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Dedication

To my parents, my sisters and my brothers.

Resumen

Al nivel más fundamental, los constituyentes fundamentales del universo interactan de cuatro formas distintas, fuerte y débilmente, por un lado a distancias subnucleares y electromagnética y gravitacionalmente a escalas que pueden ser incluso macroscpicas. Las interacciones electrodébiles y fuertes se describen en el lenguaje de la teora cuntica de campos en el Modelo Estndar de Partículas Fundamentales, que a las escalas experimentales explorada considera las interacciones gravitacionales irrelevantes. Sin embargo, en situaciones como en el Universo temprano la interacción gravitacional debe incluirse para tener una descripcin microscópica realista. A muy altas energías, las interacciones fundamentales permiten una descripción perturbativa en términos de Diagramas de Feynman. Los métodos de la segunda cuantización comúnmente usados se vuelven ineficientes rápidamente en el sentido del gran número de diagramas que deben calcularse para una descripcin más precisa de un proceso dado. Alternativamente, los métodos de primera cuantización ofrecen un escenario en el que el cálculo de amplitudes para procesos de interacciones fundamentales, con la inclusión particular de la interacción gravitacional, pueden ser tratables y susceptibles de mejoras sistemáticas donde los métodos tradicionales fallan. Si bien los métodos basados en formalismo de línea de mundo, inspirado en la teoría de cuerdas, no son recientes, su aplicabilidad y eficiencia ha alcanzado un nivel predictivo que compite con los esquemas tradicionales. En esta tesis aplicamos dichos métodos a cuatro diferentes problemas: Un análisis en el rango entero de masas de la acción efectiva de QED en un fondo con simetría $O(2) \times O(3)$ y la representación de línea de mundo del vértice tres gluones y el vértice de cuatro gluones, as como la amplitud fotón gravitón.
Esta tesis está organizada de la siguiente manera: El Capítulo 1 presenta una introducción al formalismo de línea de mundo. El Capítulo 2 está dedicado a la primera aplicación de dicho formalismo, a saber, la acción efectiva de QED en un fondo con simetría $O(2) \times O(3)$, donde se discute una comparación con otros métodos. En el capítulo 3 se introducen herramientas matemáticas que serán utilizadas en el resto de la disertación, especialmente en el caso gluónico. Los siguientes tres capítulos describen, respectivamente, la corrección a un lazo y fuera de la capa de masa de los vértices de tres y cuatro gluones as como la amplitud fotón-gravitón. Información adicional se presenta en varios apéndices.

Palabras Clave:

Formalismo de línea de mundo, Método inspirado en la teoría de cuerdas, QED, QCD, Gravedad.

Abstract

At the most fundamental level, the constituent blocks of the universe interact in four different ways as far as known, namely, strongly and weakly and subnuclear scales, as well as electromagnetically and gravitationally at scales that could be even macroscopical, of the size of the Universe itself. At the scale energies explored in large particle accelerators, it has been determined that weak and electromagnetic are but two sides of a single fundamental interaction, the electroweak interaction. On the other side, what holds nuclei together is the strong interaction of quantum chromodynamics (QDC). QCD and the electroweak theory are described in the elegant language of quantum field theory in the celebrated Standard Model of Particle Physics. At the tested scales, gravitational interactions are irrelevant for elementary particles. Nevertheless, either at the early universe or near strong gravitational fields, gravitational interaction has to be taken into account in order to have a realistic microscopic description. In high energy experiments, fundamental interactions allow a perturbative description in terms of Feynman Diagrams. The second quantization methods - often employed in such a description- soon become inefficient as the number of Feynman diagrams required for a precise description of a given process grows with the order of approximation. Alternatively, first quantization methods such as the worldline formalism offer a different framework for the calculation of scattering amplitudes for fundamental interactions, including gravitational interaction. They may help to get a systematic computation of (classes of) scattering amplitudes. This is a task that traditional methods has failed to achieve. Although the (string inspired) worldline formalism is not a recent proposal, only in the last decades it has become a powerful tool as an alternative to conventional second-quantized methods. In this thesis, we review the application of worldline methods to four different problems: The full mass range analysis of the QED effective action with an $O(2) \times O(3)$ symmetric background and the worldline representation of one-loop corrections to the off-shell three- and four-gluon vertex as well as the photon-graviton amplitude.
The thesis is organized as follows: Chapter 1 provides an introduction to the worldline formalism. Chapter 2 is devoted to the first application of the WL formalism, namely, the effective action for QED in the background of an $O(2) \times O(3)$ symmetric field, where a comparison against other methods is discussed. In Chapter 3 some mathematical tools are introduced which will be apply in the rest of this dissertation especially for the gluon case . The next three Chapters describe, respectively, the one-loop correction to the off-shell three- and four-gluon vertex as well as the photon-graviton amplitude. Some additional information is presented in different appendices.

Key Words:

Worldline formalism, String inspired method, QED, QCD, Gravity.

Chapter 1

Introduction to path integral and string inspired methods

1.1 History

The notion of path integral (sometimes also called functional integral or integral over trajectories or integral over histories) was introduced for the first time, in 1920s by Norbert Wiener [1] as a method to solve problems in the theory of diffusion and Brownian motion. This integral, which is now also called the Wiener integral, has played a central rule in the further development in the subject of path integration.
It was reinvented in a different form by Richard Feynman in 1942 [2,3], for the reformulation of the quantum mechanics (the so-called third formulation of the quantum mechanics besides the Schrödinger and Heisenberg ones). The Feynman approach was inspired by Dirac's paper [4] on the role of the Lagrangian and the least action principle in quantum mechanics. This eventually led Feynman to represent the propagator of the Schrödinger equation by the complex-valued path integral which now bears his name. At the end of 1940s Feynman [5,6], worked out on the basis of the path integrals, a new formulation of Quantum Electrodynamic (QED) and developed a well-known diagram techniques for perturbation theory.
In 1950s, path integrals were studied intensively for solving functional equations in quantum field theory (Schwinger equations). The functional formulation of quantum field theory was considered in the works of Bogoliubov [7], Gelfand and Minlos [8], Fradkin [9] and others. Starting from these works, many important applications of path integral have been found in statistical physics: in the theory of phase transition, superfluidity, superconductivity, the Ising model, in 1955, Feynman used the path integral to investigating the polaron problem and invented his variational principle of the quantum mechanics. At the same time, attempts were initiated to widen the class of exactly solvable path integrals, i.e. to expand it beyond the class of the Gaussian-like integrals. In the early 1950s, Ozaki (in unpublished lecture notes, Kyushu university) started with a short time action for a free particle written in Cartesian coordinates and transformed into the polar form. Later, Peak and Inomata [10] calculated explicitly the radial path integral for the harmonic oscillator. This opened the way for an essential broadening of the class of path integral models.
In 1960s, a new field of path integral applications appeared, namely the quantization of a gauge filed, examples of which are electromagnetic, gravitational and Yang-Mills fields. The specific

properties of the action functionals for gauge fields (their invariance with respect to gauge transformations) should be taken into account when quantizing, otherwise wrong results emerge. This was first noticed by Feynman [11] using the example of Yang-Mills and gravitational fields. He showed that quantization by straightforward use of the Fermi method, in analogy with QED, violates the unitarity condition. Latter as a result of works by De Witt [12], Faddeev and Popov [13], Mandelstam [14], Fradkin and Tyutin [15] and 't Hooft [16], the problem was solved and the path path integralintegral method turned out to be the most suitable one for this aim. In addition, in the mid-1960s, Berezin [17] took a curtail step which allowed the comprehensive use of the path integration: he introduced integration over Grassmann variables to describe fermions; actually it opened the way for a unified treatment of bosons and fermions in the path integral approach.

In the 1970s, Wilson [18] formulated the field theory of quarks and gluons (Quantum Chromodynamics (QCD)) on an Euclidean spacetime lattice. This may be considered as the discrete form of the field theoretical path integral. The lattice serves as both an ultraviolet (UV) and infrared (IR) cut-off which makes the theory well defined. At low energy, it is the most fruitful method to treat the theory of strong interactions (for example, making use of computer simulations). A few years later, Fujikawa [19] showed how the quantum anomalies emerge from the path integral. He realized that it is the "measure" in the path integral which is not invariant under a certain class of symmetry transformations and this makes the latter anomalous.

All these achievements led to the fact that the path integral methods have become an indispensable part of any construction and study of field theoretical models, including the realistic theories of unified electromagnetic and weak interaction [20–22] and QCD [23, 24]. Among other applications of path integrals in quantum field theory and elementary particle physics, it is worth mentioning the derivation of asymptotic formulas for IR and UV behavior of Green functions, the semiclassical approximation, rearrangement and partial summation in perturbation series, calculations in the presence of topologically non-trivial field configurations and extended objects (solitons and instantons), the study of cosmological models and black holes and such an advanced application as the formulation of the first-quantized theory of (super)string and branes. In some cases, the technique allows us to provide the solid foundations for the results obtained by other methods, to clarify the limits for their applicability and indicate the way of calculating the corrections. If an exact solution is possible, then the path integral technique gives a simple way to obtain it. In the case of physically realistic problems, which normally are far from being exactly solvable, the use of path integrals helps to build up the qualitative picture of the corresponding phenomenon and to develop approximate method of calculation. They represent a sufficiently flexible mathematical apparatus which can be suitably adjusted for the extraction of the essential ingredients of a complicated model for its further physical analysis, also suggesting the method for a concrete realization for such an analysis [1]. More than three decades ago the analogy between first quantized approach in string theory and worldline representation in field theory was pointed out in the ϕ^3 theory effective action [25], and similar approach was considered for Yang-Mills theory [26]. Since then considerable attempts and studies in the relation between string theory scattering amplitudes have been started and remarkable results emerged [27–32]. String theory organizes scattering amplitudes in a compact manner (by virtue of the conformal symmetry on the world-sheet), and the field theory emerges as a singular limit of string theory, inherits this useful feature, by which the summation of Feynman diagrams is already installed without need of performing loop integrals and the Dirac traces. In particular in 1992, Bern and Kosower derived a set of simple rules for one-loop gluon scattering amplitudes through analyzing the field theory limit of a heterotic string theory [33]. The set of rules were applied to five gluon amplitudes [27, 31], quantum gravity [29, 32] and super Yang-Mills theory [34, 35]. This method

may be called "string-inspired" because it is analogous to calculations in string perturbation theory, and its development was triggered by efforts [30,33,36] to use the peculiar organization of string amplitudes to improve on the efficiency of calculations in ordinary quantum field theory. However this is to some extent accidental, and the practical application of this technique requires no knowledge of string theory.
The completion of Feynman diagram summation means gauge invariance. It is well-known that the Feynman diagram calculation splits a gauge invariant amplitude into non-invariant terms, and this causes a cancellation between divergent diagrams (gauge cancellation), which brings a serious problem especially with numerical computation. In the Bern-Kosower formalism we do not have this problem, since the only divergence appears from the final integration of a universal master formula. The master formula does not depend on the simplicity of specific scatterings with small number of external legs. Hence the Bern-Kosower formalism has a great deal of potential to renovate the computational technique and efficiency in quantum field theory. The Bern-Kosower rules for one-loop cases are also attainable directly (without making use of string theory) in terms of the worldline method in quantum field theory [37,38]. In this case, we have to evaluate an effective action in some particular form, which can be written as path integral for a one-dimensional quantum mechanical action (worldline action), using the proper time and background field methods as well. Then, expanding the background field as a sum of Fourier plane wave modes, we get the same kind of objects that are called the vertex operators in string theory. One particle irreducible (1PI) Green functions can be obtained as multi-integrals of the master formula, which is a correlation function evaluated by Wick's contraction with the two-point correlator (worldline Green function) determined from the worldline action. It is very interesting that this kind of vertex operator technique resembles string theory calculations, and that all Feynman diagrams are consequently contained in a single master formula like string theory amplitudes. In fact, various field theory examples can be understood from this viewpoint: photon splitting [39], axion decay in a constant magnetic field [40], and Yukawa interactions [41–43] up to some finite values of N (the number of external legs) are explicitly verified; for photon scattering and ϕ^3 theory, the equivalence is formally proven up to the two-loop order with an arbitrary value of N [44]. This formalism is also useful for a manifestly covariant calculation of the effective action [45,46], and for decompositions into gauge invariant partial amplitudes [47]. In [48] the first proposal for generalization of the Bern-Kosower formalism for multi-loop has been made.
In the rest of this introduction we first discuss the worldline formulation of abelian and non-abelian gauge theories. Then we discuss the gravity case which is less straightforward, and in the last part we make a short review on the Bern-Kosower replacement rule.

1.2 Abelian gauge theory

1.2.1 Scalar QED

In the rest of this Chapter we follow the lines of [49, 50] for gauge theory part and [51, 52] for gravity part.

After the genuine work by Feynman to introduce path integral formulation of QED, it appears that he considered this approach less promising since he relegated the information on it to appendices of [5] and [6]. No essential use was made of those path integral representation for many years after, and even today path integrals are used in field theory mainly as integral over fields, not over particles. Excepting an elegant application by Affleck et al. [53] in 1982, the potential of this particle path integral or worldline formalism to improve on standard field theory methods, at least for certain type of computations, was recognized only in the early 90s through the work of Bern and Kosower [33] and Strassler [37] as was discussed above. In the following, we will concentrate on the case of the one-loop effective action in QED and QCD, and the associated photon/gluon amplitudes. The gravitational case which is less straightforward will be discussed later.

The free-propagator

We start with the free scalar propagator, that is, the Green's function for the Klein-Gordon equation is expressed as

$$D_0^{xx'} \equiv \langle 0|\mathcal{T}\phi(x)\phi(x')|0\rangle = \langle x|\frac{1}{-\Box + m^2}|x'\rangle \tag{1.1}$$

We work with Euclidean conventions, defined by starting in Minkowski space with the metric $(-+++)$ and performing a Wick rotation (analytic continuation) to the

$$\begin{aligned} E &= k^0 = -k_0 \to ik_4 \,, \\ t &= x^0 = -x_0 \to ix_4 \,. \end{aligned} \tag{1.2}$$

See Appendix A for our conventions. Thus the wave operator $\Box$ turns into four-dimensional Laplacian

$$\Box = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2} \,. \tag{1.3}$$

In the following we use the natural units and we set $\hbar = c = 1$.
Now, let us go back to the propagator. We exponentiate the denominator by using a Schwinger proper-time parameter T, which leads to

$$\begin{aligned} D_0^{xx'} &= \langle x|\int_0^\infty dT\, \exp\big[-T(-\Box + m^2)\big]|x'\rangle \\ &= \int_0^\infty dT\, e^{-m^2T}\langle x|\exp\big[-T(-\Box)\big]|x'\rangle \,. \end{aligned} \tag{1.4}$$

Next, we should transform the transition amplitude under the above integral into a path integral form. To do so, we use the transition amplitude of the free particle in quantum mechanics, see Appendix B

$$\langle \vec{x}, t|\vec{x}', 0\rangle \equiv \langle x|e^{-itH}|x'\rangle = \int_{x(0)=\vec{x}'}^{x(t)=\vec{x}} \mathcal{D}x\, e^{i\int_0^t d\tau \frac{m}{2}\dot{x}^2} \,, \tag{1.5}$$

where $H = -\frac{1}{2m}\nabla^2$. By comparing (1.4) and (1.5), one sees the following formal replacements are needed

$$\begin{aligned} \nabla^2 &\to \Box \,, \\ m &\to \frac{1}{2} \,, \\ \tau &\to -i\tau \,, \\ t &\to -iT \,, \end{aligned} \tag{1.6}$$

then we get

$$D_0^{xx'} = \int_0^\infty dT\, e^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\int_0^T \frac{1}{4}\dot{x}^2}\,. \tag{1.7}$$

This is the worldline path integral representation of the relativistic propagator of a scalar particle in a Euclidean spacetime that propagates from x' to x. Note that now $\dot{x}^2 = \sum_{i=1}^4 \dot{x}_i^2$. The parameter T for us will just be an integration variable but it has a deeper meaning related to one-dimensional diffeomorphism invariance.

