.
4. $\mathcal{ED}_{\mathsf{fin}} \leq_K \mathcal{I}_{\frac{1}{n}}$.
5. $\mathsf{conv} \leq_K \mathcal{Z}$.
6. $\mathcal{S} \leq_K \mathsf{nwd}$.
7. *For every* $n \in \omega$, $\mathrm{Fin}^n \leq_{\mathrm{K}} \mathrm{Fin}^{n+1}$.
8. $\mathrm{Fin} \times \mathrm{Fin} \leq_K \mathcal{G}_c$.
9. *For every* $n \in \omega$, $\mathcal{I}_n \leq_{\mathrm{K}} \mathcal{I}_{n+1}$.

1)-6) are proved in [46] and 8) is proved in [57].

In [46], Hrušák proves some negative results:

- None of $\mathcal{S}, \mathcal{R}, \mathcal{ED}, \mathcal{ED}_{\mathsf{fin}}$ and $\mathcal{I}_{\frac{1}{n}}$ is Katětov above conv (and hence also not above nwd, $\mathrm{Fin} \times \mathrm{Fin}$ and $\mathcal{Z}$).
- None of the ideals in the diagram is above $\mathrm{Fin} \times \mathrm{Fin}$.
- $\mathcal{ED} \not\leq_K \mathsf{nwd}$.
- Neither $\mathrm{Fin} \times \mathrm{Fin}$ nor $\mathcal{Z}$ are above $\mathcal{S}$. In particular, none of the ideals in the diagram is above nwd.
- $\mathcal{ED}_{\mathsf{fin}} \not\leq_K \mathrm{Fin}\times\mathrm{Fin}$. From this (and previous observations) it immediately follows that none of the ideals is above $\mathcal{Z}$ and also that $\mathcal{ED}_{\mathsf{fin}} \not\leq_K \mathcal{ED}$.
- $\mathcal{I}_{\frac{1}{n}} \not\leq_K \mathcal{ED}_{\mathsf{fin}}$.



1.5 Sacks Forcing

The following definitions and results are taken from [4] and [66].

Definition 1.37. Let $p \subseteq 2^{<\omega}$ be a tree ordered with the end-extension relation $\sqsubset$. We say that p is *perfect* if every $s \in p$ has incomparable extensions $t, u \in p$.

For a perfect tree p, let $[p] = \{f \in 2^\omega : (\forall n \in \omega)(f \restriction n \in p)\}$ be the set of all infinite branches of p. Note that $[p]$ is a perfect set.

Definition 1.38 (Sacks)**.** Sacks forcing $\mathbb{S}$ is the set of all perfect trees, ordered by $p \leq q$ if $p \subseteq q$.

Lemma 1.39 (Baumgartner, [4])**.** *If $\mathcal{G}$ is the $\mathbb{S}$-generic filter over $\mathbf{V}$, then $f_{\mathcal{G}} = \bigcup\{s \in 2^{<\omega} : \forall p \in \mathcal{G}(s \subset p)\} \in 2^\omega$, $f_{\mathcal{G}} \notin \mathbf{V}$, and $\mathbf{V}[f_{\mathcal{G}}] = \mathcal{G}$.*

The real $f_{\mathcal{G}}$ is called a Sacks real.

Let $p \in \mathbb{S}$ and $s \in p$. The *branching level* of s in p is the cardinality of

$$\{i < \lg s : (\exists t \in p)(\lg t > i, t \restriction i = s \restriction i \text{ and } t \restriction (i+1) \neq s \restriction (i+1))\}.$$

So, the branching level of s is the number of times branching has occurred below s in the tree p. The nth *branching level of* p, $l(n,p)$ is defined to be the set of all $s \in p$ which have branching level n and are minimal with that property, which means that if $t \subseteq s$ has forking level also, then $t = s$. Note that $|l(n,p)| = 2^n$. For $p, q \in \mathbb{S}$, let $p \leq_n q$ if and only if $p \leq q$ and $l(n,p) = l(n,q)$.

Lemma 1.40. *Suppose $\langle p_n : n \in \omega\rangle$ and $\langle m_n : n \in \omega\rangle$ are sequences such that $p_n \in \mathbb{S}$, the m_n are non-decreasing, $\lim_{n\to\infty} = \infty$ and for all $n \in \omega$, $p_{n+1} \leq_{m_n} p_n$. Then $q = \bigcup\{p_n : n \in \omega\} \in \mathbb{S}$ and for all $n \in \omega$, $q \leq_{m_n} p_n$.*

In general, if $\langle p_n : n \in \omega\rangle$ and $\langle m_n : n \in \omega\rangle$ are as in the previous lemma, then we refer to $\langle p_n : n \in \omega\rangle$ as a *fusion sequence*, and we call q the *fusion* of the sequence.

If $p \in \mathbb{S}$ and $s \in p$, let $p \restriction s = \{t \in p : t \subseteq s \text{ or } s \subseteq t\}$. Note that $p \restriction s \in \mathbb{S}$.

The following corollary appears in [4].

Corollary 1.41. *Suppose $p \in \mathbb{S}(\kappa)$, $F \subset domain(p)$ is finite, and $n \in \omega$. If $p \Vdash$ "$\tau \in \mathbf{V}$", then there exists $q \leq_{F,n} p$ such that $\forall \sigma \in l(F, n, p)$ there exists $a_\sigma \in \mathbf{V}$ satisfying $q \restriction \sigma \Vdash$ "$\tau = a_\sigma$".*

1.6 Pseudointersection and tower numbers

The combinatorial structure of $\mathcal{P}(\omega)/\text{Fin}$ has been deeply studied and it is closely related to the cardinal invariants of the continuum. In recent years research has focused on structures similar to $\mathcal{P}(\omega)/\text{Fin}$ and their natural version of cardinal invariants. We mention some examples of this. In [52], Majcher-Iwanow studies cardinal invariants of the lattice of partitions. In [14], Brendle studies van Douwen's diagram related to the structure $\text{Dense}(\mathbb{Q})/\mathsf{nwd}$. In [2], Balcar, Hernández-Hernández and Hrušák investigate cardinal invariants of the structure $(\text{Dense}(\mathbb{Q}), \subseteq)$.

Definition 1.42.

1. For two sets $X, Y \subseteq \omega$ we say that X is *almost contained* in Y, denoted $X \subseteq^* Y$, if $X \setminus Y$ is finite.
2. We say that a family of sets $\mathcal{F} \subseteq [\omega]^\omega$ has the *strong finite intersection property* (SFIP) if for every finite subfamily $\mathcal{X} \in [\mathcal{F}]^{<\omega}$, $\bigcap \mathcal{X}$ is an infinite subset of ω.
3. Given $\mathcal{F} \subseteq [\omega]^\omega$, a pseudointersection of the family $\mathcal{F}$ is a set $Y \in [\omega]^\omega$ such that for every $X \in \mathcal{F}$, $Y \subseteq^* X$.
4. The *pseudointersection number* $\mathfrak{p}$ is the smallest cardinality of a family $\mathcal{F} \subseteq [\omega]^\omega$ which has the SFIP but does not have a pseudointersection.
5. A *tower* is a sequence $\langle X_\alpha : \alpha < \delta\rangle$ of members of $[\omega]^\omega$ which is linearly ordered by $\supseteq^*$ and has no pseudointersection. The *tower number* $\mathfrak{t}$ is the smallest cardinality of a tower.

It is well-known that Martin's Axiom implies $\mathfrak{p} = \mathfrak{c}$; see for instance [3] for a proof. The cardinal invariant $\mathfrak{m}(\sigma\text{-centered})$ is defined to be the minimum cardinal κ for which there exists a σ-centered partial order $\mathcal{P}$ and a family $\mathcal{D}$ of κ many dense subsets of $\mathcal{P}$ which does not admit any $\mathcal{D}$-generic filter. Bell proved that $\mathfrak{m}(\sigma\text{-centered}) = \mathfrak{p}$ (see [7]). It is clear from the definitions that $\mathfrak{p} \leq \mathfrak{t}$, and one of the most important longstanding open problems in cardinal invariants was whether the two are equal. Malliaris and Shelah recently proved that, indeed, $\mathfrak{p} = \mathfrak{t}$ (see [53] and [54]).

Chapter 2

Ramsey spaces and ultrafilters

It is known that the Ellentuck space forces a Ramsey ultrafilter. Todorčević established that the connection between Ramsey spaces and Ramsey ultrafilters is stronger than expected when he proved using large cardinals that selective ultrafilters are generic over $L(\mathbb{R})$ for the poset $\mathcal{P}(\omega)/\mathrm{Fin}$, which is forcing equivalent to the Ellentuck space with the almost inclusion order ([21]). As we show in examples below, generic filters forced by topological Ramsey spaces have interesting partition properties. Connections between topological Ramsey spaces and ultrafilters with partition properties have been studied in different directions. In [19] Di Prisco, Mijares, and Nieto extend Todorčević's result by proving that semiselective ultrafilters on topological Ramsey spaces are generic over $L(\mathbb{R})$ (under large cardinals hypotheses). In [75] Zheng proved that selective ultrafilters on FIN are preserved under Sacks forcing and in [77] she proved the same result for Ramsey spaces $\mathcal{R}_n$ and $\mathcal{E}_\alpha$.

2.1 Ultrafilters forced by topological Ramsey spaces

It is known that the Ellentuck space and the Ramsey ultrafilters are closely related. Not only the Ellentuck space forces a Ramsey ultrafilter but also Ramsey ultrafilfers are generic for the Ellentuck space with the almost inclusion over $L(\mathbb{R})$. Since the Ellentuck space is the prototypical example of a topological Ramsey space it is interesting to ask if every space forces a Ramsey ultrafilter. In this section we prove that every topological Ramsey space satisfying Todorčević axioms which does not add reals forces a Ramsey ultrafilter.

For a family $\mathcal{H} \subset \mathcal{R}$ and an element $X \in \mathcal{R}$, let $\mathcal{H} \restriction X = \{Y \in \mathcal{F} : Y \leq X\}$.

Theorem 2.1. *Let $(\mathcal{R}, \leq, r)$ be a closed triple that satisfies* $\mathbf{A.1} - \mathbf{A.4}$*. Also let $\leq^*$ be an order on $\mathcal{R}$ coarsening $\leq$ such that $(\mathcal{R}, \leq)$ and $(\mathcal{R}, \leq^*)$ have isomorphic separative quotients and do not add reals. Then $(\mathcal{R}, \leq^*)$ forces a Ramsey ultrafilter.*

Proof. Let $\mathcal{G}$ be an $(\mathcal{R}, \leq^*)$-generic filter. In the extension $V[\mathcal{G}]$, fix some $Y \in \mathcal{G}$. For every $X \in \mathcal{G}$ let $D(X) = \{\mathrm{depth}_Y(a) : a \in (\mathcal{AR}_1 \restriction X) \cap (\mathcal{AR}_1 \restriction Y)\}$. Note that if $X, X' \in \mathcal{G}$, there exists some $Z \leq X, X'$ in $\mathcal{G}$. Also note that if $X \leq Z$ then $D(X) \subset D(Z)$. Therefore the collection of sets $D(X)$ with $X \in \mathcal{G}$ generates a filter. Let $\mathcal{U}$ be the filter on ω generated by sets $D(X)$. To prove that $\mathcal{U}$ is an ultrafilter, fix some $A \subseteq \omega$. Let $\mathcal{D}_0 = \{X \in \mathcal{R} : X$ is incompatible with Y or $D(X) \subseteq A$ or $D(X) \subseteq \omega \setminus A\}$. Now we prove that $\mathcal{D}_0$ is dense in $\mathcal{R}$. Fix $Z \in \mathcal{R}$. If Z is incompatible with Y then $Z \in \mathcal{D}_0$. Otherwise, define families $\mathcal{F}_0 = \{a \in (\mathcal{AR}_1 \upharpoonright Z) \cap (\mathcal{AR}_1 \upharpoonright Y) : \mathrm{depth}_Y(a) \in A\}$ and $\mathcal{F}_1 = \{a \in (\mathcal{AR}_1 \upharpoonright Z) \cap (\mathcal{AR}_1 \upharpoonright Y) : \mathrm{depth}_Y(a) \in \omega \setminus A\}$. By the Abstract Nash-Williams Theorem 1.5 there are some $X \leq Z$ and $i \in 2$ such that $\mathcal{AR}_1 \upharpoonright X \subseteq \mathcal{F}_i$. By the genericity of $\mathcal{G}$, there exists some $X \in \mathcal{G}_0 \cap \mathcal{D}$. Therefore $D(X) \in \mathcal{U}$ and $D(X) \subseteq A$ or $D(X) \subseteq \omega \setminus A$. In the first case we get that $A \in \mathcal{U}$ and in the second case $\omega \setminus A \in \mathcal{U}$.

Now we show that $\mathcal{U}$ is a Ramsey ultrafilter. Let $c : [\omega]^2 \to 2$ be a coloring. Define $\mathcal{D} = \{X \in \mathcal{R} : X$ is incompatible with Y or $|c \upharpoonright [D(X)]^2| = 1\}$. To see that $\mathcal{D}$ is dense in $\mathcal{R}$, take $X \in \mathcal{R}$, if X and Y are incompatible then $X \in \mathcal{D}$. Otherwise, for every $i \in 2$, define $\mathcal{F}_i = \{a \in \mathcal{AR}_2 \upharpoonright X : c(\mathrm{depth}_Y(r_1(a)), \mathrm{depth}_Y(a)) = i\}$. Note that axiom **A.2** part (c) guarantee that $\mathrm{depth}_Y(r_1(a)) \neq \mathrm{depth}_Y(a)$. Since $\mathcal{AR}_2 \upharpoonright X$ is a Nash-Williams family, by the Abstract Nash-Williams Theorem 1.5 there are some $Y \in \mathcal{R}$ with $Y \leq X$ and $i_0 \in 2$ such that $\mathcal{AR}_2 \upharpoonright Y \subseteq \mathcal{F}_{i_0}$. Then $D(Y)$ is homogeneous and $Y \in \mathcal{D}$. Since $\mathcal{G}$ is a generic filter, $\mathcal{G} \cap \mathcal{D} \neq \emptyset$ and there exists some $Z \in \mathcal{G} \cap \mathcal{D}$. Therefore $D(Z) \in \mathcal{U}$ and it is monochromatic. ■

Dobrinen also conjectures that every topological Ramsey space contains a copy of the Ellentuck space. We do not know if this is true or false.

2.2 Ramsey degrees

As seen in the previous section, many σ-closed forcings generating ultrafilters of interest have been shown to contain topological Ramsey spaces as dense subsets. The initial purpose for constructing those new Ramsey spaces was to find the exact Rudin-Keisler and Tukey structures below those ultrafilters. The Abstract Ellentuck Theorem proved to be vital to those investigations, which have been the subject of work in [29], [30], [26], [22], and [23]; the paper [24] provides an overview those results.

In this section we use the Abstract Ellentuck Theorem 1.5, to develope a general method to calculate Ramsey degrees for topological Ramsey spaces with certain homogeneity properties. These spaces consist of sequences of growing blocks. All the members of these spaces have the same structure. Since the Ramsey structure of the topological Ramsey spaces captures precisely the partition properties of ultrafilters forced by them, from the method mentioned above we also obtain the Ramsey degrees for those ultrafilters.

This property of homogeneity is satisfied by the Ellentuck space, the spaces $\mathcal{R}_n$, $1 \leq n < \omega$ (see Subsection 1.2.1), the spaces generated by Fraïssé classes with the Ramsey property (see Subsection 1.2.2) and the high dimensional Ellentuck spaces $\mathcal{E}_k$. Some of these spaces satisfy an additional property, called Independent Sequences of Structures. For these spaces we develop a general method to calculate their Ramsey degrees. We calculate Ramsey degrees for several ultrafilters associated to topological Ramsey spaces. This approach provides simple, direct proofs for some known Ramsey degrees, in particular the ultrafilters $\mathcal{U}_n$ of Laflamme in [50] mentioned in Subsection 1.2.1.

Definition 2.2. Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space such that for every $X \in \mathcal{R}$. We say that $\mathcal{R}$ is *weakly homogeneous* if for every $X, Y \in \mathcal{R}$ there is a one-to-one onto mapping $\varphi : \mathcal{AR}_1 \upharpoonright X \to \mathcal{AR}_1 \upharpoonright Y$ such that for every $Z \leq X$, $W \leq Y$ there are $W' \leq X$, $Z' \leq Y$ such that $\mathcal{AR}_1 \upharpoonright W' \subset \varphi[\mathcal{AR}_1 \upharpoonright Z]$ and $\mathcal{AR}_1 \upharpoonright Z' \subset \varphi[\mathcal{AR}_1 \upharpoonright W]$. $, \varphi^{-1}[\mathcal{AR}_1 \upharpoonright Z'] \in \mathcal{R}$, $\varphi(Z) \leq Y$ and $\varphi^{-1}(Z') \leq X$.

Definition 2.3. We say that a topological Ramsey space $(\mathcal{R}, \leq, r)$ is *homogeneous* if it is weakly homogeneous and $\mathcal{R}$ contains a strongest member.

Note that given a weakly homogeneous topological Ramsey space, we obtain a homogeneous topological Ramsey space by restricting below some member of the space.

Fix a topological Ramsey space $\mathcal{R}$ for which the almost reduction relation $\leq^*$ is a $\sigma-$closed partial order. Let $\mathcal{G}$ be the generic filter forced by $(\mathcal{R}, \leq^*)$ and let $\mathcal{U}_{\mathcal{R}}$ be the ultrafilter on base set $\mathcal{AR}_1$ generated by $\mathcal{G}$ which was presented in Definition 1.8. If $a \in \mathcal{AR}$ and $A \in \mathcal{R}$, we will write $[A]^n$ and $[a]^n$ to denote $[\mathcal{AR}_1 \upharpoonright A]^n$ and $[\mathcal{AR}_1 \upharpoonright a]^n$ respectively.

Definition 2.4. Given a topological Ramsey space $(\mathcal{R}, \leq, r)$, for $n \geq 1$, define

$$t(\mathcal{R}, n)$$

to be the least number t, if it exists, such that for each $l \geq 2$ and each coloring $c : [\mathcal{AR}_1]^n \to l$, there is a member $X \in \mathcal{R}$ such that the restriction of c to $[X]^n$ takes no more than t colors. Also, define

$$t(\mathcal{R}^+, n)$$

to be the least number t, if it exists, such that for each $l \geq 2$, $X \in \mathcal{R}$ and each coloring $c : [\mathcal{AR}_1]^n \to l$, there is a member $Y \in \mathcal{R}$ such that $Y \leq X$ and the restriction of c to $[Y]^n$ takes no more than t colors.

Remark. Every topological Ramsey space $(\mathcal{R}, \leq, r)$ satisfies

$$t(\mathcal{R}, n) \leq t(\mathcal{U}_{\mathcal{R}}, n) \leq t(\mathcal{R}^+, n)$$

for each $n \geq 2$.

Proof. First we will see that $t(\mathcal{R}, n) \leq t(\mathcal{U}_{\mathcal{R}}, n)$. Let $c : [\mathcal{AR}_1]^n \to m$ be a coloring. Define $\mathcal{D} = \{X \in \mathcal{R} : c \upharpoonright [\mathcal{AR}_1 \upharpoonright X]^n \leq t(\mathcal{R}^+, n)\}$. By the definition of $t(\mathcal{R}^+, n)$ it follows that $\mathcal{D}$ is dense on $\mathcal{R}$. Take $X \in \mathcal{G} \cap \mathcal{D}$. Then $\mathcal{AR}_1 \upharpoonright X \in \mathcal{U}_{\mathcal{R}}$ and $c \upharpoonright [\mathcal{AR}_1 \upharpoonright X]^n \leq t(\mathcal{R}^+, n)$. Therefore $t(\mathcal{R}, n) \leq t(\mathcal{U}_{\mathcal{R}}, n)$.

Note that $t(\mathcal{R}, n) \leq t(\mathcal{U}_{\mathcal{R}}, n)$ follows directly from the definition of $t(\mathcal{U}_{\mathcal{R}}, n)$.

∎

Lemma 2.5. *Let $(\mathcal{R}, \leq, r)$ be a weakly homogeneous topological Ramsey space and let $\mathcal{U}_{\mathcal{R}}$ be an ultrafilter generated by any generic filter $\mathcal{G}$ forced by $(\mathcal{R}, \leq)$. Then*

$$t(\mathcal{U}_{\mathcal{R}}, n) = t(\mathcal{R}^+, n)$$

.

Proof. Fix $X \in \mathcal{R}$ and let $c : [\mathcal{AR}_1 \restriction X]^n \to m$ be a coloring. Fix $Z \in \mathcal{G}$. Since $\mathcal{R}$ is weakly homogeneous, there is a mapping $\varphi : \mathcal{AR}_1 \restriction Z \to \mathcal{AR}_1 \restriction X$. Let $\tau : [\mathcal{AR}_1]^n \to m$ be such that for every $s \in [\mathcal{AR}_1 \restriction Z]^n$ it holds that $\tau(s) = c(\varphi(s))$. By the definition of $t(\mathcal{U}_{\mathcal{R}}, n)$, there is some $A \in \mathcal{U}_{\mathcal{R}}$ such that $c \restriction [A]^n$ takes no more than $t(\mathcal{U}_{\mathcal{R}}, n)$ colors. By the definition of $\mathcal{U}_{\mathcal{R}}$, there exists some $Y \leq Z$ such that $Y \in \mathcal{G}$ and $\mathcal{AR}_1 \restriction Y \subseteq A$. Take $Y' \leq X$ be such that $\mathcal{AR}_1 \restriction Y' \subset \varphi[Z]$. Therefore $c \restriction [Y']^n$ takes no more than $t(\mathcal{U}_{\mathcal{R}}, n)$ colors. By the minimality of $t(\mathcal{R}^+, n)$, it follows that $t(\mathcal{R}^+, n) \leq t(\mathcal{U}_{\mathcal{R}}, n)$. ∎

Theorem 2.6. *Let $(\mathcal{R}, \leq, r)$ be a homogeneous topological Ramsey space and let $\mathcal{U}_{\mathcal{R}}$ be an ultrafilter generated by any generic filter $\mathcal{G}$ forced by $(\mathcal{R}, \leq)$. Then*

$$t(\mathcal{R}, n) = t(\mathcal{U}_{\mathcal{R}}, n) = t(\mathcal{R}^+, n)$$

.

Proof. Since $\mathcal{R}$ is a weakly homogeneous topological Ramsey space, we know that $t(\mathcal{U}_{\mathcal{R}}, n) = t(\mathcal{R}^+, n)$. We shall prove that $t(\mathcal{R}^+, n) \leq t(\mathcal{R}, n)$. Let $\mathbb{A}$ be the strongest member of $\mathcal{R}$. Fix $X \in \mathcal{R}$ and let $c : [\mathcal{AR}_1 \restriction X]^n \to m$ be a coloring. Since $\mathcal{R}$ is weakly homogeneous, there is a mapping $\varphi : \mathcal{AR}_1 \to \mathcal{AR}_1 \restriction X$ which preserves the structure. Define a coloring $\tau : \mathcal{AR}_1 \to m$ such that for every $s \in [\mathcal{AR}_1]^n$, $\tau(s) = c(\varphi(s))$. By the definition of $t(\mathcal{R}, n)$, there exists some $Y \in \mathcal{R}$ such that τ restricted to $[Y]^n$ takes at most $t(\mathcal{R}, n)$ colors. Take $Z \leq X$ such that $\mathcal{AR}_1 \restriction Z \subseteq \varphi[\mathcal{AR}_1 \restriction Y]$. Therefore c restricted to $[Z]^n$ takes at most $t(\mathcal{R}, n)$ colors. By the minimality of $t(\mathcal{R}^+, n)$ it follows that $t(\mathcal{R}^+, n) \leq t(\mathcal{R}, n)$. ∎

Before studying the topological Ramsey space associated to the ideal conv we thought that for every topological Ramsey space all the associated Ramsey degrees $t(\mathcal{U}_{\mathcal{R}}, n)$, $t(\mathcal{R}, n)$, $t(\mathcal{R}^+, n)$ are the same. In Chapter 3 we will show that for the aforementioned topological Ramsey space it holds that $t(\mathcal{R}, n) < t(\mathcal{U}_{\mathcal{R}}, n) = t(\mathcal{R}^+, n)$.

2.2.1 A general method for Ramsey degrees for Ramsey spaces composed of independent sequences of structures

The Ramsey spaces associated with the ultrafilters of Baumgartner-Taylor, Blass, and Laflamme mentioned in Section 1.2 all have the following property.

Definition 2.7 (Independent Sequences of Structures (ISS))**.** We say that a topological Ramsey space $(\mathcal{R}, \leq, r)$ has *Independent Sequences of Structures (ISS)* if and only if the following hold: There are relations R_l, $l < L$ for some fixed finite integer L, where

R_0 is a linear order, and the domain of a structure S with these relations is denoted $\mathrm{dom}(S)$. The largest member $\mathbb{A}$ in $\mathcal{R}$ is a sequence $\langle \mathbb{A}(i) : i < \omega \rangle$ such that each $\mathbb{A}(i)$ is a finite structure with relations R_l, $l < L$. For $i < i'$, the domains of $\mathbb{A}(i)$ and $\mathbb{A}(i')$ are disjoint, and there are no relations between them, hence the *independence* of the sequence of structures. Each member $B \in \mathcal{R}$ can be identified with a sequence $\langle B(i) : i \in \omega \rangle$ where each $B(i)$ is isomorphic to $\mathbb{A}(i)$, and moreover, there is a strictly increasing sequence $(k_i)_{i<\omega}$ such that each $B(i)$ is an induced substructure of $\mathbb{A}(k_i)$. For $B, C \in \mathcal{R}$, $C \leq B$ if and only if each $B(n)$ is a substructure of some $B(i_n)$ for some strictly increasing sequence $(i_n)_{n<\omega}$. The members of $\mathcal{AR}_m$ are simply initial sequences of length m of members of $\mathcal{R}$: for $B \in \mathcal{R}$, $r_m(B) := \langle B(i) : i < m \rangle$. We require that $\mathrm{dom}(\mathbb{A}(0))$ is a singleton; hence the members of $\mathcal{AR}_1$ are singletons.

For each $m < \omega$ and $a, b \in \mathcal{AR}_m$ there exists a unique (because of R_0) isomorphism $\varphi_{a,b} : a \to b$. We will write φ to denote $\varphi_{a,b}$, when a and b are obvious. If $X \in \mathcal{R}$ and $s \in [\mathcal{AR}_1 \restriction X]^n$ for some $n \in \omega$, we think of s with the structure inherited by X. Thus, if $k_0, \ldots, k_m$ are those indices such that $x \cap X(k_i) \neq \emptyset$ for each $i \leq m$, then $s = \langle s_i : i \leq m \rangle$, where each s_i is the structure on $\mathrm{dom}(s) \cap \mathrm{dom}(X(l_i))$ with the substructure inherited from $X(l_i)$. Since each $X(l_i)$ is a substructure of some $\mathbb{A}(k_i)$, each s_i is also the structure on $\mathrm{dom}(s) \cap \mathrm{dom}(\mathbb{A}(k_i))$ with the substructure inherited from $\mathbb{A}(k_i)$. For $s, t \in [\mathcal{AR}_1]^n$ we say that s and t are *isomorphic*, and write $s \cong t$, if for some m, $s = \langle s_i : i \leq m \rangle$ and $t = \langle t_i : i \leq m \rangle$, and each s_i is isomorphic to t_i. For $t \in [\mathcal{AR}_1]^n$, the *isomorphism class* of t is the collection of substructures $s \in [\mathcal{AR}_1]^n$, such that s and t are isomorphic.

Note that spaces with the ISS are weakly homogeneous and satisfy

$$t(\mathcal{R}^+, n) = t(\mathcal{R}, n)$$

for each $n \geq 2$.

Definition 2.8 (ISS$^+$)**.** Let $\mathcal{R}$ be a space with the ISS. We say that $\mathcal{R}$ satisfies the ISS$^+$ if additionally, the following hold:

a) There is some $X \in \mathcal{R}$ such that for any two isomorphic members $u, v \in [X]^n$, there exist $m \in \omega$, $a, b \in \mathcal{AR}_m$, $s \in [a]^n$ isomorphic to u and $t \in [b]^n$ isomorphic to v such that $\varphi(s) = t$.

b) For every $n \geq 2$, there exists an $m \in \omega$ such that for every $X \in \mathcal{R}$ and for every $s \in [\mathcal{AR}_1]^n$, there exists some $t \in [r_m(X)]^n$ such that s and t are isomorphic.

For spaces with the ISS, the σ-closed partial order $\leq^*$ from Definition 1.6 is simply $\subseteq^*$.

Definition 2.9. If $(\mathcal{R}, \leq, r)$ is a topological Ramsey space satisfying the ISS$^+$, define $\mathbf{k}(\mathcal{R}, n)$ to be the number of isomorphism classes for substructures $b \in [\mathcal{AR}_1]^n$ such that b is a substructure of $\mathbb{A}(i)$ for some $i \in \omega$.

Notice that b) of Definition 2.8 guarantees that, for each n, $\mathbf{k}(\mathcal{R}, n)$ is finite.

Lemma 2.10. *If $(\mathcal{R}, \leq, r)$ is a topological Ramsey space with ISS$^+$, then for each $n \geq 1$,*

$$t(\mathcal{R}, n) \leq \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} \mathbf{k}(\mathcal{R}, j_i). \tag{2.2.1}$$

Proof. Fix $n \geq 1$ and $p \geq 2$, and let $c : [\mathcal{AR}_1]^n \to p$ be a coloring. Let $\tilde{k}$ denote the right hand side of the inequality in (2.2.3), and define

$$D = \{X \in \mathcal{R} : |c''[X]^n| \leq \tilde{k}\}.$$

We will prove that D is dense in $\mathcal{R}$. Let $\{\mathfrak{s}_k : k < \mathbf{k}(\mathcal{R}, n)\}$ be the collection of isomorphism classes for members of $[\mathcal{AR}_1]^n$. Let m be the least natural number such that for all $A \in \mathcal{R}$ and $k \in \tilde{k}$, there is some member of the isomorphism class $\mathfrak{s}_k$ contained in $r_m(A)$. Fix $a \in \mathcal{AR}_m$, and linearly order the members of $[a]^n$ as $\{u_l : l < L\}$, where each u_l is considered as a sequence of structures and where L is the number of n-sized subsets of a. By the ISS$^+$, for every $b \in \mathcal{AR}_m$ there is an isomorphism $\varphi_{a,b} : a \to b$. Note that $\{\varphi_{a,b}(u_l) : l < L\}$ is an enumeration for $[b]^n$ preserving the structure, so for each $l < L$, u_l is isomorphic to $\varphi_{a,b}(u_l)$.

Let $\mathcal{I} = {}^L p$. For every $\iota \in \mathcal{I}$, define

$$\mathcal{F}_\iota = \{b \in \mathcal{AR}_m : (\forall l \in L) c(\varphi_b(u_l)) = \iota(l)\}.$$

Let $A \in \mathcal{R}$ be given. Since $\mathcal{AR}_m$ is a Nash-Williams family and $\mathcal{AR}_m = \bigcup_{\iota \in \mathcal{I}} \mathcal{F}_\iota$, by Theorem 1.5 there are $B \leq A$ and $\iota \in \mathcal{I}$ such that $\mathcal{AR}_m \restriction B \subseteq \mathcal{F}_\iota$. Therefore, for every $b \in \mathcal{AR}_m \restriction B$ and for every $l < L$, $c(\varphi_b(u_l)) = \iota(l)$. Hence $|c''[\mathcal{AR}_m \restriction B]^n| \leq L$. Now suppose that $i < j < L$, u_i and u_j are isomorphic. By a) of ISS$^+$, there exist $b, d \in \mathcal{AR}_m \restriction B$, $s \in [b]^n$ isomorphic to u_i, and $t \in [d]^n$ isomorphic to u_j such that $\varphi_{b,d}(s) = t$. Therefore, $c(u_i) = c(u_j)$.

Thus, it remains to count the number of isomorphism classes in $[r_m(\mathbb{A})]^n$. Let t be a member of $[B]^n$ and note that for at least one $i \in \omega$, the substructure obtained by intersecting t with $B(i)$ is not empty. Let q be the cardinality of $\{i \in \omega : t \cap B(i) \neq \emptyset\}$. Note that if $q = 1$, then t belongs to one of $\mathbf{k}(\mathcal{R}, n)$ different isomorphism classes. If $q \geq 2$, let $\{l_i : i < q\}$ be an increasing enumeration of those $l \in \omega$ such that $t \cap B(l) \neq \emptyset$, and let t_i denote the substructure on $t \cap B(l_i)$ inherited from $B(l_i)$. For each $i \in [1, q]$, let j_i denote the cardinality of $\mathcal{AR}_1 \restriction (t \cap B(l_i))$. Note that $n = \sum_{1 \leq i \leq q} j_i$ and every $j_i < n$, and each t_i belongs to one of $\mathbf{k}(\mathcal{R}, j_i)$ isomorphism classes. Hence, t belongs to one of $\mathbf{k}(\mathcal{R}, j_1) \times \ldots \times \mathbf{k}(\mathcal{R}, j_q)$ many equivalence classes. Letting q range from 2 to its maximum possibility of n, there are

$$\sum_{1 < q \leq n} \sum_{l_1 + \ldots + l_q = n} \prod_{i < q} \mathbf{k}(\mathcal{R}, j_i) \tag{2.2.2}$$

different isomorphism classes, for n sized substructures of B that contain substructures from more than one block. Thus, $|c''[B]|^n \leq \tilde{k}$; hence $B \in D$. Thus, D is a dense subset of $\mathcal{R}$. ∎

The following Lemma tells us that the right hand side in equation (2.2.3) is not just an upper bound but it is the Ramsey degree, which will mean that it is enough to calculate the number of different isomorphism classes of j-sized substructures of blocks of $\mathbb{A}$ to know the exact Ramsey degree.

Lemma 2.11. *If $(\mathcal{R}, \leq, r)$ is a topological Ramsey space with ISS$^+$, then*

$$t(\mathcal{R}, n) \geq \sum_{1\leq q\leq n} \sum_{j_1+\ldots+j_q=n} \prod_{1\leq i\leq q} \mathbf{k}(\mathcal{R}, j_i). \tag{2.2.3}$$

Proof. As in the previous lemma, let $\tilde{k}$ denote the right hand side of equation (2.2.3). In the proof Lemma 2.10, we showed that there are $\tilde{k}$ isomorphism classes for members of $[\mathcal{AR}_1]^n$. Let $c : [\mathcal{AR}_1]^n \to \tilde{k}$ be a coloring such that for every $s, t \in [\mathcal{AR}_1]^n$, $c(s) = c(t)$ if and only if s and t belong to the same isomorphism class. By b) of ISS$^+$, there exists $m \in \omega$ such that for every $X \in \mathcal{R}$, and every $s \in [\mathcal{AR}_1]^n$ there is some member of $[r_m(X)]^n$ isomorphic to s. Then for every $X \in \mathcal{R}$, the set $[X]^n$ contains members of every isomorphism class, and hence, $|c''[X]^n| = \tilde{k}$. ∎

Theorem 2.12. *Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space with ISS$^+$. Then*

$$t(\mathcal{R}, n) = \sum_{1\leq q\leq n} \sum_{j_1+\ldots+j_q=n} \prod_{1\leq i\leq q} \mathbf{k}(\mathcal{R}, j_i). \tag{2.2.4}$$

Proof. This is a consequence of Lemmas 2.10 and 2.11. ∎

2.2.2 Calculations of Ramsey degrees of ultrafilters from spaces with the ISS$^+$

In this subsection, we will calculate the exact Ramsey degrees for several classes of ultrafilters which are forced by spaces with the ISS$^+$. First, we will use Theorem 2.12 to provide a streamlined calculation of the Ramsey degrees for the weakly Ramsey ultrafilters forced by Laflamme's forcing $\mathbb{P}_1$. Indeed, Laflamme calculated these in Theorem 1.10 of [50] via a three-page proof which shows its three-way equivalence with a combinatorial property that $\mathbb{P}_1$ is naturally seen to possess, and an interesting Ramsey property for analytic subsets of the Baire space in terms of the forcing $\mathbb{P}_1$ reminiscent of work of Mathias and Blass for Ramsey ultrafilters. The proof we present here is direct and short.

For understanding the following proof, first notice that $\mathcal{AR}_1$ consists of all single maximal branches in the tree $\mathbb{T}_1$, that is, a set of the form $\{\langle\rangle, \langle i\rangle, \langle i, j\rangle\}$, where $i \in \omega$ and $j \leq i$. Note that given $n \in \omega$ fixed, every two members of $\mathcal{AR}_n$ are isomorphic as subtrees of $\mathbb{T}_1$. This is because if $a, b \in \mathcal{AR}_n$, then there is an isomorphism $\varphi_{a,b} : a \to b$ which sends each node of the tree a to the node in the same position of the tree b. The following two figures show members, a and b, of $\mathcal{AR}_5$. The isomorphism $\varphi_{a,b}$ for these finite trees sends $\langle 0, 0\rangle$ to $\langle 20, 15\rangle$, $\langle 1, 0\rangle$ to $\langle 30, 23\rangle$, $\langle 1, 1\rangle$ to $\langle 30, 28\rangle$, $\langle 2, 0\rangle$ to $\langle 50, 48\rangle$, etc.

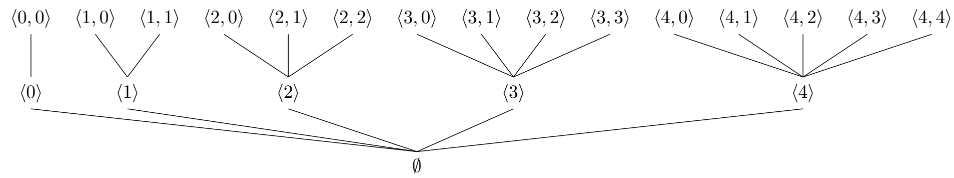


Figure 2.1: $a = r_5(\mathbb{T}_1)$

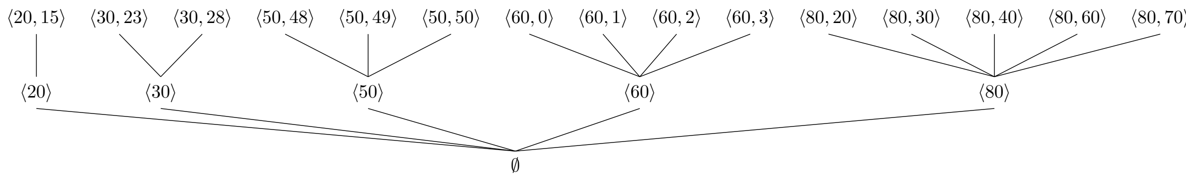


Figure 2.2: Another member, b, of $\mathcal{AR}_5$

Letting b denote the member of $\mathcal{AR}_5$ in Figure 5, note that $b(0) = \{\langle\rangle, \langle 20\rangle, \langle 20, 15\rangle\}$, $b(1) = \{\langle\rangle, \langle 30\rangle, \langle 30, 23\rangle, \langle 30, 28\rangle\}$, and so forth.

Definition 2.13. For every $n \geq 2$, let $S_n = \{x \in {}^q n : q \in [1, n], \sum_{i<q} x(i) = n$ and $\forall i \in q(x(i) \neq 0)\}$.

Lemma 2.14. *If $n \geq 2$, then $|S_n| = \sum_{p<n} \binom{n-1}{p} = 2^{n-1}$.*

Proof. For $q \in [1, n]$ and $x \in {}^q n$ satisfying $\sum_{i<q} x = n$ and for every $i \in q$, $x(i) \neq 0$, let

$$\psi(x) = \{x(0) - 1, x(0) + x(1) - 1, ..., \sum_{i<q-1} x(i) - 1\}. \tag{2.2.5}$$

Since every $x(i) \neq 0$, $\psi(x)$ is a subset of $n-1$, so $\psi(x)$ is a member of $[n-1]^{q-1}$. Notice that since $\sum_{i<q} x(i) = n$, it follows that $x(q-1) = n - \sum_{i<q-1} x(i)$ is determined.

Note that the map $\psi : S_n \to \mathcal{P}(n-1)$ is one-to-one. Actually, ψ is also an onto map. For every $p < n$ and $\{m_0, ..., m_{p-1}\} \in [n-1]^p$, a subset of $n-1$ with an increasing enumeration,

$$\{m_0, ..., m_{p-1}\} = \psi(\langle m_0 + 1, m_1 - m_0, ..., m_{p-1} - m_{p-2}, n - 1 - m_{p-1}\rangle)$$

with

$$\langle m_0 + 1, m_1 - m_0, ..., m_{p-1} - m_{p-2}, n - 1 - m_{p-1}\rangle \in ({}^{p+1}n) \cap S_n.$$

Then,

$$|S_n| = \sum_{p<n} |[n-1]^p| = \sum_{p<n} \binom{n-1}{p} = 2^{n-1}.$$

■

Corollary 2.15. *Let $\mathcal{U}_1$ be the weakly Ramsey ultrafilter forced by Laflamme's forcing $\mathbb{P}_1$, equivalently, by $(\mathcal{R}_1, \leq^*)$. Then for each $n \geq 1$, $t(\mathcal{U}_1, n) = 2^{n-1}$.*

Proof. Fix $n \geq 1$. First, note that the space $\mathcal{R}_1$ satisfies ISS$^+$. By Theorem 2.12 we have $t(\mathcal{U}_1, n) = \sum_{1\leq q\leq n}\sum_{j_1+\ldots+j_q=n}\prod_{1\leq i\leq q}\mathbf{k}(\mathcal{R}_1, j_i)$. Fix $1 \leq j \leq n$. Given any j-sized subsets s, t of $\mathcal{AR}_1$ such that $s \subset \mathbb{T}_1(i)$ and $t \subset \mathbb{T}_1(m)$ for some i, m, then s and t are isomorphic. Then for every $j \in [1, n]$, $\mathbf{k}(\mathcal{R}_1, j) = 1$. Therefore

$$t(\mathcal{U}_1, n) = \sum_{1\leq q\leq n}\ \sum_{j_1+\ldots+j_q=n}\ \prod_{1\leq i\leq q} 1 = |S_n| = 2^{n-1}. \tag{2.2.6}$$

Therefore $t(\mathcal{U}_1, n) = 2^{n-1}$. ■

Since $t(\mathcal{U}_1, 2) = 2$, for every $c : [\mathcal{AR}_1]^2 \to 3$ coloring there is an $X \in \mathcal{U}_1$ such that $|c \upharpoonright [X]^2| \leq 2$. If we identify $\mathcal{AR}_1$ with ω then $\mathcal{U}_1$ is a weakly Ramsey ultrafilter in the sense of ω. By Lemma 3, there is a coloring $c : [\mathcal{AR}_1]^2 \to 2$ such that for every $X \in \mathcal{R}_1$, $|c \upharpoonright [X]^2| = 2$. Thus, we can see in a simple way that $\mathcal{U}_1$ is not a Ramsey ultrafilter.

Next, we will calculate Ramsey degrees for ultrafilters $\mathcal{U}_k$ forced by Laflamme's forcings $\mathbb{P}_k$ from [50], $k \geq 2$. As noted in the previous section, the topological Ramsey space $\mathcal{R}_k$ forces the same ultrafilter as $\mathbb{P}_k$. The Ramsey degrees for $\mathcal{U}_k$ are stated in Theorem 2.2 of [50], but a concrete proof does not appear in that paper. Rather, Laflamme points out that the proof entirely similar to, but combinatorially more complicated than that of Theorem 1.10 in [50]. Here, we present a straightforward proof based on the Ramsey structure of $\mathcal{R}_k$. For the following proof, the set of first approximations $\{r_1(A) : A \in \mathcal{R}_k\}$ are identified with the maximal nodes of $\mathbb{T}_k$; this also applies for every member of $\mathcal{R}_k$. The downward closure of any maximal node in $\mathbb{T}_k$ recovers the tree structure below that node, so it suffices to work with the maximal nodes in $\mathbb{T}_k$.

Recall Definition 2.9 of $\mathbf{k}(\mathcal{R}, n)$ for a topological Ramsey space $\mathcal{R}$. The next lemma uses the inductive nature of the construction of $\mathcal{R}_{k+1}$ from $\mathcal{R}_k$ to show that each $\mathbf{k}(\mathcal{R}_{k+1}, n)$ can be deduced from the Ramsey degrees of $\mathcal{U}_k$.

Lemma 2.16. *For any $k, n \geq 1$, we have that $\mathbf{k}(\mathcal{R}_{k+1}, n) = t(\mathcal{U}_k, n)$.*

Proof. Let $s \in [\mathbb{T}_{k+1}]^n$ be such that $s \subset \mathbb{T}_{k+1}(l)$ for some $l \in \omega$. Recall our convention that s is a collection of maximal nodes in $\mathbb{T}_{k+1}$, so each node in s is a sequence of length $k+2$. Note that since s is contained in $\mathbb{T}_{k+1}(l)$, each node in s end-extends the sequence $\langle l\rangle$. Let t be the set of sequences resulting by taking out the first member of every sequence in s; thus, letting ${}^{k+1}\omega$ denote the set of sequences of natural numbers of length $k+1$,

$$t = \{x \in {}^{k+1}\omega : \langle l\rangle^\frown x \in s\}. \tag{2.2.7}$$

Then t is an n-sized subset of $\mathbb{T}_k$. Since there are $t(\mathcal{U}_k, n)$ isomorphism classes for $[\mathbb{T}_k]^n$, $\mathbf{k}(\mathcal{R}_{k+1}, n) = t(\mathcal{U}_k, n)$. ■

Lemma 2.17. *Let $k \geq 1$ be given, and suppose that $\mathcal{U}_{k+1}$ is an $(\mathcal{R}_{k+1}, \leq^*)$-generic filter. Then for each $n \geq 1$,*

$$t(\mathcal{U}_{k+1}, n) = \sum_{1\leq q\leq n}\ \sum_{j_1+\ldots+j_q=n}\ \prod_{1\leq i\leq q} t(\mathcal{U}_k, j_i). \tag{2.2.8}$$

Proof. This follows by Theorem 2.12 and Lemma 2.16. ■

Theorem 2.18. *Given $k, n \geq 1$, if $\mathcal{U}_k$ an ultrafilter forced by Laflamme's $\mathbb{P}_k$, or equivalently by $(\mathcal{R}_k, \leq^*)$, then $t(\mathcal{R}_k, n) = t(\mathcal{U}_k, n) = (k+1)^{n-1}$.*

Proof. The proof is by induction on k over all $n \geq 1$. The case when $k = 1$ is done by Corollary 2.15. Now we assume the conclusion for a fixed $k \geq 1$ and prove it for $k+1$. By the Lemma 2.17,

$$t(\mathcal{U}_{k+1}, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} t(\mathcal{U}_k, j_i). \tag{2.2.9}$$

By inductive hypothesis $t(\mathcal{U}_k, j_i) = (k+1)^{j_i - 1}$. Then

$$t(\mathcal{U}_{k+1}, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} (k+1)^{j_i - 1} = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} (k+1)^{n-q}. \tag{2.2.10}$$

By the proof of Lemma 2.14,

$$\sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} (k+1)^{n-q} = \sum_{1 \leq q \leq n} \binom{n-1}{q-1} (k+1)^{n-q} = \sum_{0 \leq p \leq n-1} \binom{n-1}{p} (k+1)^{n-1-p}. \tag{2.2.11}$$

By Newton's Theorem, the right hand side of equation (2.2.11) equals $((k+1)+1)^{n-1}$. Therefore $t(\mathcal{U}_{k+1}, n) = (k+2)^{n-1}$. ■

Next, we calculate the Ramsey degree for pairs for the ultrafilters forced by Blass' n-square forcing [8] and more generally, the hypercube Ramsey spaces $\mathcal{H}^n$ [26].

Corollary 2.19. *Let $\mathcal{V}_n$ be an $(\mathcal{H}^n, \leq^*)$-generic filter. Then*

$$t(\mathcal{H}^n, 2) = t(\mathcal{V}_n, 2) = 1 + \sum_{i=0}^{n-1} 3^i.$$

In particular, $t(\mathcal{V}_2, 2) = 5$, where $\mathcal{V}_2$ is the ultrafilter generated by Blass' n-square forcing.

Proof. By Theorem 2.12, we know that for each $n \geq 2$,

$$t(\mathcal{V}_n, 2) = \sum_{1 \leq q \leq 2} \sum_{j_1 + \ldots + j_q = 2} \prod_{1 \leq i \leq q} \mathbf{k}(\mathcal{H}^n, j_i) = \mathbf{k}(\mathcal{H}^n, 2) + \mathbf{k}(\mathcal{H}^n, 1).$$

Note that $\mathbf{k}(\mathcal{H}^n, 1) = 1$ because all the singletons are isomorphic. Given $n \geq 2$, let $\mathbf{A}_k$ be an n-hypercube with side length k. We will show that $\mathbf{k}(\mathcal{H}^n, 2) = \sum_{i=0}^{n-1} 3^i$.

Fix $n = 2$, and take $a = (a_0, a_1)$, $b = (b_0, b_1) \in \mathbf{A}_k$ for any large enough $k \in \omega$. Assume that a lexicographically below b, according to the lexicographical order on $\omega \times \omega$. There are 4 non-isomorphic options: Either $a_0 = b_0$ and $a_1 < b_1$, or else $a_0 < b_0$ and any of the three relations $a_1 < a_1$, $a_1 = b_1$, or $a_1 > b_1$ holds. Therefore $\mathbf{k}(\mathcal{H}^2, 2) = 4 = 3^0 + 3^1$, so $t(\mathcal{U}_2, 2) = 5$.