Having found this path integral, let us calculate it, as a consistency check and also to start developing our technical tools. We decompose the x-variable as a classical ($x^\mu_{\rm cl}$) and quantum (q^μ) parts,

$$x^\mu(\tau) = x^\mu_{\rm cl}(\tau) + q^\mu(\tau) = \big[x^\mu + \frac{\tau}{T}(x^\mu - x'^\mu)\big] + q^\mu(\tau)\,, \tag{1.8}$$

where $x_{\rm cl}$ fulfills the integral boundary conditions $x_{\rm cl}(0) = x'$ and $x_{\rm cl}(T) = x$ and $q(\tau)$ fulfills the Dirichlet boundary conditions (DBC) which is

$$q(0) = q(T) = 0 \tag{1.9}$$

By taking derivative respect to τ, from (1.8) we get

$$\dot{x}^\mu(\tau) = \dot{q}^\mu(\tau) + \frac{1}{T}(x^\mu - x'^\mu)\,. \tag{1.10}$$

Plugging this equation into the kinetic term of the path integral and using (1.9) we find

$$\int_0^T d\tau \frac{1}{4}\dot{x}^2 = \int_0^T d\tau \frac{1}{4}\dot{q}^2 + \frac{(x-x')^2}{4T}\,. \tag{1.11}$$

So, finally, for the propagator we get

$$D_0^{xx'} = \int_0^\infty dT\, e^{-m^2T}\, e^{-\frac{(x-x')^2}{4T}} \int_{DBC} \mathcal{D}q(\tau)\, e^{-\int_0^T \frac{1}{4}\dot{q}^2}\,. \tag{1.12}$$

The new path integral over the fluctuation variable $q(\tau)$ depends only on T, not on x and x'. Now, by using equation (B.23) and our formal replacement (1.6), and taking into account that our path integral has D components, we get

$$\int_{DBC} \mathcal{D}q(\tau)\, e^{-\int_0^T \frac{1}{4}\dot{q}^2} = (4\pi T)^{-\frac{D}{2}}\,. \tag{1.13}$$

Thus we have

$$D_0^{xx'} = \int_0^\infty dT (4\pi T)^{-\frac{D}{2}}\, e^{-m^2T}\, e^{-\frac{(x-x')^2}{4T}}\,. \tag{1.14}$$

This is indeed a representation of the free propagator in x-space, but let us now Fourier transform it (still in D dimensions) to get the more familiar momentum space representation:

$$
\begin{aligned}
D_0^{kk'} &= \int\int dx dx'\, e^{ik\cdot x}\, e^{ik'\cdot x'}\, D_0^{xx'} \\
&= \int_0^\infty dT (4\pi T)^{-\frac{D}{2}}\, e^{-m^2 T} \int\int dx dx'\, e^{ik\cdot x}\, e^{ik'\cdot x'}\, e^{-\frac{(x-x')^2}{4T}} \qquad (\text{if } x' \to x + x') \\
&= \int_0^\infty dT (4\pi T)^{-\frac{D}{2}}\, e^{-m^2 T} \int dx\, e^{i(k+k')\cdot x} \int dx' e^{ik'\cdot x'}\, e^{-\frac{x'^2}{4T}} \\
&= (2\pi)^D \delta^D(k+k') \int_0^\infty dT\, e^{-T(k'^2+m^2)} = (2\pi)^D \delta^D(k+k') \frac{1}{k'^2+m^2}\,,
\end{aligned}
\tag{1.15}
$$

where we have used the fact that the x-integral gives us the energy-momentum conservation and the x'-integral is Gaussian.

Coupling to the electromagnetic field

Let us begin with the simplest case of a real massive scalar field ϕ with a self-interaction potential $U(\phi)$. According to standard quantum field theory [54] the Euclidean one-loop effective action for this field theory can be written as

$$
\Gamma[\phi] = -\frac{1}{2} \mathrm{Tr} \ln \Big[\frac{-\Box + m^2 + U''(\phi)}{-\Box + m^2}\Big]\,,
\tag{1.16}
$$

where Tr denotes a functional trace. We use the following formula

$$
-\mathrm{Tr} \ln(\frac{A}{B}) = \int_0^\infty \frac{dT}{T} \mathrm{Tr}(e^{-AT} - e^{-BT})\,,
\tag{1.17}
$$

which is valid for positive definite operators A and B. Returning to eq. (1.16) after performing the functional trace in x-space leads to

$$
\Gamma[\phi] = \frac{1}{2} \int_0^\infty \frac{dT}{T} \int d^D x \Big\langle x \Big| \exp\Big\{ - T\big[- \Box + m^2 + U''(\phi(x))\big]\Big\} \Big| x \Big\rangle\,.
\tag{1.18}
$$

Now if we compare this with the Feynman's path integral formula (B.15) to be recalled as

$$
\langle x'' | e^{-i(t''-t')H} | x' \rangle = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\, e^{i \int_{t'}^{t''} dt [\frac{\tilde{m}^2}{2} \dot{x}^2 - \tilde{V}(x)]}\,,
\tag{1.19}
$$

where, we can interpret our kinetic operator as the Hamiltonian operator H for a fictitious particle moving in D dimensions,

$$
H = \frac{p^2}{2m} + V(x)
\tag{1.20}
$$

by identifying

$$
\begin{aligned}
\tilde{V}(x) &= m^2 + U''(\phi(x))\,, \\
\tilde{m} &= \frac{1}{2}\,, \\
i(t''-t') &= T\,.
\end{aligned}
\tag{1.21}
$$

Without retracing the path integral discretization procedure (Appendix B) we can immediately write

$$\Big\langle x\Big|\exp\Big\{-T\big[-\Box+m^2+U^{''}(\phi(x))\big]\Big\}\Big|x\Big\rangle=\int_{x(0)=x}^{x(T)=x}\mathcal{D}x(\tau)\,e^{-\int_0^T\left[\frac{1}{4}\dot{x}^2+m^2+U^{''}(\phi(x(\tau)))\right]}\,, \tag{1.22}$$

where $\tau=it$. Taking into account that

$$\int d^Dx\int_{x(0)=x(T)=x}\mathcal{D}x(\tau)=\int_{x(0)=x(T)}\mathcal{D}x(\tau)\,, \tag{1.23}$$

we obtain the following path integral for the effective action

$$\Gamma[\phi]=\frac{1}{2}\int_0^\infty\frac{dT}{T}\,e^{-m^2T}\int_{x(T)=x(0)}\mathcal{D}x(\tau)\,e^{-\int_0^Td\tau(\frac{1}{4}\dot{x}^2+U^{''}(\phi(x(\tau))))}\,. \tag{1.24}$$

Now, the path integral for a massive (complex) scalar filed coupled to a background Maxwell field can be also found in a similar way. The field theory kinetic operator now reads

$$(\partial+ieA)^2-m^2\,, \tag{1.25}$$

with a fictitious Hamiltonian

$$H=\frac{(p+eA)^2}{2\tilde{m}}+m^2\,. \tag{1.26}$$

This translates into

$$\Gamma[A]=\int_0^\infty\frac{dT}{T}\,e^{-m^2T}\int_{x(T)=x(0)}\mathcal{D}x\,e^{-\int_0^Td\tau\left(\frac{1}{4}\dot{x}^2+ie\dot{x}\cdot A(x(\tau))\right)}\,. \tag{1.27}$$

which is the one-loop effective action for a Maxwell background induced by a scalar loop in terms of first-quantized particle integral [38, 52]. Note that the global factor $\frac{1}{2}$ has disappeared, since in taking the trace we have to take double number of degrees of freedom of the complex scalar into account.
Note that in (1.27) we have a dT/T, and that the path integration is over closed loops; those trajectories can therefore belong only to virtual particles, not to real ones. In this formula T the usual Schwinger proper-time parameter, and m the mass of the particle circulating in the loop (virtual particle). The effective action contains the quantum effects caused by the presence of such particles in the vacuum for the background field. In particular, it causes electrodynamics to become a nonlinear theory at the one-loop level, where photons can interact with each other in an indirect fashion.

Analogously we can get the "full" or "complete" propagator $D^{xx'}[A]$ for a scalar particle, that interacts with the background field A continuously while propagating from x' to x

$$D^{x'x}[A]=\int_0^\infty dT\,e^{-m^2T}\int_{x(0)=x'}^{x(T)=x}\mathcal{D}x\,e^{-\int_0^Td\tau\left(\frac{1}{4}\dot{x}^2+ie\dot{x}\cdot A(x(\tau))\right)}\,. \tag{1.28}$$

Gaussian integral

As was already mentioned, the path integral formulas (1.28) and (1.27) were found by Feynman already in 1950 [5]. Techniques for their efficient calculation were, however, developed only much later. Presently there are three different methods available, namely

- The analytic or "string-inspired" approach, based on the use of worldline Green's functions.
- The semi-classical approximation, based on a stationary trajectory ("worldline instanton").
- A direct numerical calculation of the path integral ("worldline Monte Carlo").

In this thesis we will treat exclusively the "string-inspired" approach. In the "string-inspired" approach all path integrals are brought into Gaussian form; usually this requires some expansion and truncation. They are then calculated by a formal extension of the n-dimensional Gaussian integration formulas to infinite dimensions. This is possible because Gaussian integration involves only very crude information on operators, namely their determinants and inverses (Green's functions). We recall that, in n dimensions,

$$\begin{aligned}
\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x} &= \frac{(4\pi)^{\frac{n}{2}}}{(\det M)^{\frac{n}{2}}}\,, \\
\frac{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x + x\cdot j}}{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x}} &= e^{j\cdot M^{-1}\cdot j}\,,
\end{aligned} \tag{1.29}$$

where the $n\times n$ matrix M is assumed to be symmetric and positive definite (hence invertible). Also, by multiple differentiation of the second formula with respect to the components of the vector j one gets

$$\begin{aligned}
\frac{\int d^n x\, x_i x_j\, e^{-\frac{1}{4}x\cdot M\cdot x}}{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x}} &= 2M^{-1}_{ij}\,, \\
\frac{\int d^n x\, x_i x_j x_k x_l\, e^{-\frac{1}{4}x\cdot M\cdot x}}{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x}} &= 4\big(M^{-1}_{ij}M^{-1}_{kl} + M^{-1}_{ik}M^{-1}_{jl} + M^{-1}_{il}M^{-1}_{jk}\big)\,, \\
\vdots\qquad &\qquad \vdots
\end{aligned} \tag{1.30}$$

an odd number of x_i's gives zero by antisymmetry of the integral (the contributions from $\int_{-\infty}^{0}$ exactly cancel those from $\int_0^\infty$). Note that on the right hand sides we always have one term for each way of grouping all the x_i's into pairs; each such grouping is called a "Wick contraction". In the canonical formalism the same combinatorics arises from the canonical commutator relations. As will be seen, in flat space calculations these formulas can be generalized to the worldline path integral case in a quite naive way, while in curved space there arise considerable subtleties which will be discussed in this chapter.

N-photon amplitude

We will focus on the closed-loop case in the following, since it turns out to be simpler than the propagator one. A particle in a loop can be described as a simple quantum mechanical system existing for a finite, periodic time, or, alternatively, as a one-dimensional field theory on a compact space; external fields act as operators on the particle Hilbert space, just as in usual quantum mechanics. At any order in the external field, the effective action is a correlation function of these operators in a free and therefore soluble theory, and can be expressed in a compact form. Nevertheless, it should be emphasized that everything that we will do in the following for the effective action can also be done for the propagator. We could use (1.27) for a direct calculation of the effective action in a derivative expansion (see [50]), but let us instead apply it to the calculation of the one-loop N-photon amplitudes in scalar QED. This means that we will now consider the special case where the scalar particle, while moving along the closed trajectory in spacetime, absorbs or emits a fixed but arbitrary number N of quanta of the background field, that is, photons of fixed momentum p and polarization ε. In field theory, to implement this we first specialize the background $A(x)$, which so far was an arbitrary Maxwell field, to a sum of N plane waves,

$$A^\mu(x) = \sum_{i=1}^{N} \varepsilon_i^\mu \, e^{ik_i\cdot x}\,. \tag{1.31}$$

Now, we expand the interaction part of (1.27) as a power series, and we take term of order A^N. This term looks like

$$\frac{(ie)^N}{N!}\Big(\int_0^T d\tau \sum_{i=1}^{N} \varepsilon_i\cdot\dot{x}(\tau)\, e^{ik_i\cdot x(\tau)}\Big)^N\,. \tag{1.32}$$

We have N^N terms, but we take only $N!$ of them, the totally mixed ones that involve all N different polarization and momenta. The ordering of the polarization and momenta does not matter, so, all these $N!$ terms are equivalent, and the $1/N!$ cancels out. What remains is the following

$$\begin{aligned}&(-ie)^N \int_0^T d\tau_1 \varepsilon_1\cdot\dot{x}(\tau_1) e^{ik_1\cdot x(\tau_1)} \cdots \int_0^T d\tau_N \varepsilon_N\cdot\dot{x}(\tau_N) e^{ik_N\cdot x(\tau_N)}\\ &= (-ie)^N V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N]\,,\end{aligned} \tag{1.33}$$

where we introduce scalar *photon vertex operator* to be defined as

$$V^\gamma_{\rm scal}[k,\varepsilon] \equiv \int_0^T d\tau \varepsilon\cdot\dot{x}\, e^{ik\cdot x(\tau)}\,. \tag{1.34}$$

This is the same vertex operator which is used in (open) string theory to describe the emission or absorption of a photon by a string [50]. Now for the N-photon amplitude we have

$$\begin{aligned}\Gamma_{\rm scal}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] \;&=\; (-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2T} \int_{x(0)=x(T)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2}\\ &\quad\times V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N]\,.\end{aligned} \tag{1.35}$$

(where we now abbreviate $x(\tau_i) =: x_i$). Note that each vertex operator represents the emission or absorption of a single photon, however, the moment when this happens is arbitrary and must therefore be integrated over.