Now suppose that $\mathbf{k}(\mathcal{H}^n, 2) = \sum_{i=0}^{n-1} 3^i$. Given $a = (a_0, \dots, a_n)$ and $b = (b_0, \dots, b_n)$ in $\mathbf{A}_k$ for any large enough $k \in \omega$, there are the following possibilities. If $a_0 = b_0$, then there are $\mathbf{k}(\mathcal{H}^n, 2) = \sum_{i=0}^{n-1} 3^i$ many possible relations between $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. If $a_0 < b_0$, then for each $1 \le i \le n$, there are three possible configurations for a_i and b_i, namely $a_i < b_i$, $a_i = b_i$, or $a_i < b_i$. Thus, there are 3^n many possible configurations for $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. Therefore, $\mathbf{k}(\mathcal{H}^{n+1}, 2) = \sum_{i=0}^{n} 3^i$. ∎

In [27], Dobrinen calculated the Ramsey degrees for the k-arrow, not $(k+1)$-arrow ultrafilters of Baumgartner and Taylor.

Corollary 2.20 (Dobrinen)**.** *For $k \ge 2$, let $\mathcal{W}_k$ be the k-arrow, not $(k+1)$-arrow ultrafilter of Baumgartner and Taylor. Then*

$$t(\mathcal{W}_k, 2) = 3.$$

2.3 Ramsey degrees for $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$

In this Section we will calculate Ramsey degrees of pairs for $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, for all $k \ge 2$. As a consequence we obtain Ramsey degrees for generic ultrafilters. In this section, let $\mathcal{G}_1$ denote the ultrafilter forced by $\mathcal{P}(\omega)/\mathrm{Fin}$ and note that $\mathcal{G}_1$ is a Ramsey ultrafilter. Recall from Subsection 1.2.3 that $\mathrm{Fin}^{\otimes k}$ is a σ-closed ideal on ω^k, and that the Boolean algebras $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ force ultrafilters $\mathcal{G}_k$ such that whenever $1 \le j < k$, the projection of $\mathcal{G}_k$ to the first j coordinates of ω^k forms an ultrafilter on ω^j which is generic for $\mathcal{P}(\omega^j)/\mathrm{Fin}^{\otimes j}$. $\mathcal{G}_2$ garners much attention as it is a weak p-point which is not a p-point (see [12]). To ease notation, let $\subseteq^{*k}$ denote $\subseteq^{\mathrm{Fin}^{\otimes k}}$, and note that $(\mathcal{E}_k, \subseteq^{*k})$ is forcing equivalent to $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ (see [22]). In this section, we use the high dimensional Ellentuck spaces, $\mathcal{E}_k$, to provide concise proofs of the Ramsey degrees $t(\mathcal{G}_k, 2)$ for all $k \ge 2$. These results will appear in joint work with Natasha Dobrinen [27].

Definition 2.21. Let $k \ge 2$ and $s, t, u, v \in \omega^{\not\downarrow k}$. Define the relation $\sim_k$ on pairs of $\omega^{\not\downarrow k}$ by $\langle s, t\rangle \sim_k \langle u, v\rangle$ if and only if $s \prec t$, $u \prec v$ and for every $i, j \in k$ and $\rho \in \{=, <\}$, $s_i \,\rho\, t_j \longleftrightarrow u_i \,\rho\, v_j$.

Observe that the set of $X \in \mathcal{E}_k$ satisfying

$$\forall s, t \in X,\ \forall 1 \le i < k\ (s_i = t_i \longrightarrow s_{i-1} = t_{i-1}) \tag{2.3.1}$$

is open dense in $\mathcal{E}_k$. Let $\mathcal{D}_k$ denote the set of $X \in \mathcal{E}_k$ satisfying (2.3.1). From now on we work only with members of $\mathcal{D}_k$. Let $\mathbf{k}(\mathcal{E}_k, 2)$ be the number of equivalence classes from $\sim_k$ on pairs of $\mathcal{D}_k$. We will first calculate this number, and then show in Theorem 2.23 that it actually is the Ramsey degree of $\mathcal{G}_k$ for pairs.

Lemma 2.22. *For every $k \ge 2$, $\mathbf{k}(\mathcal{E}_k, 2) = \sum_{i<k} 3^i$.*

Proof. The proof will be by induction on $k \ge 2$. Fix $k = 2$, and fix some $X \in \mathcal{D}_2$. Fix $s, t \in X$ such that $t \prec s$. Then $s_0 < s_1$, $t_0 < t_1$ and $t_1 < s_1$, where $s = (s_0, s_1)$, $t = (t_0, t_1)$. There are four possibilities for ordering the entries of s and t:

(i) $t_0 = s_0$ and $t_1 < s_1$.

(ii) $t_0 < t_1 < s_0 < s_1$.

(iii) $t_0 < s_0 < t_1 < s_1$.

(iv) $s_0 < t_0 < s_1$ and $t_1 < s_1$.

Note that these four options are not isomorphic and for every $X \in \mathcal{D}_2$, $[X]^2$ contains all pairs in all four types. Therefore $\mathbf{k}(\mathcal{E}_2, 2) = 4 = 1 + 3$.

Now assume the conclusion holds for some fixed $k \geq 2$; we check that conclusion holds for $k+1$. Fix some $X \in \mathcal{D}_{k+1}$ and take $s, t \in X$ such that $t \prec s$. Then $s = (s_0, ..., s_{k-1}, s_k)$, $t = (t_0, ..., t_{k-1}, t_k)$ with $s_0 < ... < s_k$, $t_0 < ... < t_k$ and $t_k < s_k$. There are four options for ordering last two members of sequences s and t:

(i) $t_k < s_{k-1}$. In this case $t_{k-1} < s_{k-1}$. Then $t \restriction k \prec s \restriction k$ and the pair $\langle s \restriction k, t \restriction k\rangle$ lies in one of $\mathbf{k}(\mathcal{E}_k, 2)$ possible equivalence classes. Therefore, there are $\mathbf{k}(\mathcal{E}_k, 2)$ options for ordering members of sequences s and t.

(ii) $t_k > s_{k-1}$ and $t_{k-1} < s_{k-1}$. In this case $t \restriction k \prec s \restriction k$ and the pair $\langle s \restriction k, t \restriction k\rangle$ lies in one of $\mathbf{k}(\mathcal{E}_k, 2)$ possible equivalence classes. Therefore, there are $\mathbf{k}(\mathcal{E}_k, 2)$ options for ordering members of sequences s and t.

(iii) $t_k > s_{k-1}$ and $t_{k-1} > s_k$. In this case $s \restriction k \prec t \restriction k$ and the pair $\langle t \restriction k, s \restriction k\rangle$ lies in one of $\mathbf{k}(\mathcal{E}_k, 2)$ possible equivalence classes. Therefore, there are $\mathbf{k}(\mathcal{E}_k, 2)$ options for ordering members of sequences s and t.

(iv) $t_{k-1} = s_{k-1}$. In this case, $t_i = s_i$ for all $i \leq k-1$.

Then there are $3\mathbf{k}(\mathcal{E}_k, 2) + 1$ equivalence classes for the relation $\sim_{k+1}$. Since $\mathbf{k}(\mathcal{E}_k, 2) = \sum_{i<k} 3^i$, then

$$\mathbf{k}(\mathcal{E}_{k+1}, 2) = 3(\sum_{i<k} 3^i) + 1 = \sum_{i<k+1} 3^i. \tag{2.3.2}$$

■

Theorem 2.23. *For every $k \in \omega$ such that $k \geq 2$, $t(\mathcal{G}_k, 2) = \sum_{i<k} 3^i$.*

Proof. Let $c : [\omega^{\not\uparrow k}]^2 \to t(\mathcal{G}_k, 2)$ be such that $c(p) = c(q)$ if and only if $p \sim_k q$. For each $X \in \mathcal{E}_k$, $[X]^2$ contains members of every equivalence class. Thus, by Lemma 2.22, $|c \restriction [X]^2| = t(\mathcal{G}_k, 2) \geq \sum_{i<k} 3^i$.

Now we want to prove that $t(\mathcal{G}_k, 2) \leq \sum_{i<k} 3^i$. Let $r \geq 1$ be a natural number and $c : [\omega^{\not\uparrow k}]^2 \to r$ be a coloring. Let

$$\mathcal{D} = \{Y \in \mathcal{E}_k : |c \restriction [Y]^2| \leq \mathbf{k}(\mathcal{E}_k, 2)\}. \tag{2.3.3}$$

We will prove that $\mathcal{D}$ is a dense subset of $\mathcal{G}_k$. Let $X \in \mathcal{G}_k \cap \mathcal{D}_k$, and let m be a natural number such that $r_m(X)$ contains pairs of every equivalence relation of $\sim_k$.

Fix $a \in \mathcal{AR}_m$ and fix an order for $[a]^2 = \{p_l^a : l < L\}$ with the induced substructure, where L is the number of pairs of sequences that belong to a. For each $b \in \mathcal{AR}_m$

enumerate pairs of b as $[b]^2 = \{p_l^b : l < L\}$ such that for every $l < L$, $p_l^a \sim_k p_l^b$. Let $\mathcal{I} = {}^L r$. For every $\iota \in \mathcal{I}$, define

$$\mathcal{F}_\iota = \{b \in \mathcal{AR}_m : (\forall l < L)\, c(p_l^b) = \iota(l)\}. \tag{2.3.4}$$

Since $\mathcal{AR}_m$ is a Nash-Williams family and $\mathcal{AR}_m = \bigcup_{\iota \in \mathcal{I}} \mathcal{F}_\iota$, by Theorem 1.5 there exist $Y \leq X$ and $\iota \in \mathcal{I}$ such that $\mathcal{AR}_m \restriction Y \subseteq \mathcal{F}_\iota$. Therefore, for every $b \in \mathcal{AR}_m \restriction Y$ and for every $l < L$, $c(p_l^b) = \iota(l)$. Hence $|c''[\mathcal{AR}_m \restriction Y]^2| \leq L$.

Note that if i, j are such that $p_i^a \sim_k p_j^a$, then there exist $A, B \in \mathcal{AR}_m \restriction Y$ and $l < L$ such that $p_i^a \sim_k p_l^A$, $p_j^a \sim_k p_l^B$. This implies that if $p_i^a \sim_k p_j^a$ with $i < j \leq L$, then $c(p_i^a) = (p_j^a)$. Since there are $\mathbf{k}(\mathcal{E}_k, 2)$ different equivalence classes, and $[Y]^2$ contains pairs of every equivalence class, we obtain that $|c''[\mathcal{AR}_1 \restriction Y]^2| \leq \mathbf{k}(\mathcal{E}_k, 2)$. Therefore $Y \in D$; hence, D is a dense subset of $\mathcal{E}_k$. By Lemma 2.22 it follows that $t(\mathcal{G}_k, 2) \leq \sum_{i<k} 3^i$. ■

Chapter 3

TRS ideals

For several ideals, the quotient $\mathcal{P}(\omega)/\mathcal{I}$ is forcing equivalent to a topological Ramsey space. In this chapter we study those ideals that are related to a topological Ramsey space in the same sense that the ideal Fin is related to the Ellentuck space. These ideals are important because topological Ramsey space theory provides us with strong tools to study Tukey, Katětov and partition properties for the ideals and the filters forced by their quotients.

3.1 Definition and examples

Let $T \subseteq [\omega]^{<\omega}$ be a tree ordered with the end-extension relation $\sqsubset$. Instead of thinking of branches of T as sequences, from now the set $[T]$ consists of subsets of ω, which means

$$[T] = \{X \subset \omega : (\forall n \in \omega)(\exists s \in T) X \cap n \sqsubset s\}.$$

Recall that T_n denotes the n-th level of T. Then members of T_{n+1} are finite subsets of ω which end-extends some member of T_n. For $n \in \omega$ and $X \in [T]$, let $X||n$ denote the initial segment $t \in T_n$ such that $X||n \sqsubset X$. Denote by $X(n) = X||(n+1) \setminus X||n$ and $T[n]$ will denote the collection of all the $X(n)$ with $X \in [T]$.

Finally, we some notation from topological Ramsey space theory. If $s \in T$ and $X \in [T]$, $[s, X] = \{Y \in [T] : s \sqsubset Y \subset X\}$. If $s \in T$ and $X \in [T]$, $\mathrm{depth}_X(s)$ is the least n, if it exists, such that $s \subseteq X||n$. If such n does not exist, then write $\mathrm{depth}_X(s) = \infty$. If $\mathrm{depth}_X(s) = n < \infty$, then $[\mathrm{depth}_X(s), X]$ denotes $[X||n, X]$.

Note that for any $n \in \omega$, $X||(n+1) = X(0) \cup \ldots \cup X(n)$. Also, note that

$$X = \bigcup_{n\in\omega} X||n = \bigcup_{n\in\omega} X(n).$$

Definition 3.1. Let $T \subseteq [\omega]^{<\omega}$ be a pruned tree ordered with the end-extension relation $\sqsubset$. We say that T is *Todorčević* if $\emptyset \in T$ and

(i) If $s, t, t' \in T$ are such that $s \sqsubset t$ and $t \subset t'$, there exists $s' \in T$ such that $s \subseteq s'$ and $s' \sqsubset t'$.

(ii) If $s \in T$ and $X \in [T]$ are such that $s \subset X$, then there exists $Y \in [T] \cap [X]^\omega$ such that $s \sqsubset Y$.

(iii) If $s \in T$ and $X, Y \in [T]$ are such that $s \subset X$ and $X \subseteq Y$ then there exists $Z \in [T]$ such that $Y||n \sqsubset Z$ and $[s, Z] \subseteq [X]^\omega$ where $n = depth_Y(s)$.

(iv) (Pigeonhole condition) If $s \in T$ and $X \in [T]$ is such that $s \subseteq X$, for every $\mathcal{O} \subset succ_T(s)$ there exists $Y \in [s, X]$ such that $succ_T(s) \cap [Y]^{<\omega} \subset \mathcal{O}$ or $succ_T(s) \cap [Y]^{<\omega} \cap \mathcal{O} = \emptyset$.

Definition 3.2. Let $\mathcal{I}$ be an ideal on ω. We say that $\mathcal{I}$ is a TRS ideal if there exists a Todorčević tree $T \subseteq [\mathcal{I}]^+$ such that for every $X \in \mathcal{I}^+$ there is a $Y \in [T]$ with $Y \subseteq X$ and for every $n \in \omega$, $\{n\} \in T$.

Let $\mathcal{I}$ be a TRS ideal and $(T, \sqsubset)$ be a witness for this fact. We identify every set $X \in [T]$ with the sequences $\langle X||n : n \in \omega \rangle$ and $\langle X(n) : n \in \omega \rangle$.

Define an order $\leq$ on $[T]$. For every $X, Y \in [T]$, $X \leq Y$ if for every $n \in \omega$ there exists $m \in \omega$ such that $X(n) \subseteq Y(m)$. Let $r : \omega \times [T] \to T$ be such that $r(n, X) = X||n$. Recall that in chapter 1.1 we identified r with the sequence of mappings $r_n : [T] \to T$ such that for every $n \in \omega$ and $X \in [T]$, $r_n(X) = X||n$. Finally, define the order " $\leq_{\text{fin}}$ " on T such that for every $m < n$ and for every $s \in T_m$, $t \in T_n$ we write $s \leq_{\text{fin}} t$ if for every $l < m$ there is some $k < m$ such that $s(l) \subseteq t(k)$.

Theorem 3.3. *If $\mathcal{I}$ is a TRS ideal and $(T, \sqsubset)$ is a witness to this fact, then $([T], \leq, r)$ is a topological Ramsey space.*

Proof. We show that the topological space $[T]$ satisfies the four axioms from Todorčević.

A.1 (a) $X||0 = \emptyset$ for all $X \in [T]$.
This holds because $\emptyset \in T$ and $\emptyset$ is an initial segment of every $X \in [T]$.

(b) $X \neq Y$ implies $X||n \neq Y||n$ for some n.
This holds because T is a tree.

(c) $X||n = Y||m$ implies $n = m$ and $X||k = Y||k$ for all $k < n$.
This holds because T is a tree.

A.2 (a) $\{s \in T : s \subseteq t\}$ is finite for all $t \in T$.
This is true because every $t \in T$ is finite.

(b) $X \leq Y$ if and only if for each $n \in \omega$ there exists $m \in \omega$ such that $X||n \subseteq Y||m$.
This follows by the definition of the order " $\leq$ ".

(c) For every $s, t, t' \in T$, if $s \sqsubset t$ and $t \subseteq t'$ then there exists $t' \in T$ such that $s' \sqsubset t'$ and $s \subseteq s'$.
This holds by (i) of the definition of Todorčević tree.

A.3 (a) If $X \in [T]$ and $s \in T$ are such that $\operatorname{depth}_X(s) < \infty$ then $[s, Y] \neq \emptyset$ for all $Y \in [\operatorname{depth}_X(s), X]$.

Fix $Y \in [\operatorname{depth}_X(s), X]$. Since $s \subseteq \operatorname{depth}_X(s)$ and $\operatorname{depth}_X(s) \sqsubset Y$, it follows that $s \subset Y$. By (ii) from the definition of Todorčević tree we get that there exists $Z \in [T] \cap [Y]^\omega$ such that $s \sqsubset Z$. Therefore $Z \in [s, Y]$ and $[s, Y] \neq \emptyset$.

(b) If $X, Y \in [T]$ and $s \in T$ are such that $X \leq Y$ and $s \subset X$ then there is a $Z \in [\operatorname{depth}_Y(s), Y]$ such that $\emptyset \neq [s, Z] \subseteq [s, X]$.

Since T is a Todorčević tree, there exists $Z \in [T]$ such that $\operatorname{depth}_Y(s) \sqsubset Z$ and $[s, Z] \subset [X]^\omega$. It follows by $s \subset \operatorname{depth}_Y(s)$ and $\operatorname{depth}_Y(s) \sqsubset Z$ that there exists $Z' \in [s, Z]$. Therefore, $\emptyset \neq [s, Z] \subseteq [s, X]$.

A.4 If $\operatorname{depth}_X(s) < \infty$ with $s \in T_n$ and if $\mathcal{O} \subseteq T_{n+1}$, then there is $Y \in [\operatorname{depth}_X(s), X]$ such that $T_{n+1} \cap [Y]^{<\omega} \subseteq \mathcal{O}$ or $T_{n+1} \cap [Y]^{<\omega} \cap \mathcal{O} = \emptyset$.

Since $s \subseteq X$, $\mathcal{O} \restriction succ_T(s)$ is a partition of $succ_T(s)$ and T is a TRS ideal there exists some $Z \in [s, X]$ such that $succ_T(s) \cap [Z]^{<\omega} \subseteq \mathcal{O}$ or $succ_T(s) \cap [Z]^{<\omega} \cap \mathcal{O} = \emptyset$. By (iii), there exists $Y \in [T]$ such that $\operatorname{depth}_X(s) \sqsubset Y$ and $[s, Y] \subset [Z]^\omega$. Note that there are not successors of s contained in $\operatorname{depth}_X(s)$. Furthermore, $succ_T(s) \cap [Y]^{<\omega} \subset succ_T(s) \cap [Z]^{<\omega}$. Therefore, Z is as required.

■

Note that every TRS ideal is analytic.

Fact 3.4. *Let $\mathcal{I}$ be an ideal and let $(\mathcal{R}, \leq, r)$ be a triple that satisfies axioms* **A.1** – **A.4** *from definition 1.1. If $\mathcal{R} \subseteq \mathcal{I}^+$ and for every $X \in \mathcal{I}^+$ there is some $Y \in \mathcal{R}$ such that $Y \subseteq X$, then $\mathcal{I}$ is a TRS ideal.*

Proof. This fact is a consequence of the Axioms **A.1** – **A.4**. ■

From now on, for every TRS ideal $\mathcal{I}$, $(T, \sqsubset)$ will denote some witness tree for this fact. The following corollary is a particular case of the Abstract Nash Williams Theorem.

Corollary 3.5. *Let $\mathcal{F} \subset T$ be such that for every $s, t \in T$, $s \sqsubset t$ implies $s = t$. If $X \in [T]$ and $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ is a partition, there exist $Y \in [T] \cap [X]^\omega$ and $i \in \{0, 1\}$ such that $\mathcal{F}_i \cap [Y]^{<\omega} = \emptyset$.*

Definition 3.6. Let $\mathcal{I}$ be a TRS ideal. We say that $\mathcal{I}$ is *weakly homogeneous* if for every $X, Y \in [T]$ there exists some map $\varphi : X \to Y$ such that $Z \in [T] \cap [X]^\omega$ if and only if $\varphi(Z) \in [T]$.

Every TRS ideal in which we are interested is weakly homogeneous. It is possible to define TRS ideals that are not weakly homogeneous but these ideals do not arise in a natural way. This is the reason why from now on we assume that every ideal is weakly homogeneous.

Proposition 3.7. *If $\mathcal{I}$ is a TRS ideal then for every $X, Y \in [T]$, $\mathcal{I} \restriction X \simeq_{\mathrm{K}} \mathcal{I} \restriction Y$. In particular, if $\omega \in [T]$ then $\mathcal{I}$ is K-uniform.*

Proof. Fix $X, Y \in [T]$. Since $\mathcal{I}$ is weakly homogeneous, there exists a mapping $\varphi : X \to Y$ that satisfies that $Z \in [T] \cap [X]^\omega$ if and only if $\varphi(Z) \in [T]$. Take $X' \in \mathcal{I} \restriction X$ and fix some $Z \in [T] \cap [X']^\omega$. Since $Z \in [T] \cap [X]^\omega$, it follows that $\varphi(Z) \in [T] \cap [Y]^\omega$. Since $\varphi(Z) \in (\mathcal{I} \restriction Y)^+$ and $\varphi(Z) \subseteq \varphi(X')$, it follows that $\varphi(X') \in (\mathcal{I} \restriction Y)^+$. Therefore, $\mathcal{I} \restriction X \leq_{\mathrm{K}} \mathcal{I} \restriction Y$. By similar arguments we prove that $\mathcal{I} \restriction Y \leq_{\mathrm{K}} \mathcal{I} \restriction X$. ■

Definition 3.8. Let $\mathcal{I}$ be a TRS ideal and $s \in T$. We say that s is *independent* if for every $X \in [T]$ there is some $t \in T$ extending s such that $t \setminus s \subset X$. We say that s is *dependent* if there exists $Z \in \mathcal{I}$ such that for every $t \in T$ extending s, $t \cap Z \neq \emptyset$.

Definition 3.9. Let $\mathcal{I}$ be a TRS ideal. We say that:

- $\mathcal{I}$ has the *Weak Independence Property* (WIP) if every $s \in T$ is independent.
- $\mathcal{I}$ has the *Independent Sequence Property* (ISP) if for every $n \in \omega$ and for every $s \in T_n$ there is some $k \in \omega$ such that for every $l \geq k$ and for every $b \in T(l)$ there exists $a \subset b$ such that $s \cup a \in T_{n+1}$.
- $\mathcal{I}$ has the *Dependence Property* if every $s \in T$ is dependent.

Note that every ideal with the Independent Sequence Property has the Weak Independence Property.

Proposition 3.10. *If $\mathcal{I}$ is a TRS ideal and $\omega \in [T]$ then either $\mathcal{I}$ has the Weak Independence Property or $\mathcal{I}$ has the Dependence Property.*

Proof. We are proving this by contradiction. Suppose that there are $s_0, s_1 \in T$ such that s_0 is independent and s_1 is dependent. Take $X, Y \in [T]$ such that $s_0 \sqsubset X$, $s_1 \sqsubset Y$. Now, let $\varphi : X \to Y$ the mapping obtained because $\mathcal{I}$ is weakly homogeneous. Without loss of generality we can assume that $s_1 \subset \varphi[s_0]$. Since s_1 is dependent there is a $Z \in \mathcal{I}$ such that for any $t \in T$ extending s_1, $t \cap Z \neq \emptyset$. Since $Z \in \mathcal{I}$, $Y \setminus Z \in \mathcal{I}^+$. Therefore the preimage $\varphi^{-1}[Y \setminus Z] \in \mathcal{I}^+$. So we can extend s_0 into $\varphi^{-1}[Y \setminus Z]$. Take $X' \in [T]$ extending s_0 such that $X' \setminus s_0 \subset \varphi^{-1}[Y \setminus Z]$. So $\varphi[X'] \in [T]$ but this is a contradiction because $\varphi[X'] \cap Z = \emptyset$.

■

Proposition 3.11. *Let $\mathcal{I}$ be a TRS ideal. If $\mathcal{I}$ has the Weak Independence Property then $\mathcal{I}$ is P^+.*

Proof. Let $(T, \sqsubset)$ be a tree that witness the fact that $\mathcal{I}$ is a TRS ideal. Let $\langle X_n : n \in \omega \rangle$ be a decreasing sequence of $\mathcal{I}$-positive sets.