To perform the path integral, note that it is already of Gaussian form. Doing it for arbitrary N the way it stands would still be difficult, though, due to the factors of $\dot{x}_i$. Therefore we first use a little formal exponentiation trick, writing

$$\varepsilon_i\cdot\dot{x}_i = e^{\varepsilon_i\cdot\dot{x}_i}\big|_{\rm lin(\varepsilon_i)}\,. \tag{1.36}$$

Now the path integral is of the standard Gaussian form (1.29). Since

$$\int_0^T d\tau \dot{x}^2 = \int_0^T d\tau x(-\frac{d^2}{d\tau^2})x\,, \tag{1.37}$$

(the boundary terms vanish because of the periodic boundary condition) we have the correspondence $M \leftrightarrow -\frac{d^2}{d\tau^2}$. Thus we now need the determinant and the inverse of this operator. However, we first have to solve a little technical problem: positive definiteness does not hold for the full path integral $\mathcal{D}x$, since there is a "zero mode"; the path integral over closed trajectories includes the constant loops, $x(\tau) = const$, on which the kinetic term vanishes, corresponding to a zero eigenvalue of the matrix M [1].
To solve this problem we define the loop-center of mass (or average position) by

$$x_0^\mu := \frac{1}{T}\int_0^T d\tau x^\mu(\tau)\,. \tag{1.38}$$

We then separate off the integration over x_0, thus reducing the path integral to an integral over the relative coordinate q:

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau) \quad , \quad \int \mathcal{D}x = \int d^D x_0 \int \mathcal{D}q(\tau)\,. \tag{1.39}$$

It follows from (1.38), (1.39) that the variable $q(\tau)$ obeys, in addition to periodicity, the constraint equation

$$\int_0^T d\tau q^\mu(\tau) = 0\,. \tag{1.40}$$

The zero mode integral can be done immediately, since it factors out as (since $\dot{x} = \dot{q}$)

$$\int d^D x_0\, e^{i\sum_{i=1}^N k_i\cdot x_0} = (2\pi)^D \delta\big(\sum_{i=1}^N k_i\big)\,. \tag{1.41}$$

This is just the expected global delta function for energy-momentum conservation. Since $M = -\frac{d^2}{d\tau^2}$ has only positive eigenvalues we get

$$\mathrm{det} M = (4T)^D\,, \tag{1.42}$$

and that the Green's function of M corresponding to the above treatment of the zero mode is

$$G_B^c(\tau,\tau') \equiv 2\langle\tau|(\frac{d^2}{d\tau^2})^{-2}|\tau'\rangle_{SI} = |\tau-\tau'| - \frac{(\tau-\tau')^2}{T} - \frac{T}{6}\,. \tag{1.43}$$

Note that this Green's functions is the inverse of $\frac{1}{2}\frac{d^2}{d\tau^2}$. Note also that it is a function of the difference $\tau - \tau'$ only; the reason is that the boundary condition (1.40) does not break the translation invariance in τ. The subscript "B" stands for "bosonic" (later on we will also introduce a "fermonic" Green's function) and the subscript "SI" stands for "string-inspired", since in string theory the zero mode of the worldsheet path integral is usually fixed analogously. The superscript "c" refers to the inclusion of the constant (coincidence limit) $-T/6$. It turns

[1] No such problem arose for the propagator, since $q(0) = q(T) = 0$ and q = constant implies that $q = 0$.

out that, in flat space calculations, this constant is irrelevant and can be omitted. Thus in flat space we will usually use instead

$$G_B(\tau,\tau') \equiv |\tau-\tau'| - \frac{(\tau-\tau')^2}{T}\,. \tag{1.44}$$

This replacement does not work in curved space calculations, though. Below we will also need the first and second derivatives of this Green's function, which are

$$\begin{aligned}\dot G_B(\tau,\tau') &= \mathrm{sign}(\tau-\tau') - 2\frac{\tau-\tau'}{T}\,,\\ \ddot G_B(\tau,\tau') &= 2\delta(\tau-\tau') - \frac{2}{T}\,.\end{aligned} \tag{1.45}$$

Here and in the follwoing "dot" always means derivative respect to the first variable, and since $G_B(\tau,\tau')$ is a function of $\tau-\tau'$, we can always rewrite $\frac{\partial}{\partial\tau'} = -\frac{\partial}{\partial\tau}$. Now, we go back to the Gaussian integral formula (1.29), to use it we define

$$j(\tau) \equiv \sum_i \big(i\delta(\tau-\tau_i)k_i - \dot\delta(\tau-\tau_i)\varepsilon_i\big)\,, \tag{1.46}$$

that enable us to rewrite

$$e^{\sum_{i=1}^N (ik_i\cdot q_i + \varepsilon_i\cdot\dot q_i)} = e^{\int_0^T d\tau j(\tau)\cdot q(\tau)}\,, \tag{1.47}$$

where we used the fact that $\int_0^T \dot\delta(\tau-\tau_i)q(\tau) = -\dot q(\tau_i)$. Then by formal application of (1.29) we get

$$\begin{aligned}\frac{\int \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau \frac14 \dot q^2}\, e^{\sum_{i=1}^N (ik_i\cdot q_i + \varepsilon_i\cdot\dot q_i)}}{\int \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau \frac14 \dot q^2}} &= \exp\Big[-\frac12\int_0^T d\tau\int_0^T d\tau'\, G_B(\tau,\tau')j(\tau)\cdot j(\tau')\Big]\\ &= \exp\Big\{\sum_{i,j=1}^N \big[\frac12 G_{Bij}k_i\cdot k_j - i\dot G_{Bij}\varepsilon_i\cdot k_j + \frac12 \ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\big]\Big\}\,.\end{aligned} \tag{1.48}$$

Here we have abbreviated $G_{Bij} \equiv G_B(\tau_i,\tau_j)$, and in the second step we have used the antisymmetry of $\dot G_{Bij}$. Note that a constant added to G_B would have no effect, since it would modify only the first term in the exponent by a term that vanishes by momentum conservation. This justifies our replacement of G_B^c by G_B.
Finally, we need the absolute normalization of the free path integral which turns out to be the same as in the DBC case,

$$\int \mathcal{D}q(\tau) e^{-\int_0^T \frac14 \dot x^2} = (4\pi T)^{-\frac{D}{2}}\,. \tag{1.49}$$

Now, putting things together, we get a famous "*Bern-Kosower master formula*"

$$\begin{aligned}\Gamma_{\rm scal}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] &= (-ie)^N(2\pi)^D\delta^D(\sum_i k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i\\ &\quad\times\exp\Big\{\sum_{i,j=1}^N \big[\frac12 G_{Bij}k_i\cdot k_j - i\dot G_{Bij}\varepsilon_i\cdot k_j + \frac12 \ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\big]\Big\}\Big|_{\mathrm{lin}(\varepsilon_1\cdots\varepsilon_N)}\,.\end{aligned} \tag{1.50}$$

This formula (or rather its analogue for the QCD case, see below) was first derived by Bern and Kosower [33] and rederived in the present approach by Strassler [37]. As it stands, it represents the one-loop N- photon amplitude in scalar QED, but Bern and Kosower also derived a set of rules which allows one to construct, starting from this master formula and by purely algebraic means, parameter integral representations for the N- photon amplitudes with a fermion loop, as well as for the N-gluon amplitudes involving a scalar, spinor or gluon loop [33, 50, 55]. However, there is a part of those rules that is valid only after imposing on-shell conditions, while the master formula itself is still valid completely off-shell (i.e. one where states do not satisfy the Einstein relation $k^2 = m^2$) which is mainly the subject of this dissertation.

Vacuum polarization

In this part we will calculate the vacuum polarization of the scalar QED, i.e. the two-point amplitude. According to (1.50) this amplitude is obtained by putting $N = 2$ after expanding out the exponential

$$\begin{aligned}\Gamma_{\rm scal}[k_1,\varepsilon_1;k_2,\varepsilon_2] &= (-ie)^2(2\pi)^D\delta^D(k_1+k_2)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,e^{-m^2T} \\ &\quad\times\int_0^T d\tau_1\int_0^T d\tau_2(-i)^2\,P_2\,e^{G_{B12}k_1\cdot k_2}\,,\end{aligned} \tag{1.51}$$

where

$$P_2 = \dot G_{B12}\varepsilon_1\cdot k_2\dot G_{B21}\varepsilon_2\cdot k_1 - \ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\,. \tag{1.52}$$

We could now perform the parameter integrals but it is advantageous to first remove the $\ddot G_{B12}$ by integration by part (IBP) which will be discussed extensively in the following chapters. By adding some total derivative, P_2 transforms into Q_2

$$Q_2 = \dot G_{B12}\dot G_{B21}\big(\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1 - \varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2\big)\,. \tag{1.53}$$

We use momentum conservation to set $k_1 = -k_2 =: k$, and define

$$\Gamma_{\rm scal}[k_1,\varepsilon_1;k_2,\varepsilon_2] = (2\pi)^D\delta(k_1+k_2)\varepsilon_1\cdot\Pi_{\rm scal}\cdot\varepsilon_2\,, \tag{1.54}$$

where

$$\begin{aligned}\Pi^{\mu\nu}_{\rm scal}(k) &= e^2(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,e^{-m^2T} \\ &\quad\times\int_0^T d\tau_1\int_0^T d\tau_2\dot G_{B12}\dot G_{B21}\,e^{G_{B12}k_1\cdot k_2}\,.\end{aligned} \tag{1.55}$$

One can see the effect of the IBP from this simple amplitude, the usual transversal projector $\delta^{\mu\nu}k^2 - k^\mu k^\nu$ factors out nicely already in the integrand level. At this level we rescale to the unit circle, i.e. $\tau_i = Tu_i$ $(i = 1, 2)$, and use translational invariance in τ to fix the zero to be at the location of the second vertex operator, $u_2 = 0$ and $u_1 = u$. We then have

$$\begin{aligned}G_{B12} &= Tu(1-u)\,,\\ \dot G_{B12} &= 1-2u\,,\end{aligned} \tag{1.56}$$

then

$$\begin{aligned}\Pi^{\mu\nu}_{\rm scal}(k) &= -\frac{e^2}{(4\pi)^{\frac{D}{2}}}(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty dT\, T^{2-\frac{D}{2}}\, e^{-m^2T} \\ &\quad\times\int_0^1 du\,(1-2u)^2 e^{-Tu(1-u)k^2}\,. \end{aligned} \tag{1.57}$$

Now one needs to perform the T-integral, we use the following elementary integral

$$\int_0^\infty \frac{dx}{x}\, x^\lambda\, e^{-ax} = \Gamma(\lambda)\, a^{-\lambda} \quad \text{for} \quad a > 0\,, \tag{1.58}$$

where we finally get

$$\begin{aligned}\Pi^{\mu\nu}_{\rm scal}(k) &= -\frac{e^2}{(4\pi)^{\frac{D}{2}}}(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\,\Gamma\big(2-\frac{D}{2}\big) \\ &\quad\times\int_0^1 du \frac{(1-2u)^2}{\big[m^2+u(1-u)k^2\big]^{2-\frac{D}{2}}}\,, \end{aligned} \tag{1.59}$$

This should now be renormalized to get a physical result, this will be pursue in Chapter 2. Our result agrees, of course, with a computation of the two corresponding Feynman diagrams Fig. 1.1.

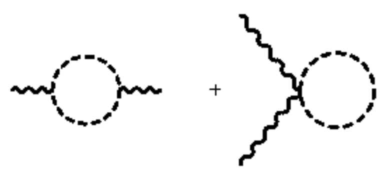


Figure 1.1: Vacuum polarization diagrams in Scalar QED.

In this simple case our integrand before the IBP, (1.51), would have still allowed a direct comparison with the Feynman diagram calculation; namely, the diagram involving the quartic vertex matches with the contribution of the $\delta(\tau_1-\tau_2)$ part of the $\ddot{G}_{B12}$.

Finally, it should be mentioned that, although any Gaussian integral can be brought to the standard form of (1.29) by formal exponentiations such as (1.36), this is not always the most efficient way to proceed. Alternatively, one can use the following set of Wick contractions involving elementary fields as well as exponentials of fields

- The basic Wick contraction of two bosonic fields is

$$\langle q^\mu(\tau_1)q^\nu(\tau_2)\rangle = -G_B(\tau_1,\tau_2)\delta^{\mu\nu}\,. \tag{1.60}$$

- Wick contract fields among themselves according to (1.29) e.g.

$$\Big\langle q^\mu(\tau_1)q^\nu(\tau_2)q^\lambda(\tau_3)q^\kappa(\tau_4)\Big\rangle = G_{B12}G_{B34}\delta^{\mu\nu}\delta^{\lambda\kappa} + G_{B13}G_{B24}\delta^{\mu\lambda}\delta^{\nu\kappa} + G_{B14}G_{B23}\delta^{\mu\kappa}\delta^{\nu\lambda}\,. \tag{1.61}$$

- Contract fields with exponentials according to

$$\langle q^\mu(\tau_1)\, e^{ik\cdot q(\tau_2)}\rangle = i\langle q^\mu\,(\tau_1) q^\nu(\tau_2)\rangle\, k_\nu\, e^{ik\cdot q(\tau_2)}\,. \tag{1.62}$$

  Note that the field disappears, but the exponential remains.

- Once all elementary fields have been eliminated, the contraction of the remaining exponentials leads to a universal factor

$$\begin{aligned} \Big\langle e^{ik_1\cdot q_1}\cdots e^{ik_N\cdot q_N}\Big\rangle &= \exp\Big[-\frac{1}{2}\sum_{i,j=1}^{N} k_{i\mu}\langle q^\mu(\tau_i) q^\nu(\tau_j)\rangle k_{j\nu}\Big] \\ &= \exp\Big[\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij} k_i\cdot k_j\Big]\,. \end{aligned} \tag{1.63}$$

It is assumed that Wick-contractions commute with derivatives.

1.2.2 Spinor QED

Feynman's vs Grassmann representation

In [6] Feynman presented the following generalization of the formula (1.27) for the effective action for the spinor QED case

$$\Gamma_{\rm spin}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_{PBC}\mathcal{D}x(\tau)\, e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x(\tau))\right)}\,{\rm Spin}[x(\tau),A]\,, \tag{1.64}$$

where the ${\rm Spin}[x(\tau),A]$ is the "spin factor" which reads as

$${\rm Spin}[x(\tau),A] = {\rm tr}_\gamma \mathcal{P}\exp\Big[i\frac{e}{4}[\gamma^\mu,\gamma^\nu]\int_0^T d\tau F_{\mu\nu}(x(\tau))\Big]\,, \tag{1.65}$$

where the ${\rm tr}_\gamma$ denotes the Dirac trace and $\mathcal{P}$ is the path ordering operator. The comparison of (1.27) with (1.64) makes it clear that the x-path integral, which is the same as we had for the scalar case, represents the contribution to the effective action due to the orbital degree of freedom of the spin $\frac{1}{2}$ particle, and that all the spin effects are indeed due to the spin factor. The minus sign in front of the path integral implements the Fermi statistics.

A modern way of writing the spin effect is in term of additional Grassmann path integral [56,57, 60,61],

$${\rm Spin}[x(\tau),A] = \int_{ABC}\mathcal{D}\psi(\tau)\,\exp\Big[-\int_0^T d\tau\big(\frac{1}{2}\psi\cdot\dot{\psi} - ie\psi^\mu F_{\mu\nu}(x(\tau))\psi^\nu\big)\Big]\,. \tag{1.66}$$

Here the path integration is over the space of anticommuting functions antiperiodic in proper-time, $\psi^\mu(\tau_1)\psi^\nu(\tau_2) = -\psi^\nu(\tau_2)\psi^\mu(\tau_1)$, $\psi^\mu(T) = -\psi^\mu(0)$ (which is indicated by the subscript 'ABC' on the path integral). The exponential in (1.66) is now an ordinary one, not a path ordered one. The ψ^μ's effectively replace the Dirac matrices γ^μ, but are functions of the proper-time, and thus will appear in all possible orderings after the expansion of the exponential. This fact is crucial for extending to the Spinor QED case the above-mentioned ability of the formalism to combine the contributions of Feynman diagrams with different orderings of the photon legs around the loop. Another advantage of introducing this second path integral is that, as it turns out, there is a "worldline" supersymmetry between the coordinate function $x(\tau)$ and the spin function $\psi(\tau)$ [61]. Namely, the total worldline Lagrangian

$$L_{\text{spin}} = \frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A + \frac{1}{2}\psi\cdot\dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu\,, \tag{1.67}$$

is invariant under

$$\begin{aligned} \delta x^\mu &= -2\eta\psi^\mu\,, \\ \delta\psi^\mu &= \eta\dot{x}^\mu\,, \end{aligned} \tag{1.68}$$

with a constant Grassmann parameter η. Although this "worldline supersymmetry" is broken by the different periodicity conditions for x and ψ, it still has a number of useful computational consequences.

1.2.3 Grassmann Gauss integrals

We need to generalize the usual Gauss integral formulas to the case of Grassmann (anticommuting) numbers. For the spinor QED we restrict ourself to real Grassmann variables. We define the Grassmann integration for a single Grassmann variable ψ by setting

$$\int d\psi\psi = 1\,. \tag{1.69}$$

Since $\psi^2 = 0$, the most general function of ψ could be defined as $f(\psi) = a + b\psi$, and

$$\int d\psi f(\psi) = b\,. \tag{1.70}$$

Now in the same way we can define the most general function for two Grassmann variables ψ_1 and ψ_2 as $f(\psi_1,\psi_2) = a + b\psi_1 + c\psi_2 + d\psi_1\psi_2$, and

$$\int d\psi_1 d\psi_2 f(\psi_1,\psi_2) = -d\,, \tag{1.71}$$

where we assume a, b, c and d are bosonic variables.
We can then also form a Gaussian integral: let $\psi = (\psi_1,\psi_2)$ and M a real antisymmetric matrix, then

$$e^{-\frac{1}{2}\psi^T\cdot M\cdot\psi} = e^{-M_{12}\psi_1\psi_2} = 1 - M_{12}\psi_1\psi_2\,, \tag{1.72}$$

(note that a symmetric part added to M would cancel out, so that we can restrict ourselves to the antisymmetric case from the beginning) and

$$\int d\psi_1 \int d\psi_2\, e^{-\frac{1}{2}\psi^T\cdot M\cdot\psi} = M_{12}\,. \tag{1.73}$$

And on the other hand since M is antisymmetric

$$\det(M) = -M_{12}M_{21} = M_{12}^2 \,, \tag{1.74}$$

then we have

$$\int d\psi_1 \int d\psi_2 \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi} = \pm(\det M)^{\frac{1}{2}} \,. \tag{1.75}$$

It is easy to show that this generalizes to any even dimension: assume $\psi_1 \cdots \psi_{2n}$ are Grassman variables and M an antisymmetric $2n \times 2n$ matrix, then

$$\int d\psi_1 \cdots \int d\psi_{2n} \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi} = \pm(\det M)^{\frac{1}{2}} \,. \tag{1.76}$$

Thus we have a determinant factor in the numerator, instead of the denominator as we had for the ordinary Gaussian integral (1.29), it can be generalized to the Grassmann case,

$$\frac{\int d\psi_1 \cdots \int d\psi_{2n} \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi + \psi \cdot j}}{\int d\psi_1 \cdots \int d\psi_{2n} \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} = e^{\frac{1}{2} j \cdot M^{-1} \cdot j} \,. \tag{1.77}$$

By differentiation with respect to the components of j one can generalize the (1.30) and find the Grassmann analogue of Wick contraction rules as

$$\begin{aligned} \frac{\int d\psi_1 \cdots \int d\psi_{2n} \psi_i \psi_j \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}}{\int d\psi_1 \cdots \int d\psi_{2n} \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} &= M_{ij}^{-1} \,, \\ \frac{\int d\psi_1 \cdots \int d\psi_{2n} \psi_i \psi_j \psi_k \psi_l \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}}{\int d\psi_1 \cdots \int d\psi_{2n} \, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} &= M_{ij}^{-1} M_{kl}^{-1} - M_{ik}^{-1} M_{jl}^{-1} + M_{il}^{-1} M_{jk}^{-1} \,, \\ \vdots \qquad & \quad \vdots \end{aligned} \tag{1.78}$$

Now, returning to our Grassmann path integral (1.66), we see that the matrix M now corresponds to the first derivative operator $\frac{d}{d\tau}$, acting in the space of antiperiodic functions, and its inverse gives the Green's function for spinor case is

$$G_F(\tau, \tau') = 2\langle \tau | \big(\frac{d}{d\tau}\big)^{-1} | \tau' \rangle = \text{sign}(\tau - \tau') \,. \tag{1.79}$$

Note that in the antiperiodic case there is no zero mode problem. Thus we find the Wick contraction rules

$$\begin{aligned} \langle \psi^\mu(\tau_1) \psi^\nu(\tau_2) \rangle &= \frac{1}{2} G_F(\tau_1, \tau_2) \delta^{\mu\nu} \,, \\ \langle \psi^\mu(\tau_1) \psi^\nu(\tau_2) \psi^\lambda(\tau_3) \psi^\kappa(\tau_4) \rangle &= \frac{1}{4} \Big[G_{F12} G_{F34} \delta^{\mu\nu} \delta^{\lambda\kappa} - G_{F13} G_{F24} \delta^{\mu\lambda} \delta^{\nu\kappa} + G_{F14} G_{F23} \delta^{\mu\kappa} \delta^{\nu\lambda} \Big] \,, \\ \vdots \qquad & \quad \vdots \end{aligned} \tag{1.80}$$

The free path integral normalization for the spinor case is derived as [50]

$$\int_{ABC} \mathcal{D}\psi \, e^{-\int_0^T d\tau \frac{1}{2} \psi \cdot \dot{\psi}} = 2^{\frac{D}{2}} \,. \tag{1.81}$$

Here D is any even spacetime dimension, and we recognize the factor $2^{\frac{D}{2}}$ as the number of real degrees of freedom of a Dirac spinor in such dimension.

N-photon amplitude

The procedure for obtaining the N-photon amplitude in spinor QED from the effective action (1.64) with (1.66) is completely analogous to the Scalar QED case treated in section 1.2.1, and we proceed straight away to the analogue of

$$\begin{aligned}\Gamma_{\text{spin}}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] &= -\frac{1}{2}(-ie)^N\int_0^\infty\frac{dT}{T}\,e^{-m^2T}\int_{x(0)=x(T)}\mathcal{D}x(\tau)\,e^{-\int_0^T d\tau\frac{1}{4}\dot{x}^2}\\ &\quad\times\int_{ABC}\mathcal{D}\psi(\tau)\,e^{-\int_0^T d\tau\frac{1}{2}\psi\cdot\dot{\psi}}\,V^{\gamma}_{\text{spin}}[k_1,\varepsilon_1]\cdots V^{\gamma}_{\text{spin}}[k_N,\varepsilon_N]\,,\end{aligned} \tag{1.82}$$

where

$$V^{\gamma}_{\text{spin}}[k,\varepsilon]\equiv\int_0^T d\tau\big[\varepsilon\cdot\dot{x}(\tau)+2i\varepsilon\cdot\psi(\tau)\,k\cdot\psi(\tau)\big]\,e^{ik\cdot x(\tau)}\,, \tag{1.83}$$

is a photon vertex operator for the emission or absorption of a photon by a spinor, the second term in (1.83) is the momentum-space version of $\psi\cdot F\cdot\psi$.
Now one would like to obtain a closed formula for general N, that is a generalization of the Bern-Kosower master formula. As we already mentioned in section 1.2.2, the Lagrangian for spinor QED (1.67) has a global super-symmetry under (1.68). One consequence of this is that we can make use of a one-dimensional superfield formalism. Introducing

$$\begin{aligned}X^\mu &= x^\mu+\sqrt{2}\theta\,\psi^\mu\,,\\ Y^\mu &= X^\mu-x_0^\mu\,,\\ D &= \frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial\tau}\,,\\ \int d\theta\theta &= 1\,,\end{aligned} \tag{1.84}$$

we can combine the x- and ψ- path integrals into the following super path integral [38,62–64]

$$\Gamma_{\text{spin}}[A] = -\frac{1}{2}\int_0^\infty\frac{dT}{T}\,e^{-m^2T}\int\mathcal{D}X\,e^{-\int_0^T d\tau\int d\theta\left[-\frac{1}{4}X\cdot D^3X-ieDX\cdot A(X)\right]}\,. \tag{1.85}$$

Written in this way, the spinor path integral becomes formally analogous to the scalar one, and can be considered as its "supersymmetrization" [50]. The supersymmetrization vertex operator can be written as

$$V^{\gamma}_{\text{spin}}[k,\varepsilon]=\int_0^T d\tau\int d\theta\varepsilon\cdot DX\,e^{ik\cdot X}\,. \tag{1.86}$$

And one can also combine the two Wick contractions for bosonic and spinor fields into a single one for the superfield as

$$\langle Y^\mu(\tau_1,\theta_1)Y^\nu(\tau_2,\theta_2)\rangle=\delta^{\mu\nu}\hat{G}(\tau_1,\theta_1;\tau_2,\theta_2)\,, \tag{1.87}$$

with a worldline superpropagator

$$\hat{G}(\tau_1,\theta_1;\tau_2,\theta_2) \equiv G_B(\tau_1,\tau_2) + \theta_1\theta_2 G_F(\tau_1,\tau_2)\,. \tag{1.88}$$

In this way one can generalize (1.85) and write down a master formula for N-photon scattering [50, 65] which is formally analogous to the one for the scalar loop

$$\begin{aligned}\Gamma_{\rm spin}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] &= -2(-ie)^N(2\pi)^D\delta(\sum_{i=1}^N k_i)\int_0^\infty \frac{dT}{T}\,(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\\ &\times\prod_{i=1}^N\int_0^T d\tau_i\int d\theta_i\exp\Big\{\sum_{i,j=1}^N\Big[\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j + iD_i\hat{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}D_iD_j\hat{G}_{ij}\varepsilon_i\cdot\varepsilon_j\Big]\Big\}\Big|_{\mathrm{lin}\varepsilon_1\cdots\varepsilon_N}\,,\end{aligned} \tag{1.89}$$

Here as well as in (1.86), we introduce the further convention that also the polarization vectors $\varepsilon_1\cdots\varepsilon_N$ are to be treated as Grassmann variables. Thus we have now all ψ's, θ's, $d\theta$'s and ε's anticommuting with each other. The overall sign of the master formula refers to the standard ordering of the polarization vector $\varepsilon_1\varepsilon_2\cdots\varepsilon_N$. Equation (1.89) is the generalization of the Bern-Kosower master formula for spinor case.
But as we will see, there is a more efficient way to proceed. We will first revisit the case of the photon propagator.

The vacuum polarization

For $N=2$, (1.82) becomes

$$\begin{aligned}\Gamma_{\rm spin}[k_1,\varepsilon_1;k_2,\varepsilon_2] &= -\frac{1}{2}(-ie)^2\int_0^\infty\frac{dT}{T}\,e^{-m^2T}\int\mathcal{D}x\int\mathcal{D}\psi\int_0^T d\tau_1\int_0^T d\tau_2\\ &\times\varepsilon_{1\mu}\big(\dot{x}_1^\mu + 2i\psi_1^\mu\psi_1\cdot k_1\big)e^{ik_1\cdot x_1}\varepsilon_{2\nu}\big(\dot{x}_2^\nu + 2i\psi_2^\nu\psi_2\cdot k_2\big)e^{ik_2\cdot x_2}\,e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi\cdot\dot{\psi}\right)}\,.\end{aligned} \tag{1.90}$$

Since the Wick contractions do not mix the x and ψ fields, the calculation of $\mathcal{D}x$ is identical with the scalar QED calculation. Only the calculation of $\mathcal{D}\psi$ is new, and amounts to a use of the four-point Wick contraction (1.78)

$$(2i)^2\big\langle\psi_1^\mu\psi_1\cdot k_1\,\psi_2^\nu\psi_2\cdot k_2\big\rangle = G_F^2\big[\delta^{\mu\nu}k_1\cdot k_2 - k_2^\mu k_1^\nu\big]\,. \tag{1.91}$$

Adding this term to the integrand for the scalar case, (1.55) one finds

$$\begin{aligned}\Pi^{\mu\nu}_{\rm spin}(k) &= -2e^2(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,e^{-m^2T}\\ &\times\int_0^T d\tau_1\int_0^T d\tau_2\big(\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\big)\,e^{G_{B12}k_1\cdot k_2}\,.\end{aligned} \tag{1.92}$$

Thus we see that, up to the normalization, the parameter integral for the spinor loop is obtained from the one for the scalar loop simply by replacing, in (1.57)

$$\dot{G}_{B12}\dot{G}_{B21} \to \dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\,, \tag{1.93}$$

which leads the final result for spinor vacuum polarization

$$\Pi^{\mu\nu}_{\text{spin}}(k) = -8\frac{e^2}{(4\pi)^{\frac{D}{2}}}\left[\delta^{\mu\nu}k^2 - k^\mu k^\nu\right]\Gamma(2-\frac{D}{2})\int_0^1 du\frac{u(1-u)}{\left[m^2+u(1-u)k^2\right]^{2-\frac{D}{2}}}\,. \tag{1.94}$$

and at this level one can easily verify the equivalence with the standard textbook calculation of the vacuum polarization diagram in spinor QED, Fig. 1.2.

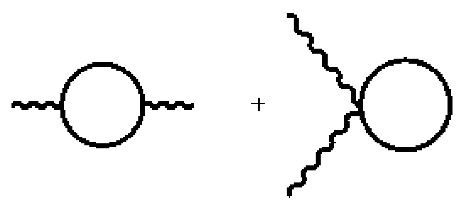


Figure 1.2: Vacuum polarization diagrams in Spinor QED.