We shall define a sequence $\langle s_i : i \in \omega \rangle$ of members of T such that for every $i \in \omega$, $s_i \sqsubset s_{i+1}$. Let $s_0 = \emptyset$. Now suppose we know the sequence s_i, $i < n$. Since $X_n \in \mathcal{I}^+$, there exists some $Y \in [T]$ such that $Y \subseteq X_n$. Since $\mathcal{I}$ satisfies the WIP there exists some $t \in T$ such that $s \sqsubset t$ and $t \setminus s \subseteq Y \subseteq X_n$. Let $s_n = t$. Let $X = \bigcup_n s_n$. Note that $X \in [T]$, $X \subseteq X_0$ and for every $n > 0$, $X \setminus X_n \subset s_{n-1}$. Therefore X is a pseudointersection for the sequence $\langle X_n : n \in \omega \rangle$ and $\mathcal{I}$ is a P^+ ideal. ■

Question 3.12. Let $\mathcal{I}$ be a TRS ideal with the Weak Independence Property. Is $(\mathcal{I}, \subseteq_{\mathcal{I}})$ σ-closed?

Recall that an ideal $\mathcal{I}$ is *locally* F_σ if for every $X \in \mathcal{I}^+$ there exists some $Y \subseteq X$ such that $Y \in \mathcal{I}^+$ and $\mathcal{I} \upharpoonright Y$ is F_σ.

Corollary 3.13. *If $\mathcal{I}$ is a TRS ideal with the the Weak Independence Property then $\mathcal{I}$ is a locally F_σ ideal.*

Proof. By Lemma 3.11 we know that $\mathcal{I}$ is P^+. By Theorem 1.33 it follows that $\mathcal{I}$ is locally F_σ. ∎

Proposition 3.14. *If $\mathcal{I}$ is a TRS ideal with the Independent Sequence Property then $\mathcal{I}$ is F_σ.*

Proof. To see that $\mathcal{I}$ is F_σ we shall show that $\mathcal{I}^+$ is G_δ. We will see that

$$\mathcal{I}^+ = \{X \subset \omega : (\forall n \in \omega)(\exists a \in T[n])(a \subset X)\}.$$

Take $X \in \mathcal{I}^+$ and $Y \in [T] \cap [X]^\omega$. For every $n \in \omega$, $Y(n) \in T[n]$ and $Y(n) \subset X$ therefore $X \in \{X \subset \omega : (\forall n \in \omega)(\exists a \in T[n])(a \subset X)\}$.

Now, take $X \in \{X \subset \omega : (\forall n \in \omega)(\exists a \in T[n])(a \subset X)\}$. We will find some increasing sequence $\langle s_m : m \in \omega\rangle$ contained in X. Let $s_0 = \emptyset$. For $n = 0$ there exists some $a \in T[0] = T_1$ such that $a \subset X$. Denote by s_1 such a. Now suppose we know $s_n \in T_n$ such that $s_n \subset X$. Since $\mathcal{I}$ has the ISP there exists k_n such that for every $l \geq k_n$ and for every $b \in T[l]$ there exists some $c \subset b$ such that $s_n \cup c \in T_{n+1}$. Take $a \in T[k_n]$ with $a \subset X$, there is some $c \subset a$ such that $s_n \cup c \in T_{n+1}$. Denote $s_n \cup c$ by s_{n+1}. Note that $s_n \sqsubset s_{n+1}$. Let $Y = \bigcup_{n\in\omega} s_n$. Note that $Y \in [T]$ and $Y \subseteq X$. Therefore $X \in \mathcal{I}^+$. ∎

Recall that $\triangle$ is the set of elements of $\omega \times \omega$ that are below the diagonal.

For every $A \subset \omega \times \omega$ and $m \in \omega$ define

$$\pi(A) = \{m \in \omega : \exists n \in \omega((m, n) \in A)\},$$

$$\pi_m(A) = \{n \in \omega : (m, n) \in A)\}.$$

Recall from Definition 1.10 of $\mathcal{R}_1$ that $\mathbb{T}_1$ denote the following infinite tree of height 2:

$$\mathbb{T}_1 = \{\langle\rangle\} \cup \{\langle n\rangle : n < \omega\} \cup \bigcup_{n<\omega} \{\langle n, i\rangle : i \leq n\}.$$

As we did before, we think of $\mathbb{T}_1$ just as the top nodes $\bigcup_{n<\omega}\{\langle n, i\rangle : i \leq n\}$. We identify every node of $\mathbb{T}_1$, $\langle m, n\rangle$ with the ordered pair $(m, n) \in \triangle$. So, for this chapter we think $\mathbb{T}_1$ and $\triangle$ and subtrees of $\mathbb{T}_1$ as subsets of $\triangle$.

Recall that in [29] Dobrinen and Todorčević prove that the topological Ramsey space $(\mathcal{R}_1, \leq, r)$ satisfy axioms **A.1** − **A.4** from definition 1.1.

Proposition 3.15. *The ideal $\mathcal{ED}_{\mathsf{fin}}$ is a TRS ideal.*

Proof. Define a tree $T \subset [\triangle]^{<\omega}$ which consist of finite sets $a \subset \triangle$ such that if $\pi(a) = \{n_0, ..., n_l\}$ is an increasing enumeration then for every $i \in [0, l]$, $|\pi_{n_i}(a)| = i + 1$. If $a, b \in T$, we write $a \sqsubset b$ if $\pi(a)$ is an initial segment of $\pi(b)$ and for every $n \in \pi(a)$, $\pi_n(a) = \pi_n(b)$. Note that $[T]$ coincides with the topological Ramsey space $\mathcal{R}_1$.

Take $X \in \mathcal{ED}^+_{\mathsf{fin}}$; for every $m, n \in \omega$ there exists some $k > n$ such that $|X(k)| > m$. We will find a sequence $(k_i)_{i\in\omega}$ such that for every $i \in \omega$, $|X(k_i)| \geq i + 1$. If $m, n = 1$ there exists some k_0 such that $k_0 > 1$ and $|X(k_0)| > 1$. Now suppose we know k_i. For $m = i + 2$ and $n = k_i$ there exists $k_{i+1} > k_i$ such that $|X(k_{i+1})| > i + 2$. For every $i \in \omega$ choose $Y(i) \subset X(k_i)$ such that $|Y(i)| = i + 1$. If $Y = \bigcup_{i\in\omega} Y(i)$, then $Y \in [T]$ and $Y \subseteq X$.

Since $\mathcal{R}_1$ is a topological Ramsey space, it follows that $\mathcal{I}$ is a TRS ideal.

■

Proposition 3.16 (Dobrinen [22], [23])**.** *For every $\alpha < \omega_1$, the ideal* Fin^α *is a TRS ideal.*

Proposition 3.17. *The ideal $\mathcal{ED}$ is a TRS ideal.*

Proof. To prove this, we define a tree T in the same way that the tree defined for $\mathcal{ED}_{fin}$ with the difference that in this proof members of T are contained in $\omega \times \omega$ instead of $\triangle$.

Members of T are finite subsets $a \subset \omega \times \omega$ such that the *n-th* fiber contains exactly $n + 1$ points. For $a, b \in T$, $a \sqsubset b$ if $a \subset b$ and $\pi(a) \sqsubset \pi(b)$.

Now we see that T is Todorčević.

(i) Let $a, b, c \in T$ be such that $a \sqsubset b$ and $b \subset c$. Let d be such that $d \sqsubset c$ and $\pi(d) = \pi(a)$. Note that d is as required.

(ii) Let $a \in T$ and $X \in [T]$ be such that $a \subset X$. Enumerate increasingly $\pi(a) = \{n_0, ..., n_m\}$ with $m \in \omega$. For every $k > m$ there exists n_k such that $\langle n_i : i < \omega\rangle$ is an increasing sequence and $|\pi_{n_k}(X)| > k + 1$. Take $Y(k) \subset \pi_{n_k}(X)$ such that $|Y(k)| = k + 1$. Define $Y = a \cup \cup_{k>m} Y(k)$. Then $Y \in [T] \cap [X]^\omega$ and $a \sqsubset Y$.

(iii) Take $a \in T$ and $X, Y \in [T]$ such that $a \subset X$ and $X \subset Y$. Let l be such that $depth_Y(a) \in T_l$. For every $k > l$ there is a n_k such that $\langle n_i : i \in \omega\rangle$ and $|X(n_k)| > k + 1$. Take $Z(k) \subset X(n_k)$ such that $|Z(k)| = k + 1$. Define $Z = depth_Y(a) \cup \cup_{k>l} Z(k)$. Then $depth_Y(a) \sqsubset Z$ and $[a, Z] \subset [X]^\omega$.

(iv) Let $n \in \omega$, $a \in T_n$ and $X \in [T]$ be such that $a \subset X$. Take $\mathcal{O} \subset succ_T(a)$. Let l be the maximum natural number such that $a \cap (\{l\} \times \omega) \neq \emptyset$. By finite Ramsey theorem, for every $i > n+1$ there is N_i big enough such that any coloring $c : [N_i]^{i+1} \to 2$ there is $F \in [N_i]^{n+1}$ which is monochromatic. For every $i > n$ there is $k_i > j$ such that $|X(k_i)| > N_i$. Note that we can choose such k_i such that $\langle k_i : i > n\rangle$ is an increasing sequence. Let $c_i : [X(k_i)]^{n+1} \to 2$ be a coloring such that $c_i(b) = 1$ if $a \cup b \in \mathcal{O}$ and $c_i(b) = 0$ if $a \cup b \notin \mathcal{O}$. Let $F_i \in [N_i]^{i+1}$ be a monochromatic for c_i. Now, let $Y = a \cup \cup_{j>n} F_j$. Note that Y satisfy the statement.

Therefore, T is Todorčević.

Take $X \in \mathcal{ED}^+$; for every $m, n \in \omega$ there exists some $k > n$ such that $|X(k)| > m$. We will find a sequence $(k_i)_{i\in\omega}$ such that for every $i \in \omega$, $|X(k_i)| \geq i+1$. If $m, n = 1$ there exists some k_0 such that $k_0 > 1$ and $|X(k_0)| > 1$. Now suppose we know k_i. For $m = i+2$ and $n = k_i$ there exists $k_{i+1} > k_i$ such that $|X(k_{i+1})| > i+2$. For every $i \in \omega$ choose $Y(i) \subset X(k_i)$ such that $|Y(i)| = i+1$. If $Y = \bigcup_{i\in\omega} Y(i)$, then $Y \in [T]$ and $Y \subseteq X$.

■

3.1.1 A non σ-closed TRS ideal

When we started working with TRS ideals $\mathcal{I}$, we wondered if every quotient $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed. In this subsection we show that there is a TRS ideal which is not $\sigma-$closed. Now we are interested in discovering what precise properties of TRS ideals guarantee that the corresponding quotient is σ-closed.

Remember that the ideal $\mathcal{ED}_{fin}$ contains subsets of $\omega \times \omega$. To simplify the following definition we will think of members of $\mathcal{ED}_{fin}$ as subsets of ω. This is possible because $\omega \times \omega$ is countable.

Definition 3.18. Let $\mathcal{ED}_\omega$ be the ideal on $\omega \times \omega$ such that $A \in \mathcal{ED}_\omega$ if for every $n \in \omega$, $\pi_n(A) \in \mathcal{ED}_{fin}$.

Let T be the Todorčević tree defined in proposition 3.15 to prove that $\mathcal{ED}_{fin}$ is a TRS ideal.

Define U to be a tree on $[\omega \times \omega]^{<\omega}$ whose members are finite sets $\{n\} \times t$ with $n \in \omega$ and $t \in T$. Define the order $\sqsubset$ such that $\{m\} \times s \sqsubset \{n\} \times t$ if $m = n$ and $s \sqsubset t$. Note that $[U]$ consist of sets $\{n\} \times X$ with $n \in \omega$ and $X \in [T]$.

Proposition 3.19. *$\mathcal{ED}_\omega$ is a TRS ideal.*

Proof. First, note that U is Todorčević because T is Todorčević. This is because whenever $X \subset Y$, $a \subset X$ and $a \sqsubset b$, all objects are contained in the same fiber which means that they belong to the same copy of $\mathcal{ED}_{fin}$.

Now, take $X \in \mathcal{ED}_\omega^+$. Since $A \notin \mathcal{ED}_\omega$ there exists some $n \in \omega$ such that $\pi_n(A) \notin \mathcal{ED}_{fin}$. There exists $X \in [T]$ such that $X \subseteq \pi_n(A)$. Therefore $\{n\} \times X \in [U]$ and $\{n\} \times X \subseteq A$.

■

Proposition 3.20. *$\mathcal{ED}_\omega$ ordered with the inclusion modulo $\mathcal{ED}_\omega$ is not σ-closed.*

Proof. For every $n \in \omega$ let $A_n = \{l \in \omega : l > n\} \times \omega$. Note that $\langle A_n : n \in \omega \rangle$ is a $\subseteq$-decreasing sequence of positive sets. Let $B \subseteq \omega \times \omega$ be such that for every $n \in \omega$, $B \subseteq_{\mathcal{ED}_\omega} A_n$. For every $n \in \omega$, $B \setminus A_n = \bigcup_{l \leq n} \pi_n(B) \in \mathcal{ED}_\omega$. Therefore $\pi_n(B) \in \mathcal{ED}_{fin}$ for every $n \in \omega$. Hence $B \in \mathcal{ED}_\omega$. ■

Note that although $\mathcal{ED}_\omega$ is not σ-closed, it is locally σ-closed. This is why $\mathcal{ED}_\omega^+$ does not add reals.

Question 3.21. Does there exist a TRS ideal $\mathcal{I}$ such that $\mathcal{I}^+$ adds reals?

3.1.2 Ideals generated by Fraïssé classes

In this section we prove that ideals generated by Fraïssé classes are TRS. In particular, the collection of ideals generated by hypercube spaces are TRS. We use the Ramsey degrees calculated in chapter 2 to better understand the Katětov order on F_σ and K-uniform ideals.

Let $\mathbb{A} = \langle (\mathbf{A}_{k,j})_{k<\omega} : j < J\rangle$ be a generating sequence and $\mathcal{R}(\mathbb{A})$ be a topological Ramsey space generated by $\mathbb{A}$ as in Subsection 1.2.2. Recall that members of $\mathcal{R}(\mathbb{A})$ are sequences $B = \langle\langle m_k, \mathbf{B}_k\rangle : k < \omega\rangle$ for each $k < \omega$, $\mathbf{B}_k$ is an J-tuple $(\mathbf{B}_{k,j})_{j<J}$, where each $\mathbf{B}_{k,j}$ is a substructure of $\mathbf{A}_{n_k,j}$ isomorphic to $\mathbf{A}_{k,j}$. To simplify our proofs we will identify sequences $B = \langle\langle m_k, \mathbf{B}_k\rangle : k < \omega\rangle$ with $\bigcup_{k\in\omega} \mathbf{B}_k$, when it makes sense. From now we will work with spaces $\mathcal{R}(\mathbb{A})$ for which $\bigcup_{k<\omega, j<J} \mathbf{A}_{k,j}$ makes sense.

Let $\mathcal{I}(\mathbb{A})$ be an ideal on $\bigcup_{k<\omega, j<J} \mathbf{A}_{k,j}$ such that $X \in \mathcal{I}(\mathbb{A})^+$ if X contains a member of $\mathcal{R}(\mathbb{A})$. Note that $\mathcal{I}(\mathbb{A})$ is homogeneous. Therefore $\mathcal{I}(\mathbb{A})$ is K-uniform.

Define the tree T as follows: Members of T are the empty set $\emptyset$ and initial segments of members of $\mathcal{R}(\mathbb{A})$, $b = r_l(B) = \bigcup_{k<l} \mathbf{B}_k$ for some $l \in \omega$ and $B \in \mathcal{R}(\mathbb{A})$. Given $a, b \in T$ we write $a \sqsubset b$ if there are some $B \in \mathcal{R}(\mathbb{A})$ and $j < l$ such that $a = r_j(B)$ and $b = r_l(B)$. Note that $[T] = \mathcal{R}(\mathbb{A})$. Also note that for every $n \in \omega$, $T[n]$ contains structures isomorphic to $\mathbb{A}_n$.

Remark. For every generating sequence $\mathbb{A}$, $\mathcal{I}(\mathbb{A})$ is a TRS ideal.

In particular for every $n \in \omega$, the n-th hypercube ideal $\mathcal{I}_n$ is a TRS ideal.

Theorem 3.22. *For every generating sequence* $\mathbb{A}$*, the ideal* $\mathcal{I}(\mathbb{A})$ *is* F_σ*.*

Proof. We prove that $\mathcal{I}(\mathbb{A})$ has the ISP. Take some $b = \langle\langle n_k, \mathbf{B}_k\rangle k < m\rangle \in T$. Let $k = n_m + 1$ and note that for every $l \geq k$ and $C \in \binom{\mathbf{A}_l}{\mathbf{A}_{m+1}}$, $b^\frown\langle l, C\rangle \in T$.

By 3.14, it follows that $\mathcal{I}(\mathbb{A})$ is F_σ. ∎

It is a consequence of this theorem that the quotient $\mathcal{I}(\mathbb{A})$

Corollary 3.23. *Ideals* $\mathcal{I}_n$ *are TRS ideals.*

In [46], Hrůsák uses the Katětov order to classify Borel and analytic ideals. In this work he proves that the eventually different ideal $\mathcal{ED}_{\mathsf{fin}}$ is tall, F_σ and K-uniform. He states the question: Is $\mathcal{ED}_{\mathsf{fin}}$ the only tall F_σ ideal which is K-uniform? In his PhD Thesis, Pelayo Gomez ([35]) answers Hrušák's question by proving the existence of an ideal which is tall, F_σ and K-uniform but it is not Katětov equivalent to $\mathcal{ED}_{\mathsf{fin}}$.

In this work we prove that the collection of hypercube ideals is a chain in the Katětov order of ideals which are tall, F_σ and K-uniform.

By Proposition 3.7 it follows that every hypercube ideal $\mathcal{I}_n$ is K-uniform. In order to prove that ideals $\mathcal{I}_n$ are not Katětov equivalent, we will work with Ramsey degrees for ideals.

Definition 3.24. Let $\mathcal{I}$ be an ideal on ω and n a natural number, the *Ramsey degree* $t(\mathcal{I}, n)$ is the least number t, if it exists, such that for every $l \geq 2$ and every coloring $c : [\omega]^n \to l$, there is some $X \in \mathcal{I}^+$ such that the restriction of c to $[X]^n$ takes no more than t colors. Similarly define the *Ramsey degree* $t(\mathcal{I}^+, n)$ as the least number t, if it exists, such that for every $l \geq 2$, for every $X \in \mathcal{I}^+$ and every coloring $c : [X]^n \to l$, there is some $Y \in \mathcal{I}^+ \cap [X]^\omega$ such that the restriction of c to $[Y]^n$ takes no more than t colors.

Note that if $\mathcal{I}$ is a TRS ideal and $(\mathcal{R}, \leq, r)$ is the topological Ramsey space that witness that, then $t(\mathcal{I}, n) = t(\mathcal{R}, n)$ and $t(\mathcal{I}^+, n) = t(\mathcal{R}^+, n)$.

Lemma 3.25. *If $\mathcal{I}$, $\mathcal{J}$ are ideals on ω such that $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ then for every $n \in \omega$, $t(\mathcal{I}, n) \leq t(\mathcal{J}, n)$.*

Proof. Fix some $n \in \omega$. Let $l \geq 2$ be a natural number and $c : [\omega]^n \to l$ be a coloring. Define $c' : [\omega]^n \to l$ such that for every $\{l_0, ..., l_{n-1}\} \in [\omega]^n$, $c'(\{l_0, ..., l_{n-1}\}) = c(\{f(l_0), ..., f(l_{n-1})\})$. By the Definition of $t(\mathcal{J}, n)$, there exists some $X \in \mathcal{J}^+$ such that $|c' \restriction [X]^n| \leq t(\mathcal{J}, n)$. Let $Y = f[X]$. Since $X \in \mathcal{J}^+$ and f is a Katětov mapping it follows that $Y \in \mathcal{I}^+$. Note that $|c \restriction [Y]^n| = |c' \restriction [X]^n| \leq t(\mathcal{J}, n)$. Therefore $t(\mathcal{I}, n) \leq t(\mathcal{J}, n)$. ∎

It follows by this Proposition that if $\mathcal{I}$, $\mathcal{J}$ are ideals on ω which are Katětov equivalent then for every $n \in \omega$, $t(\mathcal{I}, n) = t(\mathcal{J}, n)$.

Corollary 3.26. *If $m, n \geq 2$ are different natural numbers, ideals $\mathcal{I}_n, \mathcal{I}_m$ are not Katětov equivalent.*

Proof. In Corollary 2.19 we proved that for every $n \in \omega$, $t(\mathcal{H}^n, 2) = 1 + \sum_{i=0}^{n-1} 3^i$. Therefore if $n \neq m$ then the ideals $\mathcal{I}_n, \mathcal{I}_m$ are not Katětov equivalent. ∎

Typically it is not easy to prove that two ideals are not Katětov equivalent. By using Ramsey degrees it becomes easier. It is not trivial to calculate Ramsey degrees but for those ideals related to topological Ramsey spaces we have a method to calculate them. Those are some reasons why we care about ideals related to topological Ramsey spaces.

Proposition 3.27. *For every $n \in \omega$, $\mathcal{I}_n$ is an F_σ, K-uniform, tall ideal and $\mathcal{I}_n \leq_{\mathrm{K}} \mathcal{I}_{n+1}$ but they are not Katětov equivalent.*

Proof. This is a direct consequence of Theorem 3.22 and Corollary 3.26. ∎

Question 3.28 (Hrušák)**.** Is there a copy of $\mathcal{P}(\omega)/\mathrm{Fin}$ contained in the collection of all F_σ, tall and K-uniform ideals ?

3.1.3 The ideal conv

In this Subsection we prove that the ideal conv is a TRS ideal and the Ramsey degrees $t(\mathsf{conv}, 2)$ and $t(\mathsf{conv}^+, 2)$ are different. For this subsection we will say that a sequence is increasing if it is strictly increasing and we will say that a sequence is decreasing if it is strictly decreasing.

Let $\mathbf{S}$ be the collection of sets $X = \{x_{n,m} : n, m \in \omega\}$ contained in $\mathbb{Q} \cap [0, 1]$ such that for every $n \in \omega$, $\vec{x}_n = (x_{n,i})_{i\in\omega}$ is convergent in $[0, 1]$, one of the following conditions holds:

Case 1. $(\lim \vec{x}_n)_{n\in\omega}$ is an increasing sequence, every sequence $\vec{x}_n$ is an increasing sequence and for every $n \geq 1$, $x_{n,0} > \lim \vec{x}_{n-1}$.

Case 2. $(\lim \vec{x}_n)_{n\in\omega}$ is an increasing sequence, every sequence $\vec{x}_n$ is a decreasing sequence and for every $n \geq 1$, $x_{n,0} < \lim \vec{x}_{n+1}$.

Case 3. $(\lim \vec{x}_n)_{n\in\omega}$ is a decreasing sequence, every sequence $\vec{x}_n$ is an increasing sequence and for every $n \geq 1$, $x_{n,0} > \lim \vec{x}_{n+1}$.

Case 4. $(\lim \vec{x}_n)_{n\in\omega}$ is a decreasing sequence, every sequence $\vec{x}_n$ is a decreasing sequence and for every $n \geq 1$, $x_{n,0} < \lim \vec{x}_{n-1}$.

To simplify the following proof, we abuse notation and use $\mathrm{depth}_X(s)$ to denote the initial segment $t \sqsubset X$ such that $t \in T_{\mathrm{depth}_X(s)}$.

Theorem 3.29. *The ideal* conv *is a TRS ideal.*

Proof. Let $\varphi : \omega \to \omega\times\omega$ be the canonical correspondence. Denoted by $\ll$ the order on $\omega\times\omega$ inherited from φ, which means that $\{i, j\} \ll \{k, l\}$ if and only if $\varphi(\{i, j\}) < \{k, l\}$. Let T be the collection of finite sets $t \subset \mathbb{Q}\cap[0, 1]$ such that $t = \{x_{\varphi(i)} : i < n\}$ for some $n \in \omega$ and some $X = \{x_{n,m} : n, m \in \omega\}$ in the collection $\mathcal{S}$. Given $s, t \in T$, let $s \sqsubset t$ be the end-extension order considering the order $\ll$ on $\omega \times \omega$. Note that $[T] = \mathbf{S}$.

We show that T is Todorčević. First, note that $\emptyset \in T$. To prove (i), take $s, t, t' \in T$ such that $s \sqsubset t$ and $t \subset t'$. We can write $t' = \{y_{\varphi(i)} : i < m\}$. Therefore there exist $a, b \subseteq m$ such that $a \sqsubset b$, $s = \{y_{\varphi(i)} : i \in a\}$ $t = \{y_{\varphi(i)} : i \in b\}$. Let $s' = \{y_{\varphi(i)} \in s : i \leq \max a\}$ and note that $s \subseteq s'$ and $s' \sqsubset t'$.