For the worldline representation of non-abelian gauge theory which contains scalar, spinor and gluon loop contributions to the gluon scattering we refer the readers to [50] and also it will be discussed in the Section 4.4

1.3 Worldline formalism in curved space-time

In all developments of worldline formalism a difficult problem was the inclusion of gravity, even as a background. Gravity generically decouples in a naive particle limit of string theory, while in a direct worldline formulation the gravitational background leads to a path integral which necessarily requires a detailed discussion of UV regularization. Much progress has been made in [32] where string inspired rules were developed. Later Bastianelli and Zirotti in [51] addressed this issue by using the worldline formalism in the presence of background gravity starting directly from the first quantization of point particle. In the following we follow their lines. In the rest of this Section we follow the lines of [49, 51].

The Euclidean one-loop effective action $\Gamma[g]$ is the one obtained by quantizing a Klein-Gordon field ϕ coupled to gravity as

$$S[\phi;g] = \int d^Dx\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (m^2+\xi R)\phi^2)\right]\,, \tag{1.95}$$

where g is the metric ($g_{\mu\nu}$) determinant, R is the Ricci scalar, $\sqrt{g}d^Dx$ is invariant D-volume element and ξ describing an additional non-minimal coupling to a scalar curvature[2].

[2]The value $\xi = 0$ is the minimal coupling, while the value $\xi = \frac{D-2}{4(D-1)}$ gives a conformally invariant coupling in the massless case.

The corresponding one-loop effective action is formally given by

$$\begin{aligned} e^{-\Gamma[g,A]} &= \int \mathcal{D}\phi e^{-S[\phi;g]}\,, \\ \Gamma[g] &= -\log \mathrm{Det}^{-\frac{1}{2}}\, \mathrm{Tr}\log(-\Box + m^2 + \xi R) = \frac{1}{2}\mathrm{Tr}\log(-\Box + m^2 + \xi R)\,. \end{aligned} \tag{1.96}$$

where $\Box$ is the gravitationally covariant Laplacian for scalar fields. This effective action can also be obtained by considering the first quantization of a scalar point particle with coordinates x^μ and action [60]

$$S[e, x^\mu] = \int_0^T \frac{d\tau}{T}\frac{1}{2}\big[e^{-1}T^2 g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + e(m^2 + \xi R(x))\big]\,, \tag{1.97}$$

where e is an auxiliary field einbein and $\dot{x}^\mu = \partial_\tau x^\mu$ and requiring that the worldline is a closed loop (by imposing periodic boundary conditions for all fields). By a standard gauge fixing procedure [38] one can eliminate the einbein by the gauge condition $e(\tau) = 2T$ (the factor 2 is conventional), thus leaving an integration over proper time parameter T (the ghosts decouple and a factor $\frac{1}{T}$ is due to the presence of an isometry on the circle)

$$\Gamma[g] = -\int_0^\infty \frac{dT}{T}\int_{PBC} \mathcal{D}x\, e^{-S[x^\mu]}\,, \tag{1.98}$$

with worldline action

$$S[x^\mu] = \int_0^T d\tau\Big(\frac{1}{4}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2 + \xi R(x)\Big)\,. \tag{1.99}$$

and with the fields $x^\mu(\tau)$ satisfying periodic boundary conditions at $\tau = 0, T$. This system is one-dimensional quantum field theory with double derivative interactions, and hence they are not UV finite by power counting, rather the one-loop and two-loop diagrams are divergent. The UV infinities cancel in the sum of diagrams, but one needs to regularize individual diagrams which are divergent. The results of individual diagrams are then regularization-scheme dependent, and also the results for the sum of diagrams are finite but scheme dependent. One must then add finite counterterms which are also scheme dependent, and which must be chosen such that certain physical requirements are satisfied (renormalization conditions). Of course, the final physical answers should be the same, no matter which scheme one uses [52]. Three different regularization scheme have been analyzed: mode regularization (MR), time slicing (TS) and dimensional regularization (DR). The corresponding counterterms are given by

$$\begin{aligned} V_{MR} &= -\frac{1}{8}R + \frac{1}{8}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}\,, \\ V_{TS} &= -\frac{1}{8}R - \frac{1}{24}g^{\mu\nu}g^{\alpha\beta}g_{\lambda\rho}\Gamma^\lambda_{\mu\alpha}\Gamma^\rho_{\nu\beta}\,, \\ V_{DR} &= -\frac{1}{8}R\,, \end{aligned} \tag{1.100}$$

where $\Gamma^\alpha_{\nu\beta}$'s are the Christoffel symbols, for details on these counterterms see [52, 66–68].
All these regularizations require different counterterms to produce the same physical results. The

optimal choice for perturbative calculations is the DR scheme which requires a representation invariant counterterm (It does not break general coordinate invariance at intermediate stages and the counterterm V_{DR} is Einstein and local Lorentz invariant [52]),

$$\Delta S_{DR} = \int_0^T d\tau 2V_{DR}\,, \tag{1.101}$$

to be added to (1.99) with $V_{DR} = -\frac{1}{8}R$. Hence

$$S_{gf} = \int_0^T d\tau \Big(\frac{1}{4} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m^2 + \bar{\xi}R(x)\Big)\,, \tag{1.102}$$

where $\bar{\xi} = \xi - \frac{1}{4}$ takes into account the DR counterterm. We fix $\bar{\xi} = 0$, since in this thesis we will consider the vertices with one-graviton leg.
Another technical issue concerns the ways one treated a constant zero mode for path integral on a circle [51]. One option was already used in trace anomalies calculations. It consists first considering loops with a fixed based-point x_0^μ in target space, and then integrating over the position of that base-point. The coordinate $x^\mu(\tau)$ have Dirichlet boundary conditions (DBC) $x^\mu(0) = x^\mu(T) = x_0^\mu$, so that the quantum fields $y^\mu = x^\mu - x_0^\mu$ describe fluctuation around the background position x_0^μ which must vanish at $\tau = 0, T$. These quantum fields have a kinetic term without zero modes and the propagators can be derived immediately. This way of casting the path integral computation delivers a covariant effective Lagrangian density. A second option, sometimes called "string inspired", consists in directly separating out the constant zero mode $x_0^\mu = \int_0^T d\tau x^\mu(\tau)$ of the differential operator ∂_τ^2 on the circle as was discussed previously. The fields $y^\mu(\tau) = x^\mu(\tau) - x_0^\mu$ are now defined on the circle and thus satisfy periods boundary condition (PBC). The corresponding propagators are periodic and translationally invariant. This set up is simpler than the first one since in actual computations one can use translational invariance on the circle. However, it has the disadvantage that it produces an effective Lagrangian density with certain total derivative term which are non-covariant. This non-covariance invalidates any advantage of using Riemann normal coordinates [69]. The total derivative terms of the "string inspired" method are present not only in the gravitational case, but also for the standard field theories in flat space, including gauge theories, but in that case they are not bothersome, since they do not violate gauge invariance. In fact, they are beneficial since their addition leads to a more compact form of the effective action [198].
Because of derivative interactions present in this nonlinear sigma model, divergences may arise in the quantum-mechanical loop corrections. Thus regularization is not avoidable, however infinite renormalization is not necessary: the covariant path integral measure produces other infinities that cancel the original ones [70, 71]. In order to generalize the worldline formalism to curved background, naively it seems clear what to do: in a presence of a background metric field $g_{\mu\nu}(x)$ we should replace the free kinetic part of the worldline Lagrangian by the geodesic one:

$$L_{\rm free} = \frac{\dot{x}^2}{4} \longrightarrow L_{\rm geo} \equiv \frac{1}{4}\,\dot{x}^\mu\, g_{\mu\nu}(x(\tau))\dot{x}^\nu\,. \tag{1.103}$$

As we know from general relativity this action yields the classical equations of motion for a spinless particle in a background gravitational field. So we would be tempted to write down the following formula for the one-loop effective action due to a scalar particle in quantum gravity:

$$\Gamma_{\rm scal}[g] \overset{?}{=} \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int_P \mathcal{D}x\, e^{-\int_0^T d\tau L_{\rm geo}}\,, \tag{1.104}$$

where P indicates periodic boundary condition.

As is usual in quantum gravity, we could then introduce gravitons as small plane wave perturbation of the metric around flat space

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \kappa h_{\mu\nu}(x)\,, \tag{1.105}$$

where κ is the gravitational coupling constant which is proportional to the square-root of the Newton constant $\kappa = \sqrt{32G}$ and is in natural units basically the inverse Planck mass $\kappa = \sqrt{32\pi}/M_{Pl}$. $h_{\mu\nu}$ can be written as plane wave as

$$h_{\mu\nu}(x) = \varepsilon_{\mu\nu} e^{ik\cdot x}\,, \tag{1.106}$$

with a symmetric polarization tensor $\varepsilon_{\mu\nu}$. The usual perturbative evaluation of the path integral will then yield graviton amplitudes in terms of Wick contractions with the usual bosonic Green's function G_B, and each graviton presented by a vertex operator

$$V^h_{\text{scal}}[k,\varepsilon] \stackrel{?}{=} \varepsilon_{\mu\nu} \int_0^T d\tau \dot{x}^\mu \dot{x}^\nu e^{ik\cdot x(\tau)}\,. \tag{1.107}$$

However, one can immediately see that, contrary to the gauge theory case, the Wick contractions will now lead to mathematically ill-defined expressions, due to the $\delta(\tau-\tau')$ contained in $\ddot{G}_B(\tau,\tau')$. For example, a Wick contraction of two vertex operator will produce a term with $\delta(\tau_1-\tau_2)^2$, and even the one of just a single vertex operator will already contain and ill-defined $\delta(0)$. These ill-defined terms are signals of UV divergences in one dimensional worldline field theory which we have mentioned at the beginning of this section. To get rid of them we have to take the nontrivial background metric into account not only in the Lagrangian but also in the path integral measure. In general relativity the general covariance requires that each spacetime integral should contain a factor of $\sqrt{g} \equiv \sqrt{\det g_{\mu\nu}(x(\tau))}$ as in (1.95). Therefore the measure which should be used in (1.104) is of the form [71]

$$\mathcal{D}x = Dx \prod_{0\le\tau<T} \sqrt{\det g_{\mu\nu}(x(\tau))} \tag{1.108}$$

where $Dx = \prod_\tau d^D x(\tau)$ is the standard translationally invariant measure. The measure $\mathcal{D}x$ in (1.108) is formally a scalar under coordinate transformation, but the factor $\sqrt{\det g_{\mu\nu}(x(\tau))}$ is field dependent and makes the measure unsuitable to generate the perturbative expansion. In our string-inspired approach we clearly cannot use these metric factors as they stand; they ought to be exponentiated. A convenient way of doing this was proposed in [72,73]. By this proposal one needs to introduce commuting a^μ and anticommuting b^μ and c^μ worldline ghost fields with periodic boundary conditions

$$\mathcal{D}x = Dx \prod_{0\le\tau<1} \sqrt{g} = Dx \int_{PBC} DaDbDc\, e^{S_{\text{gh}}[x,a,b,c]}\,, \tag{1.109}$$

where the ghost action is given by

$$S_{\text{gh}}[x,a,b,c] = \int_0^T d\tau \frac{1}{4} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu)\,. \tag{1.110}$$

After introducing these ghost fields the final version of the path integral representation (1.104) can be written as

$$\Gamma_{\text{scal}}[g] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{PBC} DxDaDbDc\, e^{-\frac{1}{4}\int_0^T d\tau g_{\mu\nu}(x(\tau))\left(\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)+a^\mu(\tau)a^\nu(\tau)+b^\mu(\tau)c^\nu(\tau)\right)} . \tag{1.111}$$

This way one is left with quantum mechanical correlation functions on the circle of the form

$$\left\langle (\dot{x}_1^{\mu_1}\dot{x}_1^{\nu_1} + a_1^{\mu_1}a_1^{\nu_1} + b_1^{\mu_1}c_1^{\nu_1})e^{ik_1\cdot x_1}\cdots(\dot{x}_N^{\mu_N}\dot{x}_N^{\nu_N} + a_N^{\mu_N}a_N^{\nu_N} + b_N^{\mu_N}c_N^{\nu_N})e^{ip_N\cdot x_N}\right\rangle, \tag{1.112}$$

where the fields x_1, a_1 stand for $x(\tau_1)$, $a(\tau_1)$ and so on (this formula is exact for $\bar{\xi} = 0$, the general case has additional contact terms due to vertices with multiple graviton legs arising from the expansion of the $\bar{\xi}R$ term).

Final representation for gravity vertex coupled to scalar loop can be written as

$$V^h_{\text{scal}}[k,\varepsilon] = -\frac{\kappa}{4}\varepsilon_{\mu\nu}\int_0^T d\tau\Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau)\Big]\, e^{ik\cdot x} . \tag{1.113}$$

and for the case where $\bar{\xi} \neq 0$ this vertex operator has two extra terms as

$$V^h_{\text{scal}}[k,\varepsilon;\bar{\xi}] = -\frac{\kappa}{4}\varepsilon_{\mu\nu}\int_0^T d\tau\Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau) + 4\bar{\xi}(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\Big]\, e^{ik\cdot x} . \tag{1.114}$$

The ghost fields have a trivial kinetic terms, so that their Wick contractions involves only δ-functions:

$$\begin{aligned} \langle a^\mu(\tau_1)a^\nu(\tau_2)\rangle &= G_{\text{gh}}(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2)\delta^{\mu\nu} , \\ \langle b^\mu(\tau_1)c^\nu(\tau_2)\rangle &= -2G_{\text{gh}}(\tau_1,\tau_2) = -4\delta(\tau_1-\tau_2)\delta^{\mu\nu} . \end{aligned} \tag{1.115}$$

The extra terms arising from the ghost action will remove all the UV divergences. Diagrammatically this cancelations can be seen as Fig. 1.3.

Figure 1.3: Finite result by adding the ghost loop to the divergent one.

Similarly, for the fermion loop case one finds a graviton vertex operator

$$\begin{aligned} V^h_{\text{spin}}[k,\varepsilon] &= -\frac{\kappa}{4}\varepsilon_{\mu\nu}\int_0^T d\tau\Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau) \\ &\quad +2\big(\psi^\mu(\tau)\dot{\psi}^\nu(\tau) + \alpha^\mu(\tau)\alpha^\nu(\tau) + i\dot{x}^\mu(\tau)\psi^\nu(\tau)\psi(\tau)\cdot k\big)\Big]\, e^{ik\cdot x} , \end{aligned} \tag{1.116}$$

where $\alpha^\mu(\tau)$ is an additional bosonic ghosts arising from the nontrivial path integral measure for ψ^μ [74].