To prove (ii) take $s \in T$ and $X \in [T]$ such that $s \subset X$. We prove that there is some $Y \in [T]$ such that $Y \subseteq X$ and $s \sqsubset Y$. Define $F = \{n \in \omega : t \cap \vec{x}_n \neq \emptyset\}$. For every $n \in F$, let $\vec{y}_n$ be a subsequence of $\vec{x}_n$ such that for every $m \in \omega$, $y_{n,m} \in s$ or $\varphi^{-1}(\{n, m\}) > \varphi^{-1}(\{n, \max F\})$. Let

$$Y = \{y_{n,m} : n \in F, m \in \omega\} \cup \{x_{n,m} : n > \max F, m \in \omega\}.$$

Note that $Y \subseteq X$ and $s \sqsubset Y$.

To prove (iii), take $s \in T$ and $X, Y \in [T]$ such that $s \subset X$ and $X \subseteq Y$. We show that there exists $Z \in [T]$ such that $\mathrm{depth}_Y(t) \sqsubset Z$ and $[s, Z] \subseteq [s, X]$. Let $M = \{i \in \omega : (\exists j)(i, j) = \varphi(\mathrm{depth}_Y(s))\}$. Since $\mathrm{depth}_Y(s)$ is an initial segment of Y, it follows than $M \in \omega$. Define $F = \{n \in \omega : \vec{y}_n \cap s \neq \emptyset\}$ and note that $F \subset M$. Let $Z = \{z_{n,m} : n, m \in \omega\}$ be such that every sequence $\vec{z}_n$ is a subsequence of $\vec{y}_n$ and satisfies that: for $n \in M$ and $\varphi^{-1}(\{n,m\}) < \mathrm{depth}_Y(s)$, $z_{n,m} = y_{n,m}$; for $n \in F$ and $\varphi^{-1}(\{n,m\}) \geq \mathrm{depth}_Y(s)$, $z_{n,m\in X}$; for $n \geq M$, $\vec{z}_n = \vec{x}_n$. Note that Z is as required.

Now we check the Ramsey condition. Let $s \in T_n$ and $X \in [T]$ be such that $s \subset X$. Fix some $\mathcal{O} \subset T_{n+1}$. Since $s \subset X$, there exists some finite set $F \subset \omega$ such that $s = \{x_{\varphi(i)} : i \in F\}$. Let E be the subset of ω consisting of natural numbers n such that $\vec{x}_n \cap s \neq \emptyset$. Define $j = \max E$ and $l = \min E$. We have three cases for the set $\succ_T (s)$:

Case a) If $|x_j \cap s| = 1$, then $succ_T(s) = \{s \cup \{x_{l,m}\} : m \geq |\vec{x}_l \cap \mathrm{depth}_X(s)|\}$. Since $\{x_{l,m} : m \geq |\vec{x}_l \cap \mathrm{depth}_X(s)|\}$ is countable, there exists some infinite set $M \subset \omega$ such that either for every $m \in M$, $s \cup \{x_{l,m}\} \in \mathcal{O}$, or for every $m \in M$, $s \cup \{x_{l,m}\} \notin \mathcal{O}$. Define $\vec{y}_l = (s \cap \vec{x}_l) \cup \{x_{l,m} : m \in M\}$. Let $Y = \{x_{m,n} : m \neq l\} \cup \vec{y}_l$. Therefore, $Y \subset X$ and Y is homogeneous for $\mathcal{O}$.

Case b) If $|x_j \cap s| = 2$, then $succ_T(s) = \{s \cup \{x_{i,m}\} : i > j, m \in \omega\}$. For every $i > j$, there is some $\vec{y}_i$ subsequence of $\vec{x}_i$ such that for every $n \in \omega$, $s \cup \{y_{i,n}\} \in \mathcal{O}$ or for every $n \in \omega$, $s \cup \{y_{i,n}\} \in \mathcal{O}^c$. There exists some infinite set $M \subset \omega$ such that for every $i \in M$ and for every $n \in \omega$, $s \cup \{y_{i,n}\} \in \mathcal{O}$ or for every $i \in M$ and for every $n \in \omega$, $s \cup \{y_{i,n}\} \in \mathcal{O}^c$. Define $Y = \{x_{m,n} : m \leq j\} \cup \{y_{i,n} : i \in M, n \in \omega\}$. Therefore Y is monochromatic for $\mathcal{O}$.

Case c) If $|x_j \cap s| > 2$, then $succ_T(s) = \{s \cup \{x_{j+1,m}\} : m \geq |\vec{x}_{j+1} \cap \mathrm{depth}_X(s)|\}$. Since $\{x_{j+1,m} : m \geq |\vec{x}_{j+1} \cap \mathrm{depth}_X(s)|\}$ is countable, there exists some infinite set $M \subset \omega$ such that either for every $m \in M$, $s \cup \{x_{j+1,m}\} \in \mathcal{O}$, or for every $m \in M$, $s \cup \{x_{j+1,m}\} \notin \mathcal{O}$. Define $\vec{y}_{j+1} = (s \cap \vec{x}_{j+1}) \cup \{x_{j+1,m} : m \in M\}$. Let $Y = \{x_{m,n} : m \neq j+1\} \cup \vec{y}_{j+1}$. Therefore, $Y \subset X$ and Y is homogeneous for $\mathcal{O}$.

Now we prove that for every $X \in \mathsf{conv}^+$, there exists $Y \in [T]$ such that $Y \subseteq X$. Given $X \in \mathsf{conv}^+$, there is a collection of sequences $\{\vec{x}_n : n \in \omega\}$ contained in X such that for every $n \in \omega$, there exists $\lambda_n \in [0,1]$ such that $\lim \vec{x}_n = \lambda_n$, $n \neq m$ implies $\lambda_n \neq \lambda_m$ and $\bigcup_{n\in\omega} \vec{x}_n \subseteq X$. For every $n \in \omega$, the sequence $\vec{x}_n$ has either an increasing subsequence or a decreasing subsequence. Let $\vec{y}_n$ be an increasing subsequence of $\vec{x}_n$ if one exists; otherwise, let $\vec{y}_n$ be a decreasing subsequence of $\vec{x}_n$. There exists $M \in [\omega]^\omega$ such that for every $n \in M$, $\vec{y}_n$ is an increasing sequence or for every $n \in M$, $\vec{y}_n$ is a decreasing sequence. The sequence of limits $(\lambda_n)_{n\in\omega}$ has an increasing subsequence or a decreasing subsequence.

Case 1. There exists $N \in [M]^\omega$ such that $(\lambda_n)_{n\in N}$ is an increasing sequence, and for every $n \in M$, $\vec{y}_n$ is an increasing sequence. For every $n \in N$ with $n \neq \min N$, take $\vec{z}_n$ to be tail of $\vec{y}_n$ such that $z_{n,0} > \lambda_{n^*}$ where $n^* = \max\{m \in N : m < n\}$.

Case 2. There exists $N \in [M]^\omega$ such that $(\lambda_n)_{n\in N}$ is an increasing sequence, and for every $n \in M$, $\vec{y}_n$ is a decreasing sequence. For every $n \in N$, take $\vec{z}_n$ to be tail of $\vec{y}_n$ such that $z_{n,0} < \lambda_{n^*}$ where $n^* = \min\{m \in N : m > n\}$.

Case 3. There exists $N \in [M]^\omega$ such that $(\lambda_n)_{n\in N}$ is a decreasing sequence, and for every $n \in M$, $\vec{y}_n$ is an increasing sequence. For every $n \in N$, take $\vec{z}_n$ to be tail of $\vec{y}_n$ such that $z_{n,0} > \lambda_{n^*}$ where $n^* = \min\{m \in N : m > n\}$.

Case 4. There exists $N \in [M]^\omega$ such that $(\lambda_n)_{n\in N}$ is a decreasing sequence, and for every $n \in M$, $\vec{y}_n$ is a decreasing sequence. For every $n \in N$ with $n \neq \min N$, take $\vec{z}_n$

to be tail of $\vec{y}_n$ such that $z_{n,0} < \lambda_{n^*}$ where $n^* = \max\{m \in N : m < n\}$.

In each case, let $Z = \{z_{n,m} : n, m \in \omega\}$. Note that $Z \subseteq X$ and $Z \in [T]$. ∎

To calculate $t(\mathsf{conv}^+, 2)$, note that if we fix a member of the Ramsey space $X = \{x_{n,m} : n, m \in \omega\}$ then there is a natural correspondence between members of $\mathsf{conv}^+ \upharpoonright X$ and Fin × Fin-positive sets by identifying every member $x_{n,m}$ with their subscripts $\{n, m\}$.

Proposition 3.30. $t(\mathsf{conv}^+, 2) = 4$.

Proof. Fix $X \in [T]$ and let $c : [X]^2 \to m$ be a coloring. Define $c' : [\omega \times \omega]^2 \to m$ as follows, $c'((i,j),(m,n)) = c(x_{i,j}, x_{m,n})$. Since $t(\text{Fin} \times \text{Fin}, 2) = 4$, there exists $A \in \text{Fin} \times \text{Fin}^+$ such that $c' \upharpoonright [A]^2$ touches at most four colors. Let $Y = \{x_{n,m} : (n,m) \in A\}$. Therefore $c \upharpoonright [Y]^2$ touches at most four colors. Since $A \in \text{Fin} \times \text{Fin}^+$, it follows that $Y \in \mathsf{conv}^+$. Therefore, $t(\mathsf{conv}^+, 2) \leq 4$. To see that $t(\mathsf{conv}^+, 2)$ is exactly 4, fix $X \in [T]$ and define $c : [X]^2 \to 4$ as follows:

$$c(x_{i,j}, x_{m,n}) = \begin{cases} 0 & \text{if } i = m \text{ and } j < n \\ 1 & \text{if } i < m, j < n \text{ and } m - i < n - j \\ 2 & \text{if } i < m, j < n \text{ and } m - i > n - j \\ 3 & \text{if } i < m \text{ and } j > n \end{cases}$$

In the first case both points belong to the same sequence. In the second case the subscripts grow above the diagonal. In the third case the subscripts grow below the diagonal. Finally, in the fourth case the subscripts decrease. Note that for every $Y \in \mathsf{conv}^+ \upharpoonright X$, Y contains pairs with the four cases mentioned above. Therefore, $c \upharpoonright [Y]^2$ touches exactly four colors. ∎

Now we calculate $t(\mathcal{U}, 2)$ where $\mathcal{U}$ is the generic filter forced by $(\mathcal{P}(\omega)/\mathsf{conv}, \subset)$. First, note that $(\mathcal{P}(\omega)/\mathsf{conv}, \subset)$ is σ-closed so $\mathcal{U}$ is actually an ultrafilter.

Proposition 3.31. $t(\mathsf{conv}, 2) = 2$.

Proof. Let m be a natural number and let $c : [\mathbb{Q} \cap [0,1]]^2 \to m$ be a coloring. For every $s \in T_3$, let $\{p^s, q^s, r^s\}$ be the increasing enumeration of s. Then

$$[s]^2 = \{\{p^s, q^s\}, \{p^s, r^s\}, \{q^s, r^s\}\}.$$

Let $c' : T_3 \to m^3$ be a coloring such that for every $s \in T_3$,

$$c'(s) = \langle c(\{p^s, q^s\}), c(\{p^s, r^s\}), c(\{q^s, r^s\}) \rangle.$$

For every $\iota \in m^3$, let $\mathcal{F}_\iota = \{s \in T_3 : c'(s) = \iota\}$. Fix $X \in [T]$. By the Nash-Williams theorem, there exists $Y \leq X$ in $[T]$ and $\iota_0 \in m^3$ such that $T_3 \upharpoonright Y \subset \mathcal{F}_{\iota_0}$.

Note that according to what case Y belongs to, there are four options for members $s = \{z_{0,0}, z_{0,1}, z_{1,0}\}$ of $T_3 \upharpoonright Y$. If Y belongs to the **case 1**, for every $s \in T_3$, $z_{0,0} < z_{0,1} <$

$z_{1,0}$. If Y belongs to the **case 2**, for every $s \in T_3$, $z_{0,1} < z_{0,0} < z_{1,0}$. If Y belongs to the **case 3**, for every $s \in T_3$, $z_{1,0} < z_{0,0} < z_{0,1}$. If Y belongs to the **case 4**, for every $s \in T_3$, $z_{1,0} < z_{0,1} < z_{0,0}$.

Without loss of generality, assume that Y satisfies the**case 1**. Then for every $s = \{z_{0,0}, z_{0,1}, z_{1,0}\}$, it holds that $z_{0,0} < z_{0,1} < z_{1,0}$. Hence $c(\{z_{0,0}, z_{0,1}\}) = \iota(0)$, $c(\{z_{0,0}, z_{1,0}\}) = \iota(1)$ and $c(\{z_{0,1}, z_{1,0}\}) = \iota(2)$. For every pair $\{y_{i,j}, y_{i,k}\}$ with $j < k$, take $l > i$. Then $\{y_{i,j}, y_{i,k}, y_{l,0}\} \in T$ and $c(\{y_{i,j}, y_{i,k}\}) = \iota(0)$. For every pair $\{y_{i,j}, y_{m,n}\}$ with $i < m$, take $l > j$. Then $\{y_{i,j}, y_{i,l}, y_{m,n}\} \in T$ and $c(\{y_{i,j}, y_{m,n}\}) = \iota(1)$. Therefore, $|c \restriction [Y]^2| = 2$ and $t(\mathsf{conv}, 2) \leq 2$.

To see that $t(\mathsf{conv}, 2) = 2$, let $\{q_n : n \in \omega\}$ be an enumeration for $\mathbb{Q}$. Let $c : [\mathbb{Q} \cap [0,1]]^2 \to 2$ be such that if $m < n$:

$$c(\{q_m, q_n\}) = \begin{cases} 0 & if \quad q_m < q_n \\ \\ 1 & if \quad q_m > q_n \end{cases}$$

If X is monochromatic for c, either X is a increasing sequence or X is a decreasing sequence. In any case, $X \in \mathsf{conv}$.

■

Up to now, we only know a few ideals with the Dependence Property. These are the ideal conv and all the Fin^α. These ideals are σ-closed so we believe that for every TRS ideal the quotient is σ-closed.

3.1.4 Summable ideals are not TRS

In this subsection we classify summable ideals. At the beginning, we tried to classify the summable ideal by using the definition, but we did not have success. Then we used the properties of the forced ultrafilter forced to approach this question. So, we use fact that forcing with TRS ideals adds Ramsey ultrafilters Rudin Keisler below the generic filter to prove that summable ideals are not TRS.

Corollary 3.32. *Let $\mathcal{I}$ be a TRS ideal and let $\mathcal{G}$ be an $(\mathcal{I}^+, \subseteq_\mathcal{I})$-generic filter. Then $(\mathcal{I}^+, \subseteq_\mathcal{I})$ forces a Ramsey ultrafilter $\mathcal{U}$ such that $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{G}$.*

Proof. Let $\mathcal{I}$ be a TRS ideal and $(T, \sqsubset)$ some witness tree. Let $\mathcal{G}$ be an $(\mathcal{I}^+, \subseteq)$-generic filter. Fix $Y \in \mathcal{G}$. For every $X \in \mathcal{G}$ define $D(X) = \{\mathrm{depth}_Y(a) : a \in (T_1 \cap [X \cap Y]^{<\omega})\}$. By the proof of Theorem 2.1 it follows that $\mathcal{U}$ is a Ramsey ultrafilter.

Now we will prove that $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{G}$. Let $f : \omega \to \omega$ be such that for every $n \in Y$, $f(n) = k$ if $n \in Y(k) = Y||(k+1) \setminus Y||k$. Take $A \in \mathcal{U}$. There exists $X \in \mathcal{G}$ such that $D(X) \subset A$. To see that $X \cap Y \subseteq f^{-1}[A]$ take $m \in X \cap Y$. Let l be the unique natural number such that $m \in Y(l)$. Then $f(m) = l$. Since $\{m\} \in T_1$, it follows that $l \in D(X) \subset A$. Therefore $m \in f^{-1}[A]$. Since $X, Y \in \mathcal{G}$ and $\mathcal{G}$ is a filter then $f^{-1}[A] \in \mathcal{G}$. ■

Theorem 3.33 (Hrušák,Verner)**.** *(Hrušák, Verner) If $\mathcal{I}$ is a tall F_σ P-ideal, then every $\mathcal{P}(\omega)/\mathcal{I}$- generic filter is a P-point with no rapid RK-predecessor.*

Theorem 3.34. *For every function $f : \omega \to \mathbb{R}^+$ tending to zero such that $\sum_{n<\omega} f(n) = \infty$, the summable ideal $\mathcal{I}_f$ is not a TRS ideal.*

Proof. Let $\mathcal{I}_f$ be a summable ideal. Suppose that $\mathcal{I}_f$ is a TRS ideal and let $\mathcal{G}$ be an $(\mathcal{I}_f^+, \subseteq_{\mathcal{I}})$-generic filter. By Corollary 3.32, $(\mathcal{I}_f^+, \subseteq_{\mathcal{I}})$ forces a Ramsey ultrafilter $\mathcal{U}$ such that $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{G}$. Since $\mathcal{I}_f$ is a tall F_σ P-ideal it follows from Theorem 3.33 that $\mathcal{G}$ is a p-point with no rapid $RK-$predecessor, which is a contradiction because $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{G}$ and $\mathcal{U}$ is a Ramsey ultrafilter. ■

Corollary 3.35. *There are F_σ ideals which are not TRS ideals.*

The following list summarizes the known classification results of ideals we studied.

- The ideal Fin is a TRS ideal. (Ellentuck, [31])
- For every $\alpha < \omega_1$, Fin^α is a TRS ideal. (Dobrinen [22], [23])
- The *eventually different ideal* $\mathcal{ED}$ is a TRS ideal. (Proposition 3.17)
- The ideal $\mathcal{ED}_{\mathsf{fin}}$ is a TRS ideal. (Proposition 3.15)
- For every function $f : \omega \to \mathbb{R}^+$ tending to zero such that $\sum_{n<\omega} f(n) = \infty$, the summable ideal $\mathcal{I}_f$ is not a TRS ideal. (Theorem 3.34)
- Solecki's ideal $\mathcal{S}$ is not classified.
- For every $n \in \omega$, the n-th hypercube ideal $\mathcal{I}_n$ is a TRS ideal. (Corollary 3.23)
- The ideal **conv** is a TRS ideal. (Theorem 3.29)
- The asymptotic density zero ideal $\mathcal{Z}$ is not classified.
- The *random graph* ideal is not classified. We know that the collection of positive sets contains a copy of $\mathcal{ED}^+_{fin}$.
- The ideal $\mathcal{G}_c$ is not classified.

3.2 The HL property

It is known that some properties of the ground model may be destroyed after we force with a partial order. Given an ultrafilter $\mathcal{U}$, we say that $\mathcal{U}$ *is preserved by* $\mathbb{S}$ if for every infinite set $X \subseteq \omega$ in the extension there is $Y \in \mathcal{U}$ such that $Y \subseteq X$ or $Y \cap X = \emptyset$.

In [60], Miller proves that P-points are preserved by Sacks forcing. For selective ultrafilters, which are forced by the Ellentuck space, Baumgartner proved that they are preserved under Sacks forcing, product Sacks forcing and iterated Sacks forcing

([4],[5]). A natural question is if every ultrafilter generic for a topological Ramsey space is preserved under Sacks forcing. Yuan Yuan Zheng extended Baumgartner's result by proving that generic ultrafilters forced by the Milliken space of block sequences, spaces $\mathcal{R}_\alpha$ and high dimensional Ellentuck spaces are Sacks indestructible. Moreover, in her PhD thesis ([76]) she states a condition ($L4$) that guarantees that the ultrafilter forced by a topological Ramsey space is Sacks indestructible. Up to now, it is not clear whether all topological Ramsey spaces satisfy $L4$.

We are interested in extending Baumgartner's result not to every topological Ramsey space but to TRS ideals. We use a different approach. In a upcoming work, Chodounski, Guzmán and Hrušák introduce the Halpern-Läuchli property and classify several Borel ideals by using this property. In this section we show the relation of the HL property with Baumgarter's result and use this property to prove that several TRS ideals satisfy Baumgartner's result.

If p is a Sacks tree and $A \in [\omega]^\omega$, we let $p \restriction A = \bigcup_{m\in A}(p \cap 2^m)$.

Definition 3.36 (Chodounsky, Guzmán and Hrušák)**.** A family $\mathcal{H} \subseteq [\omega]^\omega$ is called a *Halpern-Läuchli family* if for every $c : 2^{<\omega} \to 2$ there are $p \in \mathbb{S}$ and $A \in \mathcal{H}$ such that $p \restriction A$ is monochromatic.

Proposition 3.37 (Chodounsky, Guzmán and Hrušák)**.** *Let $\mathcal{H} \subseteq [\omega]^\omega$ be a family closed under finitely many changes. Then the following conditions are equivalent:*

- *$\mathcal{H}$ is a Halpern-Läuchli family,*
- *for every $p \in \mathbb{S}$ and $c : p \to \{0, 1\}$ there is $q \leq p$ and $A \in \mathcal{H}$ such that $p \restriction A$ is monochromatic.*

Definition 3.38 (Chodounsky, Guzmán and Hrušák)**.** We say an ideal $\mathcal{I}$ is HL if $\mathcal{I}^+$ is a Halpern-Läuchli family.

Proposition 3.39 (Chodounsky, Guzmán and Hrušák)**.** *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω such that $\mathcal{I} \simeq_{\mathrm{K}} \mathcal{J}$. If $\mathcal{I}$ is not HL then J is not HL.*

Remark. Let $\mathcal{I}$ be a K-uniform Borel ideal. $\mathcal{I}$ is HL if and only if for every $X \in \mathcal{I}^+$, $\mathcal{I} \restriction X$ is HL.

The following results classify the most studied Borel ideals into those which are HL and those which are not.

Proposition 3.40 (Chodounsky, Guzmán and Hrušák)**.** *If $\mathcal{I}$ is not* HL *then there is $X \in \mathcal{I}^+$ such that $\mathcal{ED} \leq_{\mathrm{K}} \mathcal{I} \restriction X$.*

Proposition 3.41 (Chodounsky, Guzmán and Hrušák)**.** *If $\mathcal{I}$ is not* HL *then* $\mathsf{conv} \leq_{\mathrm{K}} \mathcal{I}$.

By this Proposition follows that ideal $\mathcal{R}$ is HL.

Proposition 3.42 (Chodounsky, Guzmán and Hrušák)**.** *Ideals* nwd *and* $\mathcal{G}_c$ *are* HL*, but the ideal* $\mathcal{Z}$ *is not HL.*

It is a consequence of this Proposition that the ideal $\mathrm{Fin} \times \mathrm{Fin}$ is HL.

Theorem 3.43. *Every TRS ideal with the Weak Independence Property is HL.*

Proof. Let $\mathcal{I}$ be a TRS ideal. Let $c : 2^{<\omega} \to 2$ be a coloring and let $\mathcal{U}$ be any ultrafilter extending the dual filter $\mathcal{I}^*$. Note that $\mathcal{U}$ is contained in $\mathcal{I}^+$. For every $x \in 2^\omega$ and $i \in 2$, let

$$H_x^i = \{n \in \omega : c(x \restriction n) = i\}.$$

Since $\mathcal{U}$ is an ultrafilter, for every $x \in 2^\omega$ there is an $i \in 2$ satisfying that $H_x^i \in \mathcal{U}$.

First, we are proving that there are $i \in 2$ and $s \in 2^{<\omega}$ satisfying that for every $t \in 2^{<\omega}$ extending s, there exists $y \in \langle t \rangle$ such that $H_t^i \in \mathcal{U}$. Fix some $s \in 2^{<\omega}$ and $i \in 2$. If they do not satisfy the statement, there exists some t extending s such that for every $x \in \langle t \rangle$, $H_x^{1-i} \in \mathcal{U}$. Therefore t and $1 - i$ satisfy the statement.

Let s and i be as before. Now we define a sequence q_n of finite subtrees of $2^{<\omega}$ such that for every $n \in \omega$ there exists some $a_n \in T$ such that:

1. $q_n \sqsubset q_{n+1}$ and $a_n \sqsubset a_{n+1}$,
2. $q_n \restriction a_n$ is monochromatic,
3. $q_n \subset 2^{\leq \max a_n}$, and
4. $|q_n \cap 2^{\max a_n}| = 2^n$.

Let q_0 be the tree that only contains s. Now suppose we know q_n and a_n satisfying statements 1-5. Enumerate $q_n \cap 2^{\max a_n}$ as $\{s_l : l < 2^n\}$. For every $s \in q_n \cap 2^{\max a_n}$ and $j \in 2$, there exists $y \in \langle s^\frown(j) \rangle$ such that $H_y^i \in \mathcal{U}$. Let $H_{s,j} = H_y^i$. Since every $H_{s,j}$ belongs to $\mathcal{U}$, there exists $H \in \mathcal{U}$ contained in all of them. Take $X \in [T]$ contained in $H \setminus \max a_n$. Since $\mathcal{I}$ has the WIP, there exists $a_{n+1} \in T$ extending a_n such that $a_{n+1} \setminus a_n \subset X$. Note that $p_{n+1} \restriction a_{n+1}$ is monochromatic.

Define $q = \bigcup_{n<\omega} q_n$ and $A = \bigcup_{n<\omega} a_n$. Note that q is a perfect tree and A is $\mathcal{I}$-positive. Moreover, $q \restriction A$ is monochromatic. ■

Definition 3.44. We say that an ideal $\mathcal{I}$ is *hereditary HL* if for every $X \in \mathcal{I}^+$, $\mathcal{I} \restriction X$ is HL.

Corollary 3.45. *Every TRS ideal with the Weak Independence Property is hereditary HL.*

Proof. By Theorem 3.43 we know that $\mathcal{I}$ is HL. If $X \in \mathcal{I}^+$ then $\mathcal{I}^+ \restriction X$ has the WIP. Therefore $\mathcal{I}^+ \restriction X$ is HL ■

It is known that the property of being HL is closely related to the preservation of ultrafilters by Sacks forcing. The reader interested in this topic is referred to [51] and [74]. First, note that since $\mathcal{I}^+$ does not add any real, it is equivalent force with $\mathcal{I}^+ \times \dot{\mathbb{S}}$ and force with $\mathcal{I}^+ \times \mathbb{S}$. Therefore, if $\dot{\mathcal{U}}$ is $\mathcal{I}^+$-generic and $\dot{\mathcal{G}}$ is $\mathbb{S}$-generic then $\mathbf{V}[\dot{\mathcal{U}} * \dot{\mathcal{G}}] = \mathbf{V}[\dot{\mathcal{U}}][\dot{\mathcal{G}}] = \mathbf{V}[\dot{\mathcal{G}}][\dot{\mathcal{U}}]$.

Theorem 3.46 (Yiparaki, [74])**.** *Let $\mathcal{U}$ be an ultrafilter. $\mathcal{U}$ is Halpern-Läuchli if and only if $\mathcal{U}$ is preserved by one step Sacks forcing.*

The following theorem can be deduced from Yiparaki's theorem by proving that an ideal is hereditary HL if and only if the ultrafilter forced by $\mathcal{I}^+$ is Sacks indestructible. We present here a direct proof.

Theorem 3.47. *Let $\mathcal{I}$ be an ideal such that $(\mathcal{I}^+, \subseteq_{\mathcal{I}})$ does not add reals, and let $\mathcal{U}$ be an $\mathcal{I}^+$-generic ultrafilter. $\mathcal{I}$ is hereditary HL if and only if $\mathcal{U}$ is Sacks indestructible.*

Proof. First, assume that $\mathcal{I}$ is hereditary HL. Let $\dot{X}$ be an $\mathbb{S}$-name for an infinite subset of ω. We are building a function $c : 2^{<\omega} \to 2$ depending on $\dot{X}$ and a condition $q \leq p$ such that for every $n \in \omega$, $i \in \{0, 1\}$ and for every $s \in l(n, q)$ it holds that $c(s^\frown(i)) = 1$ if and only if $q \restriction s^\frown(i) \Vdash$ "$n + 1 \in \dot{X}$".

To build q, define a fusion sequence $\langle q_n : n \in \omega\rangle$ such that:

(i) For every $n \in \omega$, $q_{n+1} \leq_n q_n$,

(ii) for every $s \in l(n, q)$ and $i \in \{0, 1\}$, $q_{n+1} \restriction s^\frown(i)$ decides the formula "$n + 1 \in \dot{X}$ ".

Let $q_0 = q$. Suppose we know q_n and we are defining q_{n+1}. We abuse notation and think of $\dot{X}$ as a $\mathbb{S}$-name for a characteristic function instead of a name for a set. By (ii), $q_n \Vdash$ "$(\exists i \in 2)(\dot{X}(n + 1) = i)$", and by Corollary 1.41 there exists $r \leq_n q_n$ satisfying that for every $s \in l(n, q_n)$ there exists $i_s \in 2$ such that $r \restriction s \Vdash$ "$\dot{X}(n + 1) = i_s$". Let $q_{n+1} = r$.

Let $q = \bigcup_{n\in\omega} q_n$.

By the construction it follows that for every $n \in \omega$ and $s \in l(n, q)$ and $i \in \{0, 1\}$, $q \restriction s^\frown(i)$ decides the formula "$n + 1 \in \dot{X}$". Define $c : q \to \{0, 1\}$ such that if $s \in l(n, q)$ then $c(s) = 1$ if and only if $q \restriction s^\frown(i) \Vdash$ "$n + 1 \in \dot{X}$".

Let $\mathcal{D} = \{Z \in \mathcal{I}^+ : (\exists r \leq q)(r \restriction Z \text{ is monochromatic}) \text{ and } (Z \subseteq \dot{X} \text{ or } z \cap \dot{X} = \emptyset)\}$. Let $Y \subseteq \omega$ be an $\mathcal{I}$-positive set. Without loss of generality we can assume that $Y \subseteq \dot{X}$ or $Y \cap \dot{X} = \emptyset$. Since $\mathcal{I}$ is hereditary HL there exist $r \leq q$ and $Z \subseteq Y$ $\mathcal{I}$-positive set such that $r \restriction Z$ is monochromatic.

Take $Z \in \mathcal{D} \cap \mathcal{U}$ and note that $Z \in \mathcal{U}$ and $Z \subseteq \dot{X}$ or $Z \cap \dot{X} = \emptyset$.

Now, assume that $\mathcal{U}$ is Sacks indestructible. Let $X \in \mathcal{I}^+$ and $c : 2^\omega \to 2$ be a coloring. We shall show that there are $p \in \mathbb{S}$ and $Y \subseteq X$ an I-positive satisfying that $p \restriction Y$ is monochromatic.

Note that $X \Vdash$ "$X \in \mathcal{U}$". Let $\dot{r}$ be the Sacks real and let $\dot{Y}$ be an $\mathbb{S}$-name such that $2^{<\omega} \Vdash$ "$\dot{Y} = \{n \in \omega : c(\dot{r} \restriction n) = 1\}$". Let $\dot{\mathcal{V}}$ be a name for the upward closure of

$\mathcal{U}$. Because $\mathbb{S}$ preserves $\mathcal{U}$, there are $p \in \mathbb{S}$ and $Z \in \mathcal{U}$ that satisfy that $p \Vdash$ "$\dot{Z} \subseteq \dot{Y}$" or $p \Vdash$ "$\dot{Z} \cap \dot{Y} = \emptyset$". Therefore, $p \restriction Z$ is monochromatic. Since $\mathcal{U}$ is an ultrafilter, $X \cap Z \Vdash$ "$p \restriction \dot{X} \cap \dot{Z}$ is monochromatic". Since $\mathcal{I}$ does not add reals, it follows that $p \restriction X \cap Z$ is monochromatic.

■

Theorem 3.48. *Let $\mathcal{I}$ be an ideal with the WIP and let $\mathcal{U}$ be an ultrafilter $\mathcal{I}^+$-generic. $\mathcal{U}$ is Sacks indestructible.*

Proof. This follows from Corollary 3.45 and last theorem. ■

Theorem 3.49 (Zheng)**.** *Let $\mathcal{G}_k$ be a generic ultrafilter for $\mathcal{P}(\omega^k)/\mathrm{Fin}^k$. $\mathcal{G}_k$ is preserved by both side-by-side Sacks forcing and iterated Sacks forcing.*

Corollary 3.50. *The generic ultrafilter for $\mathcal{P}(\mathbb{Q})/\mathsf{conv}$ is preserved by both side-by-side Sacks forcing and iterated Sacks forcing.*

Proof. This follows directly from Zheng's result and the fact that every conv-positive set contains a copy of $\mathrm{Fin} \times \mathrm{Fin}$. ■

Up to now, it seems that generic ultrafilters for TRS ideal with the dependence property are preserved by Sacks forcing.

Question 3.51. Is there a TRS ideal with the dependence property such that the generic ultrafilter is not preserved by Sacks forcing?

Chapter 4

Pseudointersection and tower numbers

As we mention in 1.6, the combinatorial structure of $\mathcal{P}(\omega)/\mathrm{Fin}$ has been deeply studied and it is closely related to the cardinal invariants of the continuum. In recent years research has focused on structures similar to $\mathcal{P}(\omega)/\mathrm{Fin}$ and their natural generalization of cardinal invariants. Cardinals $\mathfrak{p}$ and $\mathfrak{t}$ are invariant cardinals of the Boolean algebra $\mathcal{P}(\omega)/\mathsf{fin}$. It was a long standing open question whether they are equal or different. By work of Malliaris and Shelah [53] we now know that they are equal. In this chapter we study the natural generalization of $\mathfrak{p}$ and $\mathfrak{t}$ to Ramsey spaces and ideals. We investigate whether the associated pseudointersection and tower numbers are the same to each other, and also we compare them to $\mathfrak{p}$.

Definition 4.1. We say that $(\mathcal{R}, \leq, \leq^*, r)$ is a *σ-closed topological Ramsey space* if $(\mathcal{R}, \leq, r)$ is a topological Ramsey space and $\leq^*$ is a σ-closed order on $\mathcal{R}$ coarsening $\leq$ such that $(\mathcal{R}, \leq)$ and $(\mathcal{R}, \leq^*)$ have isomorphic separative quotients. Let $\mathcal{F}$ be a subset of $\mathcal{R}$.

- We say that $\mathcal{F}$ has the *strong finite intersection property* (SFIP) if for every finite subfamily $\mathcal{G} \subseteq \mathcal{F}$, there exists $Y \in \mathcal{R}$ such that for each $X \in \mathcal{F}$, $Y \leq^* X$.
- $Y \in \mathcal{R}$ is called a *pseudointersection* of the family $\mathcal{F}$ if for every $X \in \mathcal{F}$, $Y \leq^* X$.

Definition 4.2. Let $(\mathcal{R}, \leq, \leq^*, r)$ be a σ-closed topological Ramsey space.

- The *pseudointersection number* $\mathfrak{p}_{\mathcal{R}}$ is the smallest cardinality of a family $\mathcal{F} \subseteq \mathcal{R}$ which has the SFIP but does not have a pseudointersection.
- We say that $\mathcal{F}$ is a *tower* if it is linearly ordered by $\geq^*$ and has no pseudointersection. The *tower number* $\mathfrak{t}_{\mathcal{R}}$ is the smallest cardinality of a tower of $(\mathcal{R}, \leq^*)$.

Note that for every topological Ramsey space $\mathcal{R}$, $\mathfrak{p}_{\mathcal{R}} \leq \mathfrak{t}_{\mathcal{R}}$. A recent groundbreaking result of Malliaris and Shelah shows that $\mathfrak{p} = \mathfrak{t}$ (see [53] and [54]). For all the spaces considered in this section, we will show that they are also indeed equal.

As in previous sections, we will assume that $\mathcal{AR}$ is countable. If this is not the case, we tacitly work on $\mathcal{AR} \upharpoonright X$ for some $X \in \mathcal{R}$, which is countable by axiom **A.2**.

4.1 Pseudointersection and tower numbers for several classes of topological Ramsey spaces

The following property is satisfied by many topological Ramsey spaces, including several discussed in Section 1.2.

Definition 4.3 (IEP). We say that a topological Ramsey space $(\mathcal{R}, \leq, r)$ has the *Independent Extension Property (IEP)* if the following hold: Each $X \in \mathcal{R}$ is a sequence of the form $\langle X(n) : n \in \omega \rangle$ such that for every $n \in \omega$, $r_n(X) = \langle X(i) : i < n \rangle$, and each $X(i)$ is a finite set, possibly, but not necessarily, with some relational structures on it. Furthermore, for every $X \in \mathcal{R}$, $k \in \omega$, and $s \in \mathcal{AR}_k$, there exist $m \in \omega$ and $s(k) \subseteq X(m)$ such that $s^\frown s(k) \in \mathcal{AR}_{k+1}$ and $s \sqsubseteq s^\frown s(k)$.

Note from the definition that if a topological Ramsey space has the IEP then it has Independent Sequences of structures. Also note that Ramsey spaces $\mathcal{R}_1$, the Milliken space $\mathrm{FIN}^{[\infty]}$, and spaces generated by Fraïssé classes $\mathcal{R}(\mathbb{A})$ satisfy this property. On the other hand, high dimensional spaces $\mathcal{E}_k$, the Carlson-Simpson space $\mathcal{E}_\infty$ and spaces of strong subtrees $\mathcal{S}_\infty(U)$ do not satisfy this property.

Theorem 4.4. *Let $(\mathcal{R}, \leq, \leq^*, r)$ be a σ-closed topological Ramsey space with the IEP, and suppose that $\mathcal{R}$ closed in $\mathcal{AR}^\omega$. Then $\mathfrak{p} \leq \mathfrak{p}_\mathcal{R}$.*

Proof. Recall that by Bell's result, $\mathfrak{p} = \mathfrak{m}(\sigma\text{-centered})$. Let $\kappa < \mathfrak{m}(\sigma\text{-centered})$ be given, and let $\mathcal{F} = \{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{R}$ be a family with the SFIP. Define $\mathbb{P}$ to be the collection of all ordered pairs $\langle s, E \rangle$ such that $s \in \mathcal{AR}$ and $E \in [\kappa]^{<\omega}$. Since $\mathcal{R}$ satisfies the IEP, define some shorthand notation as follows: For $s, t \in \mathcal{AR}$ with $s \sqsubseteq t$, note that $s = \langle s(i) : i < m \rangle$ and $t = \langle t(i) : i < n \rangle$ for some $m \leq n$. Then let

$$t/s = \langle t(i) : m \leq i < n \rangle. \tag{4.1.1}$$

Define a partial order $\leq$ on $\mathbb{P}$ as follows: Given $\langle s, E \rangle, \langle t, F \rangle \in \mathbb{P}$, let $\langle t, F \rangle \leq \langle s, E \rangle$ if and only if $s \sqsubseteq t$, $E \subseteq F$, and there exists $X \in \mathcal{R}$ such that for every $\alpha \in E$, $X \leq X_\alpha$ and $t/s \subseteq X$, which means that for every $i \in [|s|, |t|)$ there exists l such that $t(i) \subseteq X(l)$.

For every $s \in \mathcal{AR}$, define $\mathbb{P}_s = \{\langle s, E \rangle : E \in [\kappa]^{<\omega}\}$. Note that $\mathbb{P}_s$ is centered. Since $\mathcal{AR}$ is countable, $\mathbb{P} = \bigcup_{s \in \mathcal{AR}} \mathbb{P}_s$ is a σ-centered partial order. Given $\alpha < \kappa$ and $m \in \omega$ let

$$D_{\alpha,m} = \{\langle s, E \rangle \in \mathbb{P} : \alpha \in E \text{ and } |s| > m\}.$$

Claim. 1. *$D_{\alpha,m}$ is dense.*

Proof. Fix $\langle t, F \rangle \in \mathbb{P}$. If $|t| > m$, then the pair $\langle t, F \cup \{\alpha\} \rangle \leq \langle t, F \rangle$ is in $D_{\alpha,m}$. If $|t| \leq m$, since $\mathcal{F}$ has the SFIP, there exists $X \in \mathcal{R}$ such that for every $\beta \in F$, $X \leq X_\beta$. Let $i = |t|$. By the IEP, there is some $u \in \mathcal{AR}_{m+1}$ such that u extends t into X. That is, there is a strictly increasing sequence $l_i < \ldots < l_m$ and substructures $u(j) \in \mathcal{R}(j) \restriction X(l_j)$, for each $j \in [i, m]$, such that $u = t^\frown \langle u(j) : i \leq j \leq m \rangle$ is a member of $\mathcal{AR}_{m+1}$. Let $E = F \cup \{\alpha\}$. By the choice of u, it follows that $\langle u, E \rangle \leq \langle t, F \rangle$ and $\langle u, E \rangle \in D_{\alpha,m}$. ■

Let $\mathcal{D} = \{D_{\alpha,m} : \alpha \in \kappa \text{ and } m \in \omega\}$ and let G be a filter on $\mathbb{P}$ meeting each dense set in $\mathcal{D}$. Let $X_G = \bigcup\{s : \exists E \in [\kappa]^{<\omega}(\langle s, E\rangle \in G)\}$. Since $\mathcal{R}$ is closed under $\mathcal{AR}^\omega$, $X_G \in \mathcal{R}$.

Claim. 2. $\forall \alpha \in \kappa$, $X_G \leq^* X_\alpha$.

Proof. Fix $\alpha < \kappa$ and some $\langle s, E\rangle \in G \cap D_{\alpha,0}$. We will prove that for every $m \geq |s|$, $r_m(X_G)\ s \subseteq X_\alpha$. Let $m > |s|$ and $\langle t, F\rangle \in G \cap D_{\alpha,m}$ be given. Since $D_{\alpha,m}$ is an open dense subset of $\mathbb{P}$ and G is a filter, there is a condition $\langle u, H\rangle$ below both $\langle s, E\rangle$ and $\langle t, F\rangle$ such that $\langle u, H\rangle \in G \cap D_{\alpha,m}$. Note that $u = r_{|u|}(X_G)$. Since $\langle u, H\rangle \leq \langle s, E\rangle$ and $\alpha \in E$, there exists $X \in \mathcal{R}$ such that $X \leq X_\alpha$ and $u/s \subseteq X \leq X_\alpha$. It follows by $|u| > m$ that $r_m(X_G)/s \subseteq X_\alpha$. ■

Therefore, $\kappa < \mathfrak{p}_\mathcal{R}$; and hence, $\mathfrak{m}(\sigma\text{-centered}) \leq \mathfrak{p}_\mathcal{R}$. ■

This immediately leads to the following corollary for all the topological Ramsey spaces from Subsections 1.2.1, 1.2.2, and 1.2.4. For each of these spaces, the relevant σ-closed partial order $\leq^*$ from Definition 1.6 is exactly the mod finite partial order, $\subseteq^*$.

Corollary 4.5. *Let $\mathcal{R}$ be any of the following topological Ramsey spaces, with the σ-closed partial order $\subseteq^*$:*

1. *$\mathcal{R}_\alpha$, where $1 \leq \alpha < \omega_1$.*
2. *$\mathcal{R}(\mathbb{A})$, where $\mathbb{A}$ is some generating sequence from a collection of $\leq \omega$ many Fraïssé classes with the Ramsey property as in [26].*
3. *$\mathrm{FIN}_k^{[\infty]}$, where $k \geq 1$.*

Then $\mathfrak{p} \leq \mathfrak{p}_\mathcal{R}$.

Proof. All of these spaces satisfy the IEP. ■

Now we show that each $\mathrm{FIN}_k^{[\infty]}$ has tower and pseudointersection numbers equal to $\mathfrak{p}$.

Theorem 4.6. $\forall k \geq 1$, $\mathfrak{t}_{\mathrm{FIN}_k^{[\infty]}} = \mathfrak{p}_{\mathrm{FIN}_k^{[\infty]}} = \mathfrak{p}$.

Proof. Let $\mathcal{F} \subseteq [\omega]^\omega$ be a family linearly ordered by $\supseteq^*$; that is, a tower. For $A \in [\omega]^\omega$, write $A = \{a_0, a_1, ..., a_n, ...\}$ with $a_n < a_{n+1}$ for each $n \in \omega$. For $n \in \omega$, define $f_n : \omega \longrightarrow \{0, ..., k\}$ to be the function such that $f_n(i) = k$ for each $i \in [a_n, a_{n+1})$, and $f_n(i) = 0$ for each $i \notin [a_n, a_{n+1})$. Note that the sequence $F_A = (f_n)_{n\in\omega}$ is a member of $\mathrm{FIN}_k^{[\infty]}$, and moreover, if $A \neq B$ then $F_A \neq F_B$. Furthermore, $A \subseteq^* B$ implies $F_A \leq^* F_B$. Let $\mathcal{G} = \{F_A : A \in \mathcal{F}\}$ and note that $\mathcal{G}$ is linearly ordered by $\geq^*$. Suppose that $\mathcal{G}$ has a pseudointersection $H = (h_n)_{n\in\omega} \in \mathrm{FIN}_k^{[\infty]}$. Let $C = \{\min(\mathrm{supp}(h_n)) : n \in \omega\}$. Note that C is a pseudointersection for $\mathcal{F}$. By Lemma 4.9, $\mathfrak{t}_{\mathrm{FIN}_k^{[\infty]}} \leq \mathfrak{t}$. Corollary 4.5 implies that $\mathfrak{p} \leq \mathfrak{p}_{\mathrm{FIN}_k^{[\infty]}}$. Thus, it follows that

$$\mathfrak{t}_{\mathrm{FIN}_k^{[\infty]}} \leq \mathfrak{t} = \mathfrak{p} \leq \mathfrak{p}_{\mathrm{FIN}_k^{[\infty]}},$$

the middle equality holding by the result of Malliaris and Shelah in [54].

Since every topological Ramsey space $\mathcal{R}$ satisfies that $\mathfrak{p}_{\mathcal{R}} \leq \mathfrak{p}_{\mathcal{R}}$ then pseudointersection and tower numbers of the space $\mathrm{FIN}_k^{[\infty]}$ are equal to $\mathfrak{p}$. ■

The next results will have proofs that follow the outline of Theorem 4.6 but will involve stronger hypthothesis in order to apply to the spaces from Subsections 1.2.1 and 1.2.2.

Recall that in Definition 2.7 we define spaces with Independent Sequences of Structures.

Definition 4.7 (ISS*)**.** Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space satisfying Independent Sequences of Structures. Recall that each finite structure $\mathbb{A}_i$ is linearly ordered. We say that $\mathcal{R}$ satisfies the *ISS** if for all $k < m$, $\mathbb{A}_k$ embeds into $\mathbb{A}_m$.

It follows from the ISS* that there are functions λ_k, $k < \omega$, such that for each $m \geq k$, $\lambda_k(\mathbb{A}_m)$ is a substructure of $\mathbb{A}_m$ which is isomorphic to $\mathbb{A}_k$. Moreover, for each triple $k < m < n$, $\lambda_k(\mathbb{A}_n)$ is a substructure of $\lambda_m(\mathbb{A}_n)$.

Lemma 4.8. *Let $\mathcal{F}$ be a family infinite subsets of ω and $\mathcal{R}$ be a topological Ramsey space with the ISS*. Then for each $B \in [\omega]^\omega$ there corresponds a unique $X_B \in \mathcal{R}$ so that given any B, $C \in \mathcal{F}$, the following hold:*

1. *$B \subseteq C$ implies $X_B \leq X_C$;*
2. *$B \subseteq^* C$ implies $X_B \leq^* X_C$; and*
3. *If $\mathcal{F} \subseteq [\omega]^\omega$ and $\mathcal{G} = \{X_A : A \in \mathcal{F}\}$ has a pseudointersection, then $\mathcal{F}$ also has a pseudointersection.*

Proof. Given $B \in [\omega]^\omega$, let $\{b_0, b_1, b_2, ...\}$ be the increasing enumeration of B. For each $n \in \omega$, let $X(n) = \lambda_n(\mathbb{A}_{b_n})$, and define $X_B = \langle X(n) : n \in \omega\rangle$. Then $X_B \in \mathcal{R}$, since $\mathcal{R}$ satisfies the ISS*. Moreover, notice that whenever $B \neq C$ are in $[\omega]^\omega$, then $X_B \neq X_C$. Suppose that $B, C \in [\omega]^\omega$ satisfy $C \subseteq B$. Let $k < \omega$ be such that $c_k \in B$. Then $c_k = b_m$ for some $m \geq k$. By our construction, $X_C(k) = \lambda_k(\mathbb{A}_{c_k})$ and $X_B(m) = \lambda_m(\mathbb{A}_{b_m})$. Since $c_k = b_m$, $X_C(n)$ is a substructure of $X_B(m)$. From these observations, (1) and (2) of the theorem immediately follow.

Fix a family $\mathcal{F} \subseteq [\omega]^\omega$, and let $\mathcal{G} = \{X_B : B \in \mathcal{F}\}$. Assume that there exists some $Y \in \mathcal{R}$ which is a pseudointersection of $\mathcal{G}$. We claim that

$$D = \{m \in \omega : (\exists i \in \omega) Y(i) \text{ is substructure of } \mathbb{A}_m\} \tag{4.1.2}$$

is a pseudointersection of $\mathcal{F}$. Let $B \in \mathcal{F}$. Since Y is a pseudointersection of $\mathcal{G}$, $Y \leq^* X_B$. Thus, there exists $p < \omega$ such that for every $i > p$, $Y(i)$ is a substructure of some $X_B(j)$ for some $j < \omega$. It follows that $D \setminus p \subseteq B$; hence, $D \subseteq^* B$. ■

Lemma 4.9. *Let $(\mathcal{R}, \leq, \leq^*, r)$ be a σ-closed topological Ramsey space such that for every $X \in [\omega]^\omega$ there exists a unique member of $\mathcal{R}$, X_A, that satisfies:*

1. *$A \subset B$ implies $X_A \leq X_B$;*

2. *$A \subset^* B$ implies $X_A \leq^* X_B$ and*

3. *if $\mathcal{F} \subseteq [\omega]^\omega$ and $\mathcal{G} = \{X_A : A \in \mathcal{F}\}$ has a pseudointersection, then $\mathcal{F}$ has also a pseudointersection.*

Then $\mathfrak{t}_\mathcal{R} \leq \mathfrak{t}$ and $\mathfrak{p}_\mathcal{R} \leq \mathfrak{p}$.