In the following sections, we will discuss several amplitudes in one-loop and tree-level. First, we present one- and two-point functions in one-loop scalar case and then we consider the main part of this chapter which is mixed electromagnetic and gravitation background. For this mixed background we will discuss graviton-photon-photon which has been calculated before in [75] then one-graviton four-photon in one loop which is in progress by the time of this dissertation. At tree-level, we will discuss electromagnetic Compton scattering and photonproduction amplitudes. The later tree-level amplitude has a reducible diagram which get combined with the irreducible ones and makes the final result in a nice form, for the one-graviton four-photon case we face with the same situation so we wish the tree-level calculations helps us to understand how combine the reducible and irreducible diagrams in loop-level in the worldline formalism.

1.4 Bern-Kosower replacement rules

1.4.1 Bern-Kosower replacement rules for spinor loop case

We have made a big point out of the substitution (1.93) because it is actually only the simplest instance of a general "replacement rule" due to Bern and Kosower [33]. Namely, performing the expansion of the exponential factor in (1.50) will yield an integrand $\sim P_N e^{(\cdot)}$, where we abbreviated

$$e^{(\cdot)} = \exp\Big\{\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij} k_i \cdot k_j\Big\}, \tag{1.117}$$

and P_N is a polynomial in $\dot G_{Bij}$, $\ddot G_{Bij}$ and the kinematic invariants. It is possible to remove all second derivatives $\ddot G_{Bij}$ appearing in P_N by suitable integrations-by-parts, leading to a new integrand $\sim Q_N e^{(\cdot)}$ depending only on the $\dot G_{Bij}$'s. Look in Q_N for "τ-cycle", that is, products of $\dot G_{Bij}$'s whose indices form a closed chain. A τ-cycle can thus be written as $\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}$ (to put it into this form may require the use of the antisymmetry of $\dot G_{Bij}$'s, e.g. $\dot G_{B12}\dot G_{B12} = -\dot G_{B12}\dot G_{B21}$). Then the integrand for the spinor loop case can be obtained from the one for the scalar loop simply by simultaneously replacing every τ-cycle appearing in Q_N by

$$\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1} \to \dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\,, \tag{1.118}$$

and supplying the global factor of -2 which we have already seen above. This "replacement rule" is very convenient, since it means that we do not have to really compute the Grassmann Wick contractions. However, the objective of removing the $\ddot G_{Bij}$'s does not fix the IBP procedure, nor the final integrand Q_N, and it is not at all obvious how to proceed in a systematic way for arbitrary N. Moreover, our two-point computations above suggest that the IBP procedure may also be useful for achieving transversality at the integrand level. Thus ideally one might want to have an algorithm for passing from P_N to (some) Q_N that, besided removing all $\ddot G_{Bij}$'s, has also the following properties:

- It should maintain the permutation symmetry between the photons.

- It should lead to a Q_N where each polarization vector ε_i is absorbed into the corresponding field strength tensor f_i, defined by

$$f_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu \,. \tag{1.119}$$

  This assures manifest transversality at the integrand level.

- It should be systematic enough to be computerizable.

Only very recently an algorithm has been developed that has all these properties [55].

1.5 Relation to Feynman diagrams for scalar and spinor QED

In this Section we show how do the integrand polynomials P_N relate to the ones encountered in an ordinary Feynman parameter calculation of the N-photon amplitude? For the scalar QED, the connection is still very direct. Consider the two Feynman diagrams for the scalar QED vacuum polarization, Fig. 1.1 the first one is a tadpole diagram involving the seagull vertex. Now the result of the worldline calculation before IBP, (1.52), contained a $\ddot{G}_{Bij}$, and thus a $\delta(\tau_1 - \tau_2)$. This δ-function also creates a quartic vertex, and comparing the parameter integrals one finds that, not surprisingly, its contribution to the amplitude matches with the tadpole diagram.
This correspondence carries over to the N-point case, if one fixes the ordering of the external legs, and transforms from τ to α parameters (Feynman parameter) according to

$$\begin{aligned} \alpha_1 &= T - \tau_1 \\ \alpha_2 &= \tau_1 - \tau_2 \\ \cdots &= \cdots \\ \alpha_N &= \tau_{N-1} \end{aligned} \tag{1.120}$$

The partially un-integrated Bern-Kosower integrand is thus obtained from the Feynman parameter integrand by a transformation of variables, and a certain regrouping of terms. This transformation has two effects:

- First, it allows one to combine into one expression an individual Feynman diagram and all the ones related to it by a permutation of the external states.

- Second, by regrouping the α-parameter expressions in terms of G_{Bij}, $\dot{G}_{Bij}$, $\ddot{G}_{Bij}$, which are functions well-adapted to the circle, the integrand is brought into a form suitable for partial integration, since now one needs, at least in the abelian case, not to worry about possible boundary terms.

In the spinor-loop case, comparison with the Feynman calculation is not quite so straightforward. The resulting parameter integrals obviously include those from the scalar loop, and thus contain contributions from diagrams including the seagull vertex. Clearly they cannot correspond to the parameter integrals obtained from the standard QED Feynman rules. It turns out that they correspond to a different break-up of those photon scattering amplitudes, a break-up according to a second-order formalism for fermions [76–79].

The Feynman rules for (Euclidean) spinor QED in the second order formalism are, up to statistics and degrees of freedom, the ones for scalar QED with the addition of a third vertex Fig.1.4

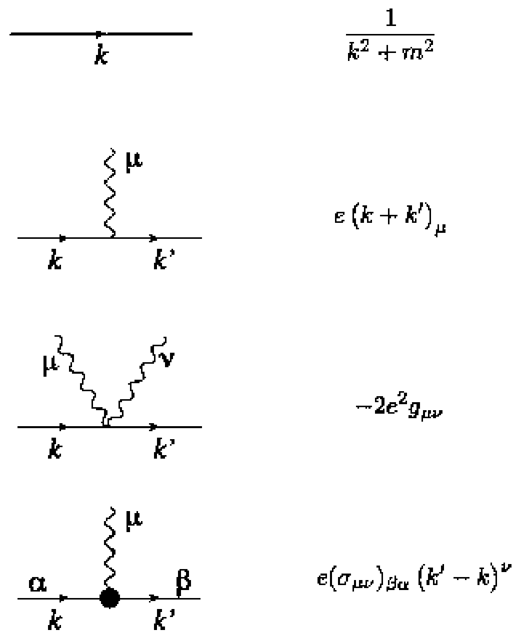


Figure 1.4: Second order Feynman rules for spinor QED.

The third vertex involves $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and corresponds to the $\psi^\mu F^{\mu\nu}\psi^\nu$-term in the worldline Lagrangian L_{spin}, (1.67). For the details and for the non-abelian case see [79]. There also an algorithm is given, based on the "Gordon identity", which transforms the sum of Feynman (momentum) integrals resulting from the first order rules into the ones generated by the second order rules. This explains the close relationship between scalar and spinor QED calculations in the worldline formalism.

Chapter 2

Full mass range analysis for the QED effective action for an $O(2) \times O(3)$ symmetric field

2.1 Introduction

[1] The one-loop effective action in a background field is an important quantity in quantum field theory. In gauge theory, the one-loop effective action has been calculated exactly and analytically only for certain cases such as constant background field [80,81], and some special inhomogeneous configurations [82].

For generic backgrounds, numerical or other approximation methods have to be used. In particular, taking the loop scalar or electron mass to be either zero or large generally leads to simplifications. At large mass, the effective action can be reduced to its heat kernel expansion, which is simple in structure and for whose computation powerful methods exist [83–86]. However, it is fair to say that, even at the numerical level, presently there is still no method available that would allow one to obtain reliable results for the effective action for arbitrary masses and in a generic background (the "worldline Monte Carlo" approach [87, 88] may ultimately provide such a formalism, although it seems too early to tell). For radially separable backgrounds, on the other hand, during the last few years the so-called "partial-wave-cutoff method" [89–93] has been developed, which seems to have all the properties one might request of such a numerical method. This class of backgrounds, although still far from generic, includes, e.g., instantons, monopoles and vortices. The method, originally invented for the case of the quark determinant in an instanton background, is based on a decomposition of the relevant one-loop operator into partial waves of definite angular momentum, and a separation into low and high angular momentum contributions, where the former are computed using the Gel'fand-Yaglom method, and the latter in a WKB expansion. It principally applies to the scalar loop, but can be extended to the spinor loop case for certain backgrounds. An example of this is the instanton where, by self-duality, the spinor effective action can be reduced to the scalar one [94–97].

In [98] G.V. Dunne et al. initiated the application of this method to the important class of $O(2) \times O(3)$ symmetric fields, first introduced by S.L. Adler in [99, 100] and studied later by a

[1] This Chapter is based on [110].

number of authors [101–104]. These backgrounds can be defined (in Euclidean metric) as

$$A_\mu(x) = \eta^3_{\mu\nu} x_\nu g(r) \; , \tag{2.1}$$

where $\eta^3_{\mu\nu}$ is a 't Hooft symbol [94], to be defined below, and $g(r)$ a radial profile function. Such backgrounds provide a good testing ground since, on one hand, they still permit a reduction from the spinor to the scalar loop case, while on the other hand, the profile function $g(r)$ is, to a good extent, arbitrary, thus leading to a large class of models. In [98], the "partial-wave-cutoff method" was applied to various profile functions, in the full mass range, and the results compared to the large mass expansion, as well as to the derivative expansion. In all cases, the method was found to be in good agreement with the large mass approximation, and in some cases, also the derivative expansion could be used to check agreement in the small mass regime. In the present Chapter, we continue the investigation started in [98] in two directions.
First, in [98], the renormalization of the effective action had been done using an unphysical renormalization condition, designed to yield a finite zero-mass limit. The asymptotic behavior for large mass then is dominated by an unphysical logarithm that made it difficult to numerically test this behavior beyond that logarithmic term. Here, we instead consider the physically renormalized effective action, which has a logarithmic divergence at small, but not at large mass. This enables us to probe the large mass behavior in deeper detail than that achieved in [98]. Specifically, we are able to numerically verify the two leading mass levels in the large mass expansion of the physical effective action. Since not all of the relevant terms in this expansion seem to be available in the literature, and moreover the coefficients depend on the chosen operator basis, we also present here their calculation from scratch, using the worldline path integral formalism along the lines of [84].
Second, this class of $O(2) \times O(3)$ symmetric backgrounds has been extensively studied by M. Fry [105–107] in a long-term effort to demonstrate that, as in the case of 1+1 dimensional QED [108, 109], also in the four-dimensional case the small m behavior of the effective action is, with a suitable renormalization choice and after subtraction of the two- and four-point contributions, dominated by a $\ln m$ coming from the chiral anomaly term $\sim \int d^4x F_{\mu\nu}\tilde{F}^{\mu\nu}$, whenever such a term is present. A simple test case for this conjecture in the class of backgrounds defined by (2.1) would be the profile function

$$g(r) = \frac{\nu}{r^2 + \rho^2} \tag{2.2}$$

where ν, ρ are positive constants. The background (2.1) with this profile function will be called the "standard $O(2)\times O(3)$ symmetric background" in the following. Our numerical method does not really allow us to treat this case as it stands, since it has insufficient radial fall-off. Even though, we will provide strong support for this hypothesis by supplying the profile function (2.2) with a radial suppression factor $e^{-\alpha r^2}$, and studying the double limit of small m and small α. Combining a perturbative and nonperturbative approach, we will show that the appropriately renormalized effective action remains finite in the small m limit for any positive value of α, and that the only obstacle preventing one to take the double limit $m, \alpha \to 0$ resides in the perturbative two-point contribution to the effective action.
The Chapter is organized as follows. In Section 2.2, we review the properties of the $O(2)\times O(3)$ background, and the predictions made in [106, 107] about its effective action. In Section (2.4) we study the perturbative N - point functions in the standard $O(2) \times O(3)$ symmetric background modified by the radial suppression factor. This Section contains also the calculation of the leading and subleading terms in the inverse mass expansion. In Section 2.5, we present our numerical

results for the effective action. Our conclusions are presented in Section 4.8. For the details of the numerical part of this work we refer the reader to [110].

2.2 The $O(2) \times O(3)$ field and its effective action

Let us start with some general facts on the effective action in four-dimensional spinor QED (we work in the Euclidean space throughout). This effective action can be written either in terms of the Dirac operator or its square as

$$\Gamma[A] = -\ln\det(\not{D} + m) = -\frac{1}{2}\ln\det\left(-\not{D}^2 + m^2\right) , \tag{2.3}$$

Here, $\not{D} = \gamma_\mu(\partial_\mu + ieA_\mu(x))$ is the Dirac operator in 4-dimensional spacetime, and $A_\mu(x)$ is the classical background gauge field. We set $e = 1$, unless explicitly indicated. We use a standard representation of the Dirac matrices [95, 98]

$$\gamma_\mu = \begin{pmatrix} 0 & \alpha_\mu \\ \bar{\alpha}_\mu & 0 \end{pmatrix} . \tag{2.4}$$

where

$$\alpha_\mu = (-i\vec{\sigma}, 1) \quad , \quad \bar{\alpha}_\mu = (\alpha_\mu)^\dagger = (i\vec{\sigma}, 1) , \tag{2.5}$$

and σ_i are the 2×2 Pauli matrices. Using this Dirac algebra one sees clearly the chiral decomposition:

$$-\not{D}^2 + m^2 = \begin{pmatrix} m^2 + DD^\dagger & 0 \\ 0 & m^2 + D^\dagger D \end{pmatrix} . \tag{2.6}$$

where

$$D \equiv \alpha_\mu D_\mu \quad , \quad D^\dagger \equiv -\bar{\alpha}_\mu D_\mu . \tag{2.7}$$

We recall the familiar properties of the α_μ matrices

$$\begin{aligned} m^2 + DD^\dagger &= m^2 - D_\mu^2 + \frac{1}{2}F_{\mu\nu}\bar{\eta}^a_{\mu\nu}\sigma_a \\ m^2 + D^\dagger D &= m^2 - D_\mu^2 + \frac{1}{2}F_{\mu\nu}\eta^a_{\mu\nu}\sigma_a \end{aligned} \tag{2.8}$$

where $\eta^a_{\mu\nu}$ is a 'tHooft symbol [94] which has the following properties

$$\begin{aligned} \eta_{a\mu\nu} &= \epsilon_{a\mu\nu} \quad , \quad \text{if} \quad \mu,\nu = 1,2,3 , \\ \eta_{a4\nu} &= -\delta_{a\nu} , \\ \eta_{a\mu4} &= \delta_{a\mu} , \\ \eta_{a44} &= 0 . \end{aligned} \tag{2.9}$$

and we also define

$$\bar{\eta}_{a\mu\nu} = (-1)^{\delta_{\mu 4}+\delta_{\nu 4}}\eta_{a\mu\nu}\ . \tag{2.10}$$

Note that here $a = 1, 2, 3$, and in deriving (2.8) the following identities will be useful

$$\bar{\alpha}_\mu \alpha_\nu = \delta_{\mu\nu} + i\eta^j_{\mu\nu}\sigma_j \quad , \quad \alpha_\mu \bar{\alpha}_\nu = \delta_{\mu\nu} + i\bar{\eta}^j_{\mu\nu}\sigma_j\ . \tag{2.11}$$

One can find more properties of this symbol in [94].