Proof. Let $\mathcal{F} \subseteq [\omega]^\omega$ be a family linearly ordered by $\subseteq^*$ such that $|\mathcal{F}| < \mathfrak{t}_\mathcal{R}$. By hypothesis there is a family $\mathcal{G} \subseteq \mathcal{R}$ linearly ordered by $\leq^*$ that satisfies that $|\mathcal{G}| = |\mathcal{F}|$. So $|\mathcal{G}| = |\mathcal{F}| < \mathfrak{t}_\mathcal{R}$. Therefore $\mathcal{G}$ has a pseudointersection and by hypothesis 2), $\mathcal{F}$ has also a pseudointersection. Hence $\mathfrak{t}_\mathcal{R} \leq \mathfrak{t}$. A similar argument proves that $\mathfrak{p}_\mathcal{R} \leq \mathfrak{p}$. ■

Theorem 4.10. *Let $\mathcal{R}$ be a topological Ramsey space that satisfies the ISS*. Then $\mathfrak{p}_\mathcal{R} = \mathfrak{t}_\mathcal{R} = \mathfrak{p}$.*

Proof. Suppose $\mathcal{R}$ satisfies the ISS*. Then by Lemmas 4.8 and 4.9, $\mathfrak{t}_\mathcal{R} \leq \mathfrak{t}$. Note that $\mathcal{R}$ satisfies the IEP, since this follows from the ISS*. then Theorem 4.4 implies that $\mathfrak{m}(\sigma$-centered$)\leq \mathfrak{p}_\mathcal{R}$. Since $\mathfrak{p} = \mathfrak{m}(\sigma$-centered$)$, we have $\mathfrak{p} \leq \mathfrak{p}_\mathcal{R} \leq \mathfrak{t}_\mathcal{R} \leq \mathfrak{t}$. The equality follows from the result of Malliaris and Shelah, that $\mathfrak{p} = \mathfrak{t}$. ■

Corollary 4.11. *1. For all $1 \leq \alpha < \omega_1$, $\mathfrak{t}_{\mathcal{R}_\alpha} = \mathfrak{p}_{\mathcal{R}_\alpha} = \mathfrak{p}$.*

2. If $\mathcal{R}$ is a topological Ramsey space generated by a collection of Fraïssé classes with the Ramsey property, then $\mathfrak{t}_\mathcal{R} = \mathfrak{p}_\mathcal{R} = \mathfrak{p}$.

Proof. The topological Ramsey space in the hypothesis satisfy the ISS*, so the corollary follows from Theorem 4.10. ■

In particular, $\forall n \geq 1$, the pseudointersection number and tower number for the n-hypercube space $\mathcal{H}^n$ all equal $\mathfrak{p}$, since these are special cases of (2) in Corollary 4.11.

4.2 Pseudointersection and tower numbers for the forcings $\mathcal{P}(\omega^\alpha/\mathrm{Fin}^{\otimes\alpha})$

Next, we look at the pseudointersection and tower numbers for the high dimensional Ellentuck spaces $\mathcal{E}_\alpha$, for $2 \leq \alpha < \omega_1$. Recall that $(\mathcal{E}_\alpha, \subseteq^{*\alpha})$ is forcing equivalent to $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$. Hence, for the high and infinite dimensional Ellentuck spaces, the partial order $\leq^*$ denotes $\subseteq^{*\alpha}$. We point out that for the spaces $\mathcal{E}_\alpha$, the σ-closed partial order defined by Mijares (recall Definition 1.6) is intermediate between $\leq$ and $\subseteq^{*\alpha}$ and hence produces the same separative quotient.

The following theorem is proved in [70].

Theorem 4.12 (Szymański and Zhou, [70])**.** $\mathfrak{t}(\mathrm{Fin} \otimes \mathrm{Fin}) = \omega_1$.

Proposition 4.13. $\mathfrak{p}_{\mathcal{E}_2} = \mathfrak{t}_{\mathcal{E}_2} = \omega_1$.

Proof. This is a consequence of the last theorem. ■

In what follows, we show that for each $\alpha \in [2, \omega_1)$, the pseudointersection and tower numbers of $\mathcal{E}_\alpha$ are equal to ω_1. In fact, this is true for each space $\mathcal{E}_B$ in [23], where B is a uniform barrier of infinite rank. We point out that for $2 \leq k < \omega$, the following results were found by Kurilić in [49], though we were unaware of those results at the time that our results were found. It is important to note that the forcings $\mathbb{P}(\alpha)$ in [49] are different from the forcings $\mathcal{P}(\alpha)/\mathrm{Fin}^{\otimes\alpha}$, so for infinite countable ordinals, the results below are new.

Notation. For every $k \geq 2$, $1 < l < k$, $X \in \mathcal{E}_k$ and $x \in \omega^{\not\downarrow k}$:

1. Let $\max x$ denote the last member of the finite sequence x.
2. Let $\pi_1(X)$ denote the set $\{x_0 : x \in X\}$.
3. Denote by $\pi_l(X)$ the set $\{x \restriction l : x \in X\}$.

Note that because of the definition of $\mathcal{E}_k$ spaces, $\pi_l(X) \in \mathcal{E}_l$.

Definition 4.14. Let X be a member of $\mathcal{E}_k$ and s a finite approximation of X. Write $\pi_1[X]$ as an increasing sequence $\{n_0, n_1, ..., n_j, ...\}$. We will say that s is the i-th *full finite approximation* of X if s is the $\sqsubseteq$-least finite approximation of X such that there exists $x \in s$ such that $\min x = n_i$. We will denote by $\mathfrak{a}^k_i(X)$ the i-th full finite approximation of X.

Note that for every $i \in \omega$, if x is the $\prec$-least member of $\mathfrak{a}^k_i(X)/\mathfrak{a}^k_{i-1}(X)$ then $x_0 = n_i$. Also note that for every $x \in X$ such that $x_0 = n_i$, there exists $Y \leq X$ such that x is the $\prec$-least member of $\mathfrak{a}^k_i(Y)/\mathfrak{a}^k_{i-1}(Y)$.

Lemma 4.15. *For every $k \in \omega$ and $X \in \mathcal{E}_k$, there exists a $X' \in \mathcal{E}_{k+1}$ such that for every $X, Y \in \mathcal{E}_k$ and $Z \in \mathcal{E}_{k+1}$:*

1. *$\pi_k(X') = X$,*
2. *$Y \leq X$ implies $Y' \leq X'$,*
3. *$Y \leq^* X$ implies $Y' \leq^* X'$ and*
4. *$Z \leq^* X'$ implies $\pi_k(Z) \leq^* X$.*

Proof. The proof will be by induction on $k \in \omega$. Fix $k = 2$. Let X be a member of $\mathcal{E}_2$, write $X = \{x_0, x_1, ..., x_i, ...\}$, with $x_i \prec x_{i+1}$ for every $i \in \omega$. Write $\pi_1(X) = \{n_m : m \in \omega\}$ where $n_m < n_{m+1}$ for every $m \in \omega$. We want to construct a member of $\mathcal{E}_3$ such that $\pi_2[X'] = X$. We will construct such X' step by step by extending members of full finite approximations. Note that $\mathfrak{a}^2_1(X) = \{x_0\}$. In this case we extend x_0 with $\overline{x}^0_0 = x_0{}^\frown \max x_0$. Now fix $i > 0$. Let $l_i = \sum_{j \leq i+1} j$, then $\mathfrak{a}^2_i(X) = \{x_0, x_1, ..., x_{l_i - 1}\}$ and $\mathfrak{a}^2_i(X)/\mathfrak{a}^2_{i-1}(X) = \{x_{l_i - i - 1}, ..., x_{l_i - 1}\}$. Note that for every $m \in [0, i+1)$, $\pi_1(x_{l_i - i - 1 + m}) = n_m$. Given $x \in \mathfrak{a}^2_i(X)$, there exists $m \in [0, i+1)$ such that $\pi_1(x) = n_m$. We extend x to $\overline{x}^i = x^\frown(\max x_{l_i - i - 1 + m})$. Note that for every $i \in \omega$ and $j \in [0, l_i)$, $\pi_2(\overline{x}^i_j) = x_j$. Let $X' = \{\overline{x}^i_j : i \in \omega, j \in [0, l_i)\}$. Note that by the definition of X', $X' \in \mathcal{E}_3$ and $\pi_2(X') =$

X. We will prove that if $Y \le X$ then $Y' \le X'$. Take $y \in Y'$ then there are $j,k \in \omega$ with $j \le k$ such that $y = y_j{}^\frown(\max y_k)$. Note that by construction, $\pi_1(y_j) = \pi_1(y_k)$. Since $Y \le X$, there are some $j', k' \in \omega$ with $j' < k'$ such that $x_{j'} = y_j$ and $x_{k'} = y_k$. There exists an $i \in \omega$ such that $x_{k'} \in \mathfrak{a}^2_i(X)/\mathfrak{a}^2_{i-1}(X)$. Then $\overline{x}^i_{j'} = x_{j'}{}^\frown(\max x_{k'}) = y$ and $\overline{x}^i_{j'} \in X'$. So, $Y' \le X'$. By similar arguments, $Y \le^* X$ implies $Y' \le^* X'$. Now suppose that $Z \le^* X'$, then $|X'/Z| < \omega$. Note that if $|X/\pi_2(Z)| > \omega$, by the construction of X', $|X'/Z| > \omega$. Therefore $\pi_2(Z) \le^* X$.

Let $k > 2$ be a natural number. Now assume we have the conclusion for every member of $[2,k]$ and we want to prove it for $k+1$. Fix $X \in \mathcal{E}_{k+1}$ and write $\pi_1(X) = \{n_0, n_1, ..., n_m, ...\}$, with $n_m < n_{m+1}$ for every $m \in \omega$. Write $X(m)$ to denote the subtree of X such that for every $x \in X(m)$, $\pi_1(x) = n_m$. Note that every $X(m)$ contains a member of $\mathcal{E}_k$. Now, for every $m \in \omega$ let $X(m) \restriction (0,...,k]$ denote the collection of $x \restriction (0,...,k]$ where x is a member of $X(m)$. Note that $X(m) \restriction (0,...,k] \in \mathcal{E}_k$. By induction hypothesis, for every $m \in \omega$ we can extend $X(m) \restriction (0,...,k]$ to a $X(m)' \in \mathcal{E}_{k+1}$ such that $\pi_k(X(m)') = X(m) \restriction (0,...,k]$. Let $X' = \bigcup_{m\in\omega}\{(n_m)^\frown x : x \in X(m)'\}$. Note that $X' \in \mathcal{E}_{k+2}$. Since for every $m \in \omega$, $\pi_k(X(m)') = X(m) \restriction (0,...,k]$, $\pi_{k+1}(X') = X$. Take $X, Y \in \mathcal{E}_{k+1}$ such that $Y \le X$. Take $y \in Y'$ then exists $n_m \in \omega$ such that $y = (n_m)^\frown z$ with $z \in Y(m)'$. Then $(z \restriction (0,k] \in Y(m)$ and since $Y \le X$ there exists an $l \in \omega$ such that $Y(m) \subset X(l)$. Hence by hypothesis induction $Y(m)' \le X(l)'$ and $y \in X'$. By similar arguments, $Y \le^* X$ implies $Y' \le^* X'$. Now suppose that $Z \le^* X'$, then $|X'/Z| < \omega$. Note that if $|X/\pi_{k+}(Z)| > \omega$, by the construction of X', $|X'/Z| > \omega$. Therefore $\pi_{k+1}(Z) \le^* X$. ■

Proposition 4.16. *For every $k \in \omega$ such that $k > 1$, $\mathfrak{t}_{\mathcal{E}_{k+1}} \le \mathfrak{t}_{\mathcal{E}_k}$.*

Proof. Fix $k \in \omega$. Let $\kappa < \mathfrak{t}_{\mathcal{E}_{k+1}}$ be a cardinal, we will prove that $\kappa < \mathfrak{t}_{\mathcal{E}_k}$. Let $\mathcal{F} \subseteq \mathcal{E}_k$ be a family linearly ordered by $\subseteq^*$. By the last Lemma for every $X \in \mathcal{E}_k$ there exists an $X' \in \mathcal{E}_{k+1}$ with properties 1,2 and 3. Let $\mathcal{G} = \{X' : X \in \mathcal{F}\}$ and note that $\mathcal{G}$ is linearly ordered by $\le^*$. Since $\kappa < \mathfrak{t}_{\mathcal{E}_{k+1}}$, $\mathcal{G}$ has a pseudointersection Z. By 3 of the last Lemma, $\pi_k(Z)$ is a pseudointersection of $\mathcal{F}$. Then $\kappa \le \mathfrak{t}_{\mathcal{E}_k}$. Therefore $\mathfrak{t}_{\mathcal{E}_{k+1}} \le \mathfrak{t}_{\mathcal{E}_k}$. ■

Theorem 4.17. *For every $k > 2$, $\mathfrak{t}_{\mathcal{E}_k} = \mathfrak{p}_{\mathcal{E}_k} = \omega_1$.*

Proof. Let $k > 2$ be given. Since $(\mathcal{E}_k, \le^*)$ is a σ-closed partial order $\omega_1 \le \mathfrak{t}_{\mathcal{E}_k}$. By the last proposition, $\mathfrak{p}_{\mathcal{E}_k} \le \mathfrak{t}_{\mathcal{E}_k} \le \mathfrak{t}_{\mathcal{E}_2} = \omega_1$. Therefore $\mathfrak{t}_{\mathcal{E}_k} = \mathfrak{p}_{\mathcal{E}_k} = \omega_1$. ■

In fact, by a similar proof, we obtain the following:

Theorem 4.18 (Dobrinen)**.** *Let B be a uniform barrier of countable rank at least 2. Then $\mathfrak{t}_{\mathcal{E}_B} \le \mathfrak{t}_{\mathcal{E}_2}$.*

4.3 The Carlson-Simpson Space

The Carlson-Simpson space is another space which does not have the property ISS. This space also has tower number equal to ω_1. Recall from Subsection 4.3 the Carlson-Simpson space $\mathcal{E}_\infty$, which is not related with high dimensional Ellentuck spaces. The following proposition is a consequence of a Proposition from Carlson that appears in [55], where Carlson prove that there is a family $\{X_\alpha : \alpha < \omega_1\} \subseteq \mathcal{E}_\infty$ such that there are not $X \in \mathcal{E}_\infty$ such that for every $\alpha < \omega_1$, $X \leq^* X_\alpha$.

Proposition 4.19. $\mathfrak{p}_{\mathcal{E}_\infty} = \mathfrak{t}_{\mathcal{E}_\infty} = \omega_1$.

4.4 The space of strong subtrees

When we investigate the pseudointersection number for the space of strong subtrees $S_\infty(U)$ we discover that the almost reduction relation defined by Mijares is not an order on this space.

Proposition 4.20. *The almost reduction relation $\leq^*$ on $S_\infty(U)$ is not transitive.*

Proof. We work on the space $S_\infty(2^{<\omega})$. Let T, U be the maximal subtrees of $2^{<\omega}$ with root (0) and (1), respectively. Note that $T \leq 2^{<\omega}$ and $2^{<\omega} \leq^* U$ but T is not $\leq^*$ below U. ■

4.5 Subgroups of $[\omega]^{<\omega}$

$([\omega]^{<\omega}, \triangle)$ is a group. Let $\mathfrak{G}$ be the collection of all infinite subgroups of $([\omega]^{<\omega}, \triangle)$. We want to study the properties of $\mathfrak{G}$ as a partial order. Note that $(\mathfrak{G}, \subseteq)$ is not a separative partial order. To see that, note that $[\omega\setminus\{0\}]^{<\omega} \in \mathfrak{G}$ and $[\omega]^{<\omega} \nsubseteq [\omega\setminus\{0\}]^{<\omega}$. If there exists an $H \in \mathfrak{G}$ such that $H \perp [\omega\setminus\{0\}]^{<\omega}$, let B be a basis for H. Take $s, t \in B$ with $s \neq t$ and $0 \in s \cap t$. Therefore $s \triangle t \subseteq \omega \setminus \{0\}$ and $s \triangle t \in H \cap [\omega \setminus \{0\}]^{<\omega}$ which is a contradiction. So there is no an infinite group $H < [\omega]^{<\omega}$ such that $H \perp [\omega \setminus \{0\}]^{<\omega}$.

Define the order $\prec$ on $[\omega]^{<\omega}$ as follows: if $s, t \in [\omega]^{<\omega}$, $s \prec t$ if $\max s < \max t$ or $\max s = \max t$ and $s <_{lex} t$ where $<_{lex}$ is the lexicographical order. For every $G \in \mathfrak{G}$, fix a basis $\mathcal{B}_G = \{g_i : i \in \omega\}$ for G such that for every $i < j$, $g_i \cap g_j = \emptyset$ and $g_i \prec g_j$.

Claim. *The basis $\mathcal{B}_G$ is unique.*

Let $\mathcal{B}'_G = \{g'_i : i \in \omega\}$ be a basis for G such that for every $i < j$, $g'_i \cap g'_j = \emptyset$ and $g'_i \prec g'_j$. Fix g'_k. Since $\mathcal{B}_G$ is a basis for G we can write $g'_k = g_{i_0} \triangle \ldots \triangle g_{i_n}$ for some $n \in \omega$. Since the members of $\mathcal{B}_G$ are pairwise disjoint then $g'_k = \bigcup_{l<n} g_{i_l}$. If $n > 1$ then $g_{l_0} \subsetneq g'_k$. That is not possible because $\mathcal{B}'_G$ is a basis for G and it is not possible to write g_{l_0} as a linear combination of members of $\mathcal{B}'_G$. Hence $g'_k \in \mathcal{B}_G$ for every $k \in \omega$.

In order to get a separative order define the equivalence relation $\sim \subset \mathfrak{G} \times \mathfrak{G}$ such that $G \sim H$ if $\{g_i : i \in \omega\} \subset^* H$ and $\{h_i : i \in \omega\} \subset^* G$. We will denote by $\leq^*$ the inclusion modulo $\sim$. Note that $(\mathfrak{G}/\sim, \subset) = (\mathfrak{G}, \leq^*)$ is a separative order. Note that the order $\leq^*$ is different from the almost inclusion relation.

Remark. Note that if $G \in \mathfrak{G}$, then $\bigcup G \in [\omega]^\omega$. If $G, H \in \mathfrak{G}$ and $|\bigcup G \cap \bigcup H| < \omega$ then $|G \cap H| < \omega$, because $G \cap H \subseteq [\bigcup G \cap \bigcup H]^{<\omega}$. Also note that if $G, H \in \mathfrak{G}$ are such that $|(\bigcup G) \cap (\bigcup H)| = \omega$, it does not imply that $|G \cap H| = \omega$. To see this, let $G = \langle\{\{n, n+1\} : n \equiv 0 \bmod 2\}\rangle$ and $H = \langle\{\{n, n+1\} : n \equiv 1 \bmod 2\}\rangle$, and

$(\bigcup G) \cap (\bigcup H) = \omega \setminus \{0\}$. If $G \cap H \neq \emptyset$ take $s \in G \cap H$. There exist $F_G, F_H \in [\omega]^{<\omega}$ such that $s = \bigcup_{i \in F_G} \{n_i, n_i + 1\} = \bigcup_{i \in F_H} \{m_j, m_j + 1\}$ and for every $i \in F_G, j \in F_H$, $n_i \equiv 0 \bmod 2$ and $m_j \equiv 1 \bmod 2$. Then $\max(s) \equiv 0 \bmod 2$ and $\max(s) \equiv 1 \bmod 2$ which is a contradiction.

If $X \in [\omega]^\omega$, $[X]^{<\omega}$ is an infinite subgroup of $([\omega]^{<\omega}, \triangle)$. Note that if $X, Y \in [\omega]^\omega$, $[X \cap Y] < \omega = [X]^{<\omega} \cap [Y]^{<\omega}$. Recall that if $\mathcal{X} \subseteq [\omega]^{<\omega}$, $\langle \mathcal{X} \rangle$ denotes the subgroup of $[\omega]^{<\omega}$ generated by $\mathcal{X}$. If $X \subseteq \omega$, $\langle X \rangle$ denotes the subgroup $\langle \{\{x\} : x \in X\} \rangle$. Note that if $\{s_n : n \in \omega\}, \{t_n : n \in \omega\} \subset [\omega]^{<\omega}$, $\langle \{s_n : n \in \omega\} \rangle \cap \langle \{t_n : n \in \omega\} \rangle = \langle \{s_n : n \in \omega\} \cap \{t_n : n \in \omega\} \rangle$

For every $H \in \mathfrak{G}$, let $\{h_i : i \in \omega\}$ be the basis for H fixed above. Define the mapping r from $\omega \times \mathfrak{G}$ to the collection of finite subgroups of $[\omega]^{<\omega}$ such that $r(n, H) = \langle \{h_i : i < n\} \rangle$. For both, finite and infinite subgroups of $[\omega]^{<\omega}$, $<$ is the relation of being subgroup.

Theorem 4.21. *The triple $(\mathfrak{G}, <, r)$ is a topological Ramsey space.*

Proof. To see that $(\mathfrak{G}, <, r)$ satisfies **A.1** first note that for every $H \in \mathfrak{G}$, $r_0(H) = \langle \emptyset \rangle = \emptyset$. If $G, H \in \mathfrak{G}$ are different infinite subgroups then $G \triangle H \neq \emptyset$. Without loss of generality suppose that there exists some $g \in G \setminus H$. There exist g_{n_i} for $i < l$ such that $g = \bigcup_{i<l} g_i$. Since $g \notin H$, there is some l such that $g_{n_l} \notin H$. Therefore $r_{n_l+1}(G) \neq r_{n_l+1}(H)$. Note that **A.1**(c) follows from the definition of r.

Note that **A.2**(a) holds because finite approximations are actually finite. Now take $G, H \in \mathfrak{G}$ with $G < H$ and $n \in \omega$. Since $r_n(G) \subseteq H$, there exist $m \in \omega$ such that $r_n(G) \subset \langle \{h_i : i < m\} \rangle$. Hence $r_n(G) < r_m(H)$. To prove **A.2**(c), take $a = \langle \{g_i : i < j\} \rangle, b = \langle \{g_i : i < l\} \rangle, c = \langle \{h_i : i < k\} \rangle \in \mathcal{A}\mathfrak{G}$ such that $a \sqsubset b$ and $b \subset c$. Since $g_j \in \langle \{h_i : i < k\} \rangle$, there exists i' such that $g_j = h_{i'}$. Therefore $d = \langle \{h_i : i < i'\} \rangle$ satisfies that $a \subseteq d$ and $d \sqsubset c$.

To prove **A.3**(a), take a a finite approximation and $H \in \mathfrak{G}$ such that $a \subset H$. If $G \in [\mathrm{depth}_H(a), H]$ there is some $m \in \omega$ such that $a = \langle \{g_{n_i} : i < m\} \rangle$. Note that any subgroup of $\langle \{g_{n_i} : i < m\} \cup \{g_i : i > n_m\} \rangle$ belong to $[a, G]$. Now we show that the statement**A.3**(b) holds. Let $G \leq H$ and $a \subset G$. Write $a = \langle \{g_{n_i} : i < m\} \rangle$. Let $G' = \langle \mathrm{depth}_H(a) \cup \{g_i : i > n_m\} \rangle$ and note that $\langle a \cup \{g_i : i > n_m\} \rangle \in [a, G'] \subset [a, G]$.

Finally, we prove axiom **A.4**. Let $a \in \mathcal{A}\mathfrak{G}_n$, $H \in \mathfrak{G}$ such that $a < H$ and $\mathcal{O} \subset \mathcal{A}\mathfrak{G}_{n+1}$. Write $a = \langle \{h_{n_i} : i < m\} \rangle$ with $m \in \omega$. Let $c : \langle \{h_i : i \geq n_m\} \rangle \to 2$ be a coloring such that $c(h) = 0$ if $\langle a \cup \{h\} \rangle \in \mathcal{O}$ and $c(h) = 0$ otherwise. By Hindman's Theorem [0.5], there is an infinite set $M \subset \langle \{h_i : i \geq n_m\} \rangle$ such that all its elements are pairwise disjoint and the set of all finite unions of members of M is monochromatic. Let $G = \langle \mathrm{depth}_H(a) \cup M \rangle$. Note that $G \in [\mathrm{depth}_H(a), H]$ and $r_{n+1}[a, G] \subset \mathcal{O}$ or $r_{n+1}[a, G] \subset \mathcal{O}^c$.