Finally, one obtains the following chiral form for the squared Dirac operator:

$$-\not{D}^2 + m^2 = \begin{pmatrix} m^2 - D^2_\mu + \frac{1}{2}F_{\mu\nu}\bar{\eta}^a_{\mu\nu}\,\sigma_a & 0 \\ 0 & m^2 - D^2_\mu + \frac{1}{2}F_{\mu\nu}\eta^a_{\mu\nu}\,\sigma_a \end{pmatrix}\ . \tag{2.12}$$

Thus, we have a chiral decomposition of the effective action,

$$\Gamma[A] \quad = \quad -\frac{1}{2}\ln\det\left(DD^\dagger + m^2\right) - \frac{1}{2}\ln\det\left(D^\dagger D + m^2\right)\ , \tag{2.13}$$

which can be written as

$$\begin{aligned} \Gamma[A] \quad &= \quad -\frac{1}{2}\ln\det\left(-D^2 + m^2 + \frac{1}{2}F_{\mu\nu}\bar{\eta}^a_{\mu\nu}\,\sigma_a\right) - \frac{1}{2}\ln\det\left(-D^2 + m^2 + \frac{1}{2}F_{\mu\nu}\eta^a_{\mu\nu}\,\sigma_a\right) \\ &=: \quad \Gamma^{(+)}[A] + \Gamma^{(-)}[A]\ . \end{aligned} \tag{2.14}$$

We can also write

$$\Gamma[A] \quad = \quad 2\Gamma^{(\pm)}[A] \mp \left(\Gamma^{(+)}[A] - \Gamma^{(-)}[A]\right)\ . \tag{2.15}$$

Furthermore, we know that the difference of the renormalized effective action for the two chiralities takes a special form, related to the chiral anomaly [111]:

$$\Delta\Gamma_{\rm ren}[A] \quad \equiv \quad \left(\Gamma^{(+)}_{\rm ren}[A] - \Gamma^{(-)}_{\rm ren}[A]\right) = \frac{1}{2}\frac{1}{(4\pi)^2}\ln\left(\frac{m^2}{\mu^2}\right)\int d^4x\, F_{\mu\nu}\tilde{F}_{\mu\nu}\ . \tag{2.16}$$

Therefore, for the computation of the full spinor effective action, it is sufficient to evaluate either $\Gamma^{(+)}_{\rm ren}[A]$ or $\Gamma^{(-)}_{\rm ren}[A]$ and the chiral anomaly term [111]. This fact is computationally quite relevant, since the contribution of one chirality might be significantly easier to compute than the other one.
We consider a field of the form (2.1), where $g(r)$ is a radial profile function. For this background, the negative chirality part of the Dirac operator takes a simple form,

$$m^2 - D^2_\mu + \frac{1}{2}F_{\mu\nu}\eta^a_{\mu\nu}\,\sigma_a = m^2 - D^2_\mu + (4g(r) + r\,g'(r))\,\sigma_3\ . \tag{2.17}$$

Hence, we use (2.15) in the form

$$\Gamma_{\rm ren}[A] \quad = \quad 2\Gamma^{(-)}_{\rm ren}[A] + \Delta\Gamma_{\rm ren}[A]\,. \tag{2.18}$$

The field strength tensor for the background (2.1) is

$$F_{\mu\nu}(x) = -2\eta^3_{\mu\nu}g(r) - \frac{g'(r)}{r}\left(\eta^3_{\mu\sigma}x_\nu x_\sigma - \eta^3_{\nu\sigma}x_\mu x_\sigma\right)\ . \tag{2.19}$$

Within the class of background gauge field (2.1), we still have the freedom to specify the radial function $g(r)$. Here we note that the large r behavior of this radial function, determines the presence or absence of zero mode. The $F_{\mu\nu}\tilde{F}_{\mu\nu}$ is

$$\begin{aligned}
F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad &= \quad \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Big(-2\eta^3_{\mu\nu}g(r) - \frac{g'(r)}{r}\left(\eta^3_{\mu\sigma}x_\nu x_\sigma - \eta^3_{\nu\sigma}x_\mu x_\sigma\right)\Big) \\
&\quad\ \times\Big(-2\eta^3_{\alpha\beta}g(r) - \frac{g'(r)}{r}\left(\eta^3_{\alpha\gamma}x_\beta x_\gamma - \eta^3_{\beta\gamma}x_\alpha x_\gamma\right)\Big) \\
&= \quad \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Big(4g(r)^2\eta^3_{\mu\nu}\eta^3_{\alpha\beta} + 2g(r)g'(r)(\eta^3_{\mu\nu}\eta^3_{\alpha\gamma}x_\beta x_\gamma - \eta^3_{\mu\nu}\eta^3_{\beta\gamma}x_\alpha x_\gamma)\Big) \\
&\quad\ +2g(r)g'(r)(\eta^3_{\alpha\beta}\eta^3_{\mu\sigma}x_\nu x_\sigma - \eta^3_{\alpha\beta}\eta^3_{\nu\sigma}x_\mu x_\sigma) \\
&\quad\ +\frac{g'(r)^2}{r^2}\left(\eta^3_{\mu\sigma}x_\nu x_\sigma - \eta^3_{\nu\sigma}x_\mu x_\sigma\right)\left(\eta^3_{\alpha\gamma}x_\beta x_\gamma - \eta^3_{\beta\gamma}x_\alpha x_\gamma\right)\Big) \\
&= \quad 16g(r)^2 + 8g(r)g'(r)r\,,
\end{aligned} \tag{2.20}$$

and similarly one gets

$$F_{\mu\nu}F_{\mu\nu} = 16g(r)^2 + 8g(r)g'(r)r + 2g'(r)^2r^2\,, \tag{2.21}$$

where we have used the following identities for the $\epsilon_{\mu\nu\alpha\beta}$ and $\eta^a_{\mu\nu}$

$$\begin{aligned}
\eta_{a\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\eta_{a\alpha\beta} \quad , \quad \bar{\eta}_{a\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\eta}_{a\alpha\beta}\,, \\
\eta_{a\mu\nu} &= -\eta_{a\nu\mu}\,, \\
\eta_{a\mu\nu}\eta_{b\mu\nu} &= 4\delta_{ab}\,, \\
\eta_{a\mu\nu}\eta_{a\mu\lambda} &= 3\delta_{\nu\lambda}\,, \\
\eta_{a\mu\nu}\eta_{a\mu\nu} &= 12\,, \\
\eta_{a\mu\nu}\eta_{a\kappa\lambda} &= \delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa} + \epsilon_{\mu\nu\kappa\lambda}\,, \\
\eta_{a\mu\nu}\eta_{b\mu\alpha} &= \delta_{ab}\delta_{\nu\alpha} + \epsilon_{abc}\eta_{c\nu\alpha}\,, \\
\eta_{a\mu\nu}\bar{\eta}_{a\mu\nu} &= 0\,, \\
\eta_{a\alpha\mu}\bar{\eta}_{b\alpha\nu} &= \eta_{a\alpha\nu}\bar{\eta}_{b\alpha\mu}\,, \\
\epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta} &= 4!\,, \\
\epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\gamma} &= 3!\delta_{\beta\gamma}\,, \\
\epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu\nu\zeta\gamma} &= 2!(\delta_{\alpha\zeta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\zeta})\,.
\end{aligned} \tag{2.22}$$

(Since in our case $a = 3$, note that whenever there is a sum over a in (2.22) we need to include a factor of 1/3).

To be specific, the integral $F_{\mu\nu}\tilde{F}_{\mu\nu}$ counts the number of zero modes:

$$\frac{1}{2}\int_0^\infty dr\ r^3(F_{\mu\nu}\tilde{F}_{\mu\nu}) \quad = \quad \int_0^\infty dr\ r^3(8g(r)^2 + 4g(r)g'(r)r) = 2(g(r)r^2)^2\Big|_0^\infty . \tag{2.23}$$

Therefore, as long as $g(r)$ falls faster than $1/r^2$, there are no zero-modes. As was already mentioned in the introduction, the partial-wave-cutoff method –which we wish to use– is not guaranteed to work well in the standard case $g(r) = 1/(r^2+\rho^2)$ due to insufficient radial fall-off. This suggests to study the following more general family of profile functions [98]:

$$g(r) \equiv \nu\,\frac{e^{-\alpha r^2}}{\rho^2 + r^2}\,, \tag{2.24}$$

where ν, ρ and α are parameters that control the amplitude, steepness, and range of the potential. For this profile function, the choice of α produces one of the following cases:

$$\alpha > 0 \quad \Longrightarrow \quad \int d^4x F_{\mu\nu}F_{\mu\nu} < \infty \quad , \quad \int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu} = 0\,, \tag{2.25}$$

$$\alpha = 0 \quad \Longrightarrow \quad \int d^4x F_{\mu\nu}F_{\mu\nu} \to \infty \quad , \quad 0 < \Big|\int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu}\Big| < \infty\,. \tag{2.26}$$

Thus, we have zero modes only for $\alpha = 0$, corresponding to the standard case. It will also be useful to note that then

$$\begin{aligned}\frac{1}{2}\int_0^\infty dr\ r^3 F_{\mu\nu}F_{\mu\nu} \quad &= \quad \int_0^\infty dr\ r^3(8g(r)^2 + 4g(r)g'(r)r + g'(r)^2r^2)\\ &= \quad \int_0^\infty dr\ r^3 F_{\mu\nu}\tilde{F}_{\mu\nu} + \int_0^\infty dr\ r^3(g'(r)^2r^2)\,,\end{aligned} \tag{2.27}$$

where the second integral diverges logarithmically. In the present Chapter, we will set $\rho = 1$ throughout, and also $\nu = 1$, unless explicitly stated otherwise. α will be a small positive number effectively serving as an IR cutoff.
Finally, let us summarize the conclusions reached at in [106, 107] about the small mass limit of the spinor QED effective action for the standard case $\alpha = 0$:

1. Let $\mathcal{R}$ denote the (scheme independent) effective action obtained after subtraction of the two-point contribution.

2. There is evidence that $\mathcal{R}$ behaves for small m as

$$\mathcal{R} \sim \frac{\nu^2}{4}\ln m^2 + \text{less singular in } m^2\,. \tag{2.28}$$

3. The logarithmic term in (2.28) is determined entirely by the chiral anomaly, given for our background by

$$-\frac{1}{(4\pi)^2}\int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{\nu^2}{2} \ . \tag{2.29}$$

In the following Sections, we provide strong support for these statements. Further, an important application of (2.28) is the search for a nontrivial zero of the determinant

$$\mathrm{lndet}_5 := \mathcal{R} - \Pi_4 \ , \tag{2.30}$$

where Π_4 is the four-point contribution to $\mathcal{R}$ [105–107]. In this connection, it is useful to know whether the four-point contribution itself adds to the logarithmic singularity of the massless limit, $\Pi_4 \sim C \ln m$. As a corollary of our study of the N - point functions in Section (2.4) we will settle this detail, that had been left open in the analysis of [106], by showing that $C = 0$.

2.3 The partial-wave-cutoff method

Recently a new numerical approach has been developed for exact computing the gauge theory effective action in a class of background fields that permit a separation of variables reducing to a set of one-dimensional operators. This has been explored in details for radially separable backgrounds, where the method has been called the "partial-wave cutoff method" [89–92]. The first application of this method was to the full mass dependence of the quark determinant in an instanton background [89], and later studies have concentrated on scalar theories. The self dual background has a special property that implies that the spinor effective action can expressed directly in term of the scalar effective action as we have mentioned in Section 2.2.
The basic idea of "partial-wave cutoff method" is simple: the one loop effective action requires the logarithm of determinant of an operator, and there is a method known as "Gel'fand-Yaglom method" [112] which computes the determinant of an ordinary differential operator, without computing its eigenvalues. For a partial differential operator, if the problem is separable down to a set of ordinary differential operators, we can formally sum over the Gel'fand-Yaglom results for each term in the separation sum. There is a technical difficulty which is the divergent of this sum so suitable regularization and renormalization is needed. This problem has been addressed by G. Dunne et al for scalar case in [89–92] and for spinor case in [98].

2.4 Perturbative results

Before starting on our numerical analysis of the effective action, which will be intrinsically non-perturbative, in this Section we perform a number of perturbative computations that will help us to interpret those nonperturbative results, as well as to verify their numerical accuracy. In these computations we use the worldline formalism along the lines of [37,50,84], and as is usual in that formalism as a byproduct of our spinor QED calculations we will obtain also the corresponding quantities for Scalar QED. The latter will be included here, for their own interest as we all as with a view on future extensions of this work to the Scalar QED case (Scalar QED quantities will be given a subscript 'scal').

2.4.1 Large mass expansion of the effective action

In this Subsection we calculate the leading and subleading terms in the inverse mass (= heat kernel) expansion of the one-loop scalar and spinor QED effective actions, following the approach of [84, 113]. The starting point is Feynman's worldline path integral representation of the scalar loop effective action [5, 50] (note that in our present conventions the effective action is defined with the opposite sign relative to the conventions of [50]),

$$\Gamma_{\rm scal}[A] = -\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x(\tau) e^{-\int_0^T d\tau \dot{x}^2/4 - ie\int_0^T d\tau \dot{x}^\mu A_\mu(x)} \,. \tag{2.31}$$

Here, at fixed proper-time T, the path integral runs over all closed loops in spacetime with periodicity T. We will generally Taylor expand the Maxwell field $A_\mu(x)$ at the loop center-of-mass x_0, defined by

$$x_0^\mu \;\; \equiv \;\; \frac{1}{T}\int_0^T d\tau x^\mu(\tau) \,, \tag{2.32}$$

and then use Fock-Schwinger gauge to write the coefficients of this expansion in terms of the field strength tensor $F_{\mu\nu}(x_0)$ and its derivatives [84]. The first few terms in this expansion are

$$A_\mu(x = x_0 + y) = -\frac{1}{2}F_{\mu\nu}(x_0)y^\nu - \frac{1}{3}F_{\mu\nu,\alpha}(x_0)y^\nu y^\alpha - \frac{1}{8}F_{\mu\nu,\alpha\beta}(x_0)y^\nu y^\alpha y^\beta + \cdots \,, \tag{2.33}$$

where $F_{\mu\nu,\alpha} = \frac{\partial}{\partial\alpha}F_{\mu\nu}$.