■

Claim 1. $\mathfrak{p}_{\mathfrak{G}} = \mathfrak{p}$.

Proof. First we check that $\mathfrak{p}_{\mathfrak{G}} \leq \mathfrak{p}$. For every infinite set $X \subseteq \omega$, let $G_X = [X]^{<\omega}$. Note that if $X \subseteq Y$, then $[X]^{<\omega} \leq [Y]^{<\omega}$. Furthermore, since $[X]^{<\omega} = \langle \{\{x\} : x \in X\} \rangle$, $X \subseteq^* Y$ implies $[X]^{<\omega} \leq^* [Y]^{<\omega}$. Let $\mathcal{F}$ be a family of infinite subsets of ω and

$\mathcal{F}' = \{[X]^{<\omega} : X \in \mathcal{F}\}$. Suppose $\mathcal{F}'$ has a pseudointersection $G \in \mathfrak{G}$. For every $X \in \mathcal{F}$, $G \leq^* [X]^{<\omega}$. Therefore there exists $k \in \omega$ such that $\{g_i : i > k\} \subset^* [X]^{<\omega}$. Hence $Y = \bigcup G$, is a pseudointersection for $\mathcal{F}$. By Lemma 4.8, $\mathfrak{p}_{\mathfrak{G}} \leq \mathfrak{p}$.

From now on we will identify every group G_α with its basis $\mathcal{B}_\alpha$. Note that because of the properties of the basis $\mathcal{B}_G$ it holds that $H < G$ if and only if $\mathcal{B}_G \subset \mathcal{B}_H$. Even though $\mathfrak{G}$ does not have the Independent Extension Property, the family $\{\mathcal{B}_G : G \in \mathfrak{G}\}$ has the IEP. Therefore $\mathfrak{p} \leq \mathfrak{p}_{\mathfrak{G}}$.

■

Let $\mathcal{R}$ be an infinite family of members of $\mathfrak{G}$. We will say that $\mathcal{R}$ is a *reaping family* if for every $G \in \mathfrak{G}$ there exists an $H \in \mathcal{R}$ such that $|G \cap H| < \omega$ or $|G \setminus H| < \omega$. $\mathfrak{r}_{\mathfrak{G}}$ is the least size of a reaping family.

Claim 2. $\mathfrak{r}_{\mathfrak{G}} \leq \mathfrak{r}$.

Proof. Let $\mathcal{F}$ be a reaping family of $[\omega]^\omega$ of size $\mathfrak{r}$. Define $\mathcal{F}' = \{\langle X \rangle : X \in \mathcal{F}\}$. We check that $\mathcal{F}'$is a reaping family for $\mathfrak{G}$. Let G be a member of $\mathfrak{G}$, then $\bigcup G \in [\omega]^\omega$ and since $\mathcal{F}$ is a reaping family there exists $Y \in \mathcal{F}$ such that $|(\bigcup G) \cap Y| < \omega$ or $|(\bigcup G) \setminus Y| < \omega$, then $|G \cap [Y]^{<\omega}| < \omega$ or $|G \setminus [Y]^{<\omega}| < \omega$. Therefore $\mathcal{F}'$ is a reaping family for $\mathfrak{G}$ and $|\mathcal{F}'| = |\mathcal{F}| = \mathfrak{r}$. ■

4.6 Pseudointersection and tower numbers for ideals

There are several works to investigate Van Dowen's diagram for ideals. In [40], [15], [32], Hernandez-Hernandez and Hrušák, Brendle and Farkas respectively investigate cardinal invariants for analytic ideals. In [42], Hrušák and Meza investigate cardinal invariants of F_σ ideals. We finalize this Thesis by showing some advances done to investigate pseudointersection and tower numbers for F_σ and TRS ideals.

First we extend naturally the notions of strong finite intersection property, pseudointersection and tower for ideals. Let $\mathcal{I}$ be an ideal and $\mathcal{F}$ a collection of $\mathcal{I}$-positive sets. We say that $\mathcal{F}$ has the *strong finite intersection property* (SFIP) if for every finite collection $\{X_1, ..., X_n\} \subseteq \mathcal{F}$, there is some $Y \in \mathcal{I}^+$ such that for every $i \in \{1, ..., n\}$, $Y \subseteq X_i$ holds. We say that some $Y \in \mathcal{I}^+$ is a *pseudointersection* for the family $\mathcal{F}$ if for every $X \in \mathcal{F}$, $Y \subset X$. We say that $\mathcal{F}$ is a *tower* if it is linearly ordered by $\subset_{\mathcal{I}}$ and does not have a pseudointersection.

For every ideal $\mathcal{I}$ define:

- The *pseudointersection number* $\mathfrak{p}_{\mathcal{I}}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}^+$ that has the SFIP but does not have a pseudointersection.
- The *tower number* $\mathfrak{t}_{\mathcal{I}}$ is the least size of a tower $\mathcal{F} \subseteq \mathcal{I}^+$.

Note that for every ideal $\mathcal{I}$, $\mathfrak{p}_{\mathcal{I}} < \mathfrak{t}_{\mathcal{I}}$ holds.

Lemma 4.22. *If $\mathcal{I}$ is an ideal such that there is some tree $(T, \sqsubseteq)$ with the WIP such that $[T]$ dense in $\mathcal{I}^+$, then $\mathfrak{p} \leq \mathfrak{p}_{\mathcal{I}}$.*

Proof. This proof is essentially the same as the proof of Theorem 4.4. ■

Theorem 4.23. *If $\mathcal{I}$ is an F_σ ideal or a TRS ideal then $\mathfrak{p} = \mathfrak{p}_\mathcal{I}$.*

Proof. Since every $\mathcal{I}$-positive set is infinite, it holds that $\mathfrak{p}_\mathcal{I} \leq \mathfrak{p}$. To see that $\mathfrak{p} \leq \mathfrak{p}_\mathcal{I}$ we build a tree $T \subset [\omega]^{<\omega}$ satisfying the hypotheses of Lemma 4.22.

If $\mathcal{I}$ is a TRS ideal let T be its related Todorčević tree. If $\mathcal{I}$ is an F_σ ideal, let $\prec$ be the order on $[\omega]^{<\omega}$ such that if $a, b \in [\omega]^{<\omega}$ then $a \prec b$ whenever $\min(a \triangle b) \in a$. Since $\mathcal{I}$ is an F_σ ideal, by Mazur's Theorem there is some lscsm φ such that $\mathcal{I} = Fin(\varphi)$. For every $X \in \mathcal{I}^+$ let a_0^X be the $\prec$-minimal $a \subset X$ with $\varphi(a) > 1$. And for every $n \in \omega$, let a_{n+1}^X be the $\prec$-minimal $a \subset (X \setminus \max(a_n^X))$ with $\varphi(a) > n$. T consists of finite sets of the form $\cup_{i<m} a_i^Y$ with $m \in \omega$ and $Y \in \mathcal{I}^+$. Given $s, t \in T$ we say that $s \prec t$ if $s = \cup_{i<m} a_i^Y$ and $t = \cup_{i<n} a_i^Y$ for $m < n$ and $Y \in \mathcal{I}^+$. Note that T has the Weak Independence Property. From Lemma 4.22 it follows that $\mathfrak{p} \leq \mathfrak{p}_\mathcal{I}$. ■

In general, we do not know if pseudointersection and tower numbers for F_σ ideals are always the same.

Recall that an F_σ ideal $\mathcal{I}$ is *fragmented* if there is a lscm φ with $\mathcal{I} = Fin(\varphi)$ and a sequence $\{t_i : i \in \omega\} \subseteq [\omega]^{<\omega}$ such that $\sup \varphi(t_n) = \infty$ and for every $X \subseteq \bigcup_{i\in\omega} t_i$, $\varphi(t) = \sup \varphi(X \cap t_n)$.

Lemma 4.24. *If $\mathcal{I}$ is a fragmented ideal then $\mathfrak{t}_\mathcal{I} \leq \mathfrak{t}$.*

Proof. Since $\mathcal{I}$ is a fragmented ideal there is a sequence $\{t_i : i \in \omega\} \subseteq [\omega]^{<\omega}$ such that $sup\varphi(t_n) = \infty$ and for every $X \subseteq \bigcup_{i\in\omega} t_i$, $\varphi(t) = sup\varphi(X \cap t_n)$ holds. Let $\kappa < \mathfrak{t}$ be a cardinal and $\mathcal{F} = \{X_\alpha : \alpha < \kappa\}$ some family linearly ordered by $\subseteq^*$. For every $\alpha < \kappa$ let $X'_\alpha = \bigcup_{i\in X_\alpha} t_i$. Note that the family $\mathcal{H} = \{X'_\alpha : \alpha \in \kappa\} \subseteq \mathcal{I}^+$ is linearly ordered by $\subseteq_\mathcal{I}$. Also note that is $Y \in \mathcal{I}^+$ is a pseudointersection for $\mathcal{H}$, then $\{n > 1 : \varphi(Y \cap t_n) > \varphi(Y \cap t_{n-1})\}$ is a pseudointersection for $\mathcal{F}$. ■

Theorem 4.25. *If $\mathcal{I}$ is a fragmented ideal then $\mathfrak{p}_\mathcal{I} = \mathfrak{t}_\mathcal{I} = \mathfrak{t}$.*

Proof. This theorem follows from Theorem 4.23 because fragmented ideals are F_σ. ■

Corollary 4.26.
1. $\mathfrak{p}_{\mathcal{ED}_{\mathsf{fin}}} = \mathfrak{t}_{\mathcal{ED}_{\mathsf{fin}}} = \mathfrak{p}$.
2. *If $\mathcal{I}$ is an ideal generated by Fraïssé classes then $\mathfrak{p}_\mathcal{I} = \mathfrak{t}_\mathcal{I} = \mathfrak{p}$.*

We know the summable ideals are not fragmented, so investigating pseudointersection and tower numbers for these ideals could help us to better understand the behavior of this cardinals.

Question 4.27. Are $\mathfrak{p}_\mathcal{I}$ and $\mathfrak{t}_\mathcal{I}$ equal when $\mathcal{I}$ is a summable ideal?

For ideals Fin^n we know from [49] that:

Theorem 4.28 (Kurilić)**.** *For every $n \in \omega$, $\mathfrak{p}_{\mathrm{Fin}^n} = \mathfrak{t}_{\mathrm{Fin}^n} = \omega_1$.*

Corollary 4.29. $\mathfrak{p}_{\mathsf{conv}} = \mathfrak{t}_{\mathsf{conv}} = \omega_1$.

Proof. This follows from the fact that conv-positive sets can be thought as $\mathrm{Fin} \times \mathrm{Fin}$-positive sets. ■

From the results presented in this work we conjecture that all TRS ideals with the WIP satisfy that pseudointersection and tower numbers are equal to $\mathfrak{p}$. Also, we conjecture that for TRS ideals with the Dependence Property pseudointersection and tower numbers are equal to ω_1.

Bibliography

[1] Alvaro Arias, Natasha Dobrinen, Gabriel Girón-Garnica, and José G Mijares. Banach spaces from barriers in high dimensional Ellentuck spaces. *Journal of Logic and Analysis*, 10(5), 2018. 10

[2] Bohuslav Balcar, Fernando Hernández-Hernández, and Michael Hrušák. Combinatorics of dense subsets of the rationals. *Fund. Math*, 183(1):59–80, 2004. xiii, xxi, 22

[3] Tomek Bartoszyński and Haim Judah. *Set Theory on the Structure of the Real Line*. A. K. Peters, Ltd., 1995. 19, 22

[4] James E Baumgartner. Sacks forcing and the total failure of Martin's axiom. *Topology and its applications*, 19(3):211–225, 1985. 21, 22, 50

[5] James E Baumgartner and Richard Laver. Iterated perfect-set forcing. *Annals of Mathematical Logic*, 17(3):271–288, 1979. xiii, xxi, 50

[6] James E. Baumgartner and Alan D. Taylor. Partition Theorems and Ultrafilters. *Tansactions of the American Mathematical Society*, 241:283–309, 1978. xi, xix, 7, 9

[7] Murray Bell. On the combinatorial principle p(c). *Fundations of Mathematics*, (114):149–157, 1981. 22

[8] Andreas Blass. The Rudin-Keisler ordering of P-Points. *Transactions of the American Mathematical Society*, 179:145–166, 1973. xii, xix, 7, 9, 32

[9] Andreas Blass. Ultrafilter mappings and their Dedekind cuts. *Transactions of the American Mathematical Society*, 188(2):327–340, 1974. 5

[10] Andreas Blass. Ultrafilters related to Hindman's finite-unions theorem and its extensions. *Contemporary Mathematics*, 65:89–124, 1987. 13, 14

[11] Andreas Blass. Selective ultrafilters and homogeneity. *Annals of Pure and Applied Logic*, 38:215–255, 1988. 5

[12] Andreas Blass, Natasha Dobrinen, and Dilip Raghavan. The next best thing to a p-point. *Journal of Symbolic Logic*, 80(3):866–900. xii, xix, 33

[13] Andreas Blass and Neil Hindman. On strongly summable ultrafilters and union ultrafilters. *Transactions of the American Mathematical Society*, 304(1):83–97, 1987. xi, xix

[14] Jörg Brendle. Van douwen's diagram for dense sets of rationals. *Annals of pure and applied logic*, 143(1-3):54–69, 2006. xiii, xxi, 22

[15] Jörg Brendle. Cardinal invariants of analytic quotient. In *Slides for ESI workshop on large cardinals and descriptive set theory, Vienna*, 2009. 63

[16] Timothy J. Carlson. Some unifying principles in Ramsey theory. *Discrete Mathematics*, 68(2-3):117–169, 1988. x, xviii

[17] Timothy J. Carlson and Stephen G. Simpson. A dual form of Ramsey's theorem. *Advances in Mathematics*, 53(3):265–290, 1984. 2, 14

[18] Timothy J. Carlson and Stephen G. Simpson. Topological Ramsey theory. In *Mathematics of Ramsey theory*, volume 5 of *Algorithms and Combinatorics*, pages 172–183. Springer, 1990. x, xviii

[19] Carlos Di Prisco, José G Mijares, and Jesus Nieto. Local Ramsey theory: an abstract approach. *Mathematical Logic Quarterly*, 63(5):384–396, 2017. xi, xix, 5, 23

[20] Carlos Di Prisco, José G Mijares, and Jesus Nieto. Local ramsey theory: an abstract approach. *Mathematical Logic Quarterly*, 63(5):384–396, 2017. 4

[21] Carlos Augusto Di Prisco and Stevo Todorčević. Perfect-set properties in l (r)[u]. *Advances in Mathematics*, 139(2):240–259, 1998. xi, xix, 4, 16, 23

[22] Natasha Dobrinen. High dimensional Ellentuck spaces and initial chains in the Tukey structure of non-p-points. *Journal of Symbolic Logic*, 81(1):237–263, 2016. xi, xii, xviii, xx, 10, 11, 24, 33, 41, 49

[23] Natasha Dobrinen. Infinite dimensional Ellentuck spaces and Ramsey-classification theorems. *Journal of Mathematical Logic*, 16(1):37 pp, 2016. xi, xii, xviii, xx, 24, 41, 49, 59

[24] Natasha Dobrinen. Topological Ramsey spaces dense in forcings. In *Structure and Randomness in Computability and Set Theory*, page 32 pp. World Scientific, 2020. To appear; arXiv:1702.02668v2. xi, xix, 5, 24

[25] Natasha Dobrinen and Daniel Hathaway. Classes of barren extensions. page 32 pp, 2019. To appear in JSL. xi, xix

[26] Natasha Dobrinen, José G Mijares, and Timothy Trujillo. Topological Ramsey spaces from Fraïssé classes, ramsey-classification theorems, and initial structures in the tukey types of p-points. *Archive for Mathematical Logic*, 56(7-8):733–782, 2017. xi, xviii, 7, 8, 9, 10, 24, 32, 56

[27] Natasha Dobrinen and Sonia Navarro. Ramsey degrees of ultrafilters, pseudointersection numbers, and the tools of topological Ramsey spaces. *Submitted*, 2019. xv, xxii, 33

[28] Natasha Dobrinen and Stevo Todorcevic. Tukey types of ultrafilters. *Illinois Journal of Mathematics*, 55(3):907–951, 2011. 14

[29] Natasha Dobrinen and Stevo Todorčević. A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 1. *Transactions of the American Mathematical Society*, 366(3):1659–1684, 2014. x, xviii, 5, 6, 24, 40

[30] Natasha Dobrinen and Stevo Todorčević. A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 2. *Transactions of the American Mathematical Society*, 367(7):4627–4659, 2015. xi, xviii, 7, 24

[31] Erik Ellentuck. A new proof that analytic sets are Ramsey. *Journal of Symbolic Logic*, 39(1):163–165, 1974. x, xviii, 49

[32] Barnabás Farkas and Lajos Soukup. More on cardinal invariants of analytic p-ideals. *arXiv preprint arXiv:1002.2192*, 2010. 63

[33] David Fernández-Bretón and Michael Hrušák. A parametrized diamond principle and union ultrafilters. *arXiv preprint arXiv:1706.00830*, 2017. xi, xix

[34] Fred Galvin and Karel Prikry. Borel sets and Ramsey's theorem 1. *The Journal of Symbolic Logic*, 38(2):193–198, 1973. ix, xvii

[35] José de Jesús Pelayo Gómez. Combinatorics of Borel ideals. *arXiv preprint arXiv:1808.09088*, 2018. 20, 43

[36] W Timothy Gowers. Lipschitz functions on classical spaces. *European Journal of Combinatorics*, 13(3):141–151, 1992. x, xviii

[37] W. Timothy Gowers. An infinite Ramsey theorem and some Banach-space dichotomies. *Annals of Mathematics*, 156(3):797–833, 2002. x, xviii, 13

[38] Ronald L Graham and Bruce L Rothschild. A short proof of van der Waerden's theorem on arithmetic progressions. *Proceedings of the American Mathematical Society*, 42(2):385–386, 1974. viii, xvi

[39] James M. Henle, Adrian R. D. Mathias, and W. Hugh Woodin. A barren extension. In *Methods in mathematical logic*, pages 195–207. Lecture Notes in Mathematics, 1130, Springer, 1985. xi, xix

[40] Fernando Hernández-Hernández and Michael Hrušák. Cardinal invariants of analytic p-ideals. *Canadian Journal of Mathematics*, 59(3):575–595, 2007. 63

[41] Neil Hindman. Finite sums from sequences within cells of a partition of n. *Journal of Combinatorial Theory. Series A*, 17:1–11, 1974. 13

[42] Michael Hrušák, David Meza-Alcántara, and Hiroaki Minami. Pair-splitting, pair-reaping and cardinal invariants of f σ-ideals. *The Journal of Symbolic Logic*, pages 661–677, 2010. 63

[43] Michael Hrušák, David Meza-Alcántara, Egbert Thümmel, and Carlos Uzcátegui. Ramsey type properties of ideals. *Annals of Pure and Applied Logic*, 168(11):2022–2049, 2017. xii, xx, 19

[44] Michael Hrušák and Jonathan L Verner. Adding ultrafilters by definable quotients. *Rendiconti del Circolo Matematico di Palermo*, 60(3):445–454, 2011. 16, 19

[45] Michael Hrušák and Jindřich Zapletal. Forcing with quotients. *Archive for Mathematical Logic*, 47(7):719–739, 2008. 16

[46] Michael Hrušák. Katětov order on borel ideals. *Archive for Mathematical Logic*, 56(7-8):831–847, 2017. xii, xx, 16, 19, 20, 43

[47] Winfried Just and Adam Krawczyk. On Certain Boolean Algebras p(ω)/i. *Transactions of the American Mathematical Society*, pages 411–429, 1984. 18

[48] Alexander S Kechris, Vladimir G Pestov, and Stevo Todorcevic. Fraïssé limits, ramsey theory, and topological dynamics of automorphism groups. *Geometric & Functional Analysis GAFA*, 15(1):106–189, 2005. 7

[49] Miloš S. Kurilić. Forcing with copies of countable ordinals. *Proceedings of the American Mathematical Society*, 143(4):1771–1784, 2015. xii, xx, 59, 64

[50] Claude Laflamme. Forcing with filters and complete combinatorics. *Annals of Pure and Applied Logic*, 49(2):125–163, 1989. xi, xviii, xix, 5, 7, 25, 29, 31

[51] Richard Laver. Products of infinitely many perfect trees. *Journal of the London Mathematical Society*, 2(3):385–396, 1984. xiii, xxi, 52

[52] Barbara Majcher-Iwanow. Cardinal invariants of the lattice of partitions. *Commentationes Mathematicae Universitatis Carolinae*, 41(3):543–558, 2000. xiii, xxi, 22

[53] Maryanthe Malliaris and Saharon Shelah. General topology meets on model theory, on p and t. *Proceedings of the National Academy of Sciences of the United States of America*, 110(3):13300–13305, 2013. xiii, xxi, 22, 54

[54] Maryanthe Malliaris and Saharon Shelah. Cofinality spectrum theorems in model theory, set theory, and general topology. *Journal of the American Mathematical Society*, 29(1):237–297, 2016. xiii, xxi, 22, 54, 57

[55] P. Matet. Partitions and Filters. *J. Symbolic Logic*, 51(1):12–21, 03 1986. xiv, 15, 61

[56] A. R. D. Mathias. Happy families. *Annals of Mathematical Logic*, 12(1):59–111, 1977. 5

[57] David Meza-Alcántara. *Ideals and filters on countable sets*. PhD thesis, Universidad Nacional Autónoma México, México, 2009. 20

[58] José G. Mijares. A notion of selective ultrafilter corresponding to topological Ramsey spaces. *Mathematical Logic Quarterly*, 53(3):255–267, 2007. 3, 4

[59] Heike Mildenberger. On Milliken-Taylor ultrafilters. *Notre Dame Journal of Formal Logic*, 52(4):381–394, 2011. 14

[60] A Miller. Rational perfect set forcing. *Contemporary Mathematics*, 31(143-159):70, 1984. 49

[61] Keith R Milliken. Ramsey's theorem with sums or unions. *Journal of Combinatorial Theory, Series A*, 18(3):276–290, 1975. x, xviii

[62] Keith R. Milliken. A partition theorem for the infinite subtrees of a tree. *Transactions of the American Mathematical Society*, 263(1):137–148, 1981. 13

[63] C St JA Nash-Williams. On well-quasi-ordering transfinite sequences. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 61, pages 33–39. Cambridge University Press, 1965. ix, xvii

[64] Jaroslav Nešetřil and Vojtěch Rödl. Ramsey classes of set systems. *Journal of Combinatorial Theory, Series A*, 34(2):183–201, 1983. 8

[65] Frank P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30:264–296, 1929. viii, xvi

[66] Gerald E Sacks. Forcing with perfect closed sets. In *Axiomatic set theory*, volume 1, pages 331–355. Amer. Math. Soc Providence, RI, 1971. 21

[67] Issai Schur. Uber die kongruenz x"+ y"= z"(mod p). *Jahresber. Deutsch. Math.-Verein*, 25:114–117, 1916. viii, xvi

[68] Jack Silver. Every analytic set is Ramsey. *The Journal of Symbolic Logic*, 35(1):60–64, 1970. ix, xvii

[69] Miodrag Sokić. *Ramsey porperty of posets and related structures*. PhD thesis, Phd thesis University of Toronto, 2010. 8

[70] Andrzej Szymanski and Zhou Hao Xua. The behaviour of $\omega 2$ under some consequences of Martin's axiom. *General Topology and Its Relations to Modern Analysis and Algebra. V,(Prague, 1981),(ed. J. Novák) Sigma Ser. Pure Math*, 3:577–584, 1981. xii, xiv, xix, xxi, 58

[71] Stevo Todorčević. *Introduction to Ramsey Spaces*. Princeton University Press, 2010. ix, x, xvii, xviii, 2, 3, 13, 14

[72] Timothy Trujillo. *Topological Ramsey spaces, associated ultrafilters, and their applications to the Tukey theory of ultrafilters and Dedekind cuts of nonstandard arithmetic*. PhD thesis, University of Denver, 2014. 4, 10

[73] Bartel Leendert van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wisk*, 15:212–216, 1927. viii, xvi

[74] Olga Yiparaki. *On some tree partitions.* PhD thesis, The University of Michigan, 1996. 52

[75] Yuan Yuan Zheng. Selective ultrafilters on FIN. *Proceedings of the American Mathematical Society*, 145:5071–5086, 2017. xiii, xxi, 23

[76] Yuan Yuan Zheng. *Parametrizing topological Ramsey spaces.* PhD thesis, University of Toronto, 2018. 5, 50

[77] Yuan Yuan Zheng. Preserved under Sacks forcing again? *Acta Mathematica Hungarica*, 154(1):1–28, 2018. xiii, xxi, 23