Combining the expansion of the interaction exponential in the path integral (2.31) with the Fock-Schwinger expansion (2.33), the path integration can be reduced to gaussian form. So, its performance requires only the knowledge of the free path integral normalization factor

$$\int \mathcal{D}y \, \exp\left[-\int_0^T d\tau \frac{1}{4}\dot{y}^2\right] = (4\pi T)^{-D/2} \,, \tag{2.34}$$

and the two-point correlator $G_B(\tau,\tau')$.
One then collects the terms with a fixed power of T, and obtains the inverse mass expansion of the effective action in the form

$$\Gamma_{\rm scal}[F] = \int_0^\infty \frac{dT}{T}\, \frac{e^{-m^2 T}}{(4\pi T)^{D/2}}\, {\rm tr} \int dx_0 \sum_{n=1}^N \frac{(-T)^n}{n!}\, O_n[F] \,, \tag{2.35}$$

where $O_n(F)$ contains the operators in the effective action of mass dimension $2n$. For the physically renormalized effective action $\Gamma^{\rm OS}_{\rm scal,spin}$, the lowest non-vanishing mass level is $n = 3$, which therefore dominates in the large mass limit. In the following, we calculate this leading contribution and also the subleading $n = 4$ terms, for both scalar and spinor QED. Since we work on the finite part of the effective action only, we can set $D = 4$ from now onward.
Starting with the leading order, this is given by the term in the effective action involving two copies of the second term in the expansion (2.33). Denoting this term by $\Gamma^{\partial F \partial F}$, we have (in an obvious notation and omitting the argument x_0 of the field strength tensors)

$$
\begin{aligned}
\Gamma^{\partial F \partial F}[A] &= -\frac{(-i)^2}{2!}\int_0^\infty \frac{dT}{T} e^{-m^2 T}\int d^4x_0 \int_0^T d\tau_1 \int_0^T d\tau_2 \\
&\times \int \mathcal{D}y \left[\frac{1}{9}\ \dot{y}^{\mu_1} F_{\mu_1\nu_1,\alpha_1} y^{\nu_1} y^{\alpha_1} \dot{y}^{\mu_2} F_{\mu_2\nu_2,\alpha_2} y^{\nu_2} y^{\alpha_2}\ \right] e^{-\int_0^T d\tau \dot{y}^2/4}\ ,
\end{aligned}
\tag{2.36}
$$

where we have set $e = 1$ as usual.
In this step one needs to do all possible Wick contractions,

$$
\begin{aligned}
\mathcal{M} &= \Big\langle \dot{y}^{\mu_1}(\tau_1)\, \dot{y}^{\mu_2}(\tau_2)\, y^{\nu_1}(\tau_1)\, y^{\nu_2}(\tau_2)\, y^{\alpha_1}(\tau_1)\, y^{\alpha_2}(\tau_2) \Big\rangle \\
&= \big\langle \dot{y}^{\mu_1}(\tau_1)\, \dot{y}^{\mu_2}(\tau_2)\big\rangle \Big\{ \big\langle y^{\nu_1}(\tau_1)\, y^{\nu_2}(\tau_2)\big\rangle \big\langle y^{\alpha_1}(\tau_1)\, y^{\alpha_2}(\tau_2)\big\rangle \\
&\qquad + \big\langle y^{\nu_1}(\tau_1)\, y^{\alpha_2}(\tau_2)\big\rangle \big\langle y^{\alpha_1}(\tau_1)\, y^{\nu_2}(\tau_2)\big\rangle \Big\} \\
&\quad + \big\langle \dot{y}^{\mu_1}(\tau_1)\, y^{\nu_2}(\tau_2)\big\rangle \Big\{ \big\langle y^{\nu_1}(\tau_1)\, \dot{y}^{\mu_2}(\tau_2)\big\rangle \big\langle y^{\alpha_1}(\tau_1)\, y^{\alpha_2}(\tau_2)\big\rangle \\
&\qquad + \big\langle y^{\nu_1}(\tau_1)\, y^{\alpha_2}(\tau_2)\big\rangle \big\langle y^{\alpha_1}(\tau_1)\, \dot{y}^{\mu_2}(\tau_2)\big\rangle \Big\} \\
&\quad + \big\langle \dot{y}^{\mu_1}(\tau_1)\, y^{\alpha_2}(\tau_2)\big\rangle \Big\{ \big\langle y^{\nu_1}(\tau_1)\, \dot{y}^{\mu_2}(\tau_2)\big\rangle \big\langle y^{\alpha_1}(\tau_1)\, y^{\nu_2}(\tau_2)\big\rangle \\
&\qquad + \big\langle y^{\nu_1}(\tau_1)\, y^{\nu_2}(\tau_2)\big\rangle \big\langle y^{\alpha_1}(\tau_1)\, \dot{y}^{\mu_2}(\tau_2)\big\rangle \Big\}\ .
\end{aligned}
\tag{2.37}
$$

We need to consider the following contractions

$$
\begin{aligned}
\big\langle y^{\mu}(\tau_1) y^{\nu}(\tau_2)\big\rangle &= -G_B(\tau_1,\tau_2)\delta^{\mu\nu}\ , \\
\big\langle \dot{y}^{\mu}(\tau_1) y^{\nu}(\tau_2)\big\rangle &= -\dot{G}_B(\tau_1,\tau_2)\delta^{\mu\nu}\ , \\
\big\langle y^{\mu}(\tau_1) \dot{y}^{\nu}(\tau_2)\big\rangle &= \dot{G}_B(\tau_1,\tau_2)\delta^{\mu\nu}\ , \\
\big\langle \dot{y}^{\mu}(\tau_1) \dot{y}^{\nu}(\tau_2)\big\rangle &= \ddot{G}_B(\tau_1,\tau_2)\delta^{\mu\nu}\ ,
\end{aligned}
\tag{2.38}
$$

so according to our convention all the derivatives are respect to the first variable.
By considering all the above contraction (6.73) would be

$$
\begin{aligned}
\mathcal{M} &= \ddot{G}_B(\tau_1,\tau_2) G_B^2(\tau_1,\tau_2) \Big\{ \delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}\delta^{\alpha_1\alpha_2} + \delta^{\mu_1\mu_2}\delta^{\nu_1\alpha_2}\delta^{\alpha_1\nu_2} \Big\} \\
&\quad + \dot{G}_B^2(\tau_1,\tau_2) G_B(\tau_1,\tau_2) \Big\{ \delta^{\mu_1\nu_2}\delta^{\nu_1\mu_2}\delta^{\alpha_1\alpha_2} + \delta^{\mu_1\nu_2}\delta^{\nu_1\alpha_2}\delta^{\alpha_1\mu_2} \\
&\qquad + \delta^{\mu_1\alpha_2}\delta^{\nu_1\mu_2}\delta^{\alpha_1\nu_2} + \delta^{\mu_1\alpha_2}\delta^{\nu_1\nu_2}\delta^{\alpha_1\mu_2} \Big\} \\
&\equiv \ddot{G}_B(\tau_1,\tau_2) G_B^2(\tau_1,\tau_2)\delta_1 + \dot{G}_B^2(\tau_1,\tau_2) G_B(\tau_1,\tau_2)\delta_2\ ,
\end{aligned}
\tag{2.39}
$$

where

$$\begin{aligned}\delta_1 &= \delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}\delta^{\alpha_1\alpha_2} + \delta^{\mu_1\mu_2}\delta^{\nu_1\alpha_2}\delta^{\alpha_1\nu_2}\,,\\ \delta_2 &= \delta^{\mu_1\nu_2}\delta^{\nu_1\mu_2}\delta^{\alpha_1\alpha_2} + \delta^{\mu_1\nu_2}\delta^{\nu_1\alpha_2}\delta^{\alpha_1\mu_2} + \delta^{\mu_1\alpha_2}\delta^{\nu_1\mu_2}\delta^{\alpha_1\nu_2} + \delta^{\mu_1\alpha_2}\delta^{\nu_1\nu_2}\delta^{\alpha_1\mu_2}\,.\end{aligned} \tag{2.40}$$

Through an integration-by-parts, we can replace

$$\ddot{G}_B(\tau_1,\tau_2)G_B^2(\tau_1,\tau_2) \;\to\; -2\dot{G}_B^2(\tau_1,\tau_2)G_B(\tau_1,\tau_2)\,,$$

in the first term of $\mathcal{M}$,

$$\mathcal{M} = \dot{G}_B^2(\tau_1,\tau_2)G_B(\tau_1,\tau_2)\Big(-2\delta_1+\delta_2\Big)\,. \tag{2.41}$$

Next, we use the Bianchi identity to show that

$$F_{\mu_1\nu_1,\alpha_1}F_{\mu_2\nu_2,\alpha_2}\delta_1 = -F_{\mu_1\nu_1,\alpha_1}F_{\mu_2\nu_2,\alpha_2}\delta_2 = \frac{3}{2}F^2_{\mu\nu,\alpha}\,. \tag{2.42}$$

Thus,

$$F_{\mu_1\nu_1,\alpha_1}F_{\mu_2\nu_2,\alpha_2}\mathcal{M} = -\frac{9}{2}\dot{G}_B^2(\tau_1,\tau_2)G_B(\tau_1,\tau_2)F^2_{\mu\nu,\alpha}\,. \tag{2.43}$$

Next, we perform the τ_i integrals. Here, as usual, one can use the unbroken reparametrization invariance to set $\tau_2=0$ and rescale $\tau_1 = Tu$, with the result

$$\int_0^T d\tau_1\int_0^T d\tau_2\,\dot{G}_B^2(\tau_1,\tau_2)G_B(\tau_1,\tau_2) = T^3\int_0^1 du(1-2u)^2u(1-u) = \frac{T^3}{30}\,. \tag{2.44}$$

where we have used the rescaled Green function and its first derivative

$$\begin{aligned}G_B(u,u') &= |u-u'|-(u-u')^2\,,\\ \dot{G}(u,u') &= \mathrm{sign}(u-u')-2(u-u')\,.\end{aligned} \tag{2.45}$$

Going back to the (2.36) one gets

$$\Gamma^{\partial F\partial F}_{\mathrm{scal}}[A] \;=\; -\frac{1}{120}\frac{1}{(4\pi)^{\frac{4}{2}}}\int_0^\infty \frac{T^3}{T}dTe^{-m^2T}(T)^{-\frac{4}{2}}\int d^4x_0\; F^2_{\mu_2\nu_2,\alpha_2}(x_0)\,, \tag{2.46}$$

in the following we set $D=4$.
Performing the final T - integration and putting things together we get our final result,

$$\Gamma^{\partial F\partial F}_{\mathrm{scal}}[A] = -\frac{1}{1920\pi^2m^2}\int d^4x_0\; F^2_{\mu_2\nu_2,\alpha_2}(x_0)\,. \tag{2.47}$$

To get the corresponding result for the spinor QED case, one can start simply by working on the effective action with the Grassmanian operator and as the scalar case expand the interacting part and perform all the Wick contractions for bosonic and fermionic fields (x and ψ) and find the final effective action,

$$\Gamma = -\frac{1}{2}\int_0^\infty \frac{dT}{T}e^{-m^2T}\int \mathcal{D}x\int_{ABC}\mathcal{D}\psi e^{-\int_0^T d\tau(\frac{1}{4}\dot{x}^2+\frac{1}{2}\dot{\psi}_\mu\psi^\mu+ie\dot{x}_\mu A^\mu(x)-ie\psi^\mu F_{\mu\nu}\psi^\nu)}\ , \tag{2.48}$$

where the ψ integral must be performed with antiperiodic boundary condition (ABC). Here again one can expand the interacting part of the bosoinc and fermionic fields to the appropriate mass level and Wick contract all possible ways of the contractions.
The free path integral normalization factor

$$\int_{\text{ABC}}\mathcal{D}\psi\text{exp}\Big[\frac{1}{2}\dot{\psi}_\mu\psi^\mu\Big] = 4\ , \tag{2.49}$$

and the two point correlator for fermionic fields $G_F(\tau,\tau')$.

Another way to get the corresponding results for the spinor QED case is to make use of the "Bern-Kosower replacement rule" [33], according to which the result for the spinor loop is inferred from the scalar result by using (1.118) which for two-point is by replacing

$$\dot{G}_B^2(\tau_1,\tau_2) \to \dot{G}_B^2(\tau_1,\tau_2) - G_F^2(\tau_1,\tau_2)\ . \tag{2.50}$$

This changes the integral (2.44) into

$$\int_0^T d\tau_1\int_0^T d\tau_2\,\big[\dot{G}_B^2(\tau_1,\tau_2) - G_F^2(\tau_1,\tau_2)\big]G_B(\tau_1,\tau_2) \quad = \quad -\frac{2}{15}T^3\ . \tag{2.51}$$

Also, the global normalization has to be changed by a factor of -2. Thus, our result for the leading term in the spinor QED large mass expansion is

$$\Gamma^{\partial F\partial F}[A] = 8\Gamma^{\partial F\partial F}_{\text{scal}}[A] = -\frac{1}{240\pi^2m^2}\int d^4x_0\ F^2_{\mu_2\nu_2,\alpha_2}(x_0)\ . \tag{2.52}$$

For the spinor QED case, this term was also computed in [86].
It should be noted that, at the same mass level, there could have been a contribution involving the product of the first and third terms in the Fock-Schwinger expansion (2.33). However, it drops out due to the vanishing of the coincidence limits $G_B(\tau,\tau) = \dot{G}_B(\tau,\tau) = 0$.
At the next mass level, which is of mass dimension eight, one again finds that the non-vanishing contributions with only two fields involve two copies of the third term in the Fock-Schwinger expansion (2.33). Here in the following we present more details of these calculations.

Let us starts with the scalar case:

$$
\begin{aligned}
\Gamma^{\partial\partial F\partial\partial F}_{\rm scal}[A] &= -\frac{(-i)^2}{128}\int_0^\infty \frac{dT}{T}e^{-m^2T}\int d^4x_0\int_0^T d\tau_1\int_0^T d\tau_2 \\
&\quad\times\int \mathcal{D}y\Big[\dot{y}^{\mu_1}F_{\mu_1\nu_1,\alpha_1\beta_1}y^{\nu_1}y^{\alpha_1}y^{\beta_1}\,\dot{y}^{\mu_2}F_{\mu_2\nu_2,\alpha_2\beta_2}y^{\nu_2}y^{\alpha_2}y^{\beta_2}\Big]e^{-\int_0^T d\tau\dot{y}^2/4} \\
&= +\frac{1}{128}\int_0^\infty \frac{dT}{T}e^{-m^2T}\int d^4x_0\int_0^T d\tau_1\int_0^T d\tau_2 \\
&\quad\times\int \mathcal{D}yF_{\mu_1\nu_1,\alpha_1\beta_1}\,F_{\mu_2\nu_2,\alpha_2\beta_2}\mathcal{M}'e^{-\int_0^T d\tau\dot{y}^2/4}\ .
\end{aligned}
\tag{2.53}
$$

Again one needs to consider all the Wick contractions of the bosonic fields,

$$
\begin{aligned}
\mathcal{M}' &= \Big\langle \dot{y}^{\mu_1}(\tau_1)y^{\nu_1}(\tau_1)y^{\alpha_1}(\tau_1)y^{\beta_1}(\tau_1)\,\dot{y}^{\mu_2}(\tau_2)y^{\nu_2}(\tau_2)y^{\alpha_2}(\tau_2)y^{\beta_2}(\tau_2)\Big\rangle \\
&= -\ddot{G}_B(\tau_1,\tau_2)G_B^3(\tau_1,\tau_2)\delta_1-\dot{G}_B^2(\tau_1,\tau_2)G_B^2(\tau_1,\tau_2)\delta_2\ ,
\end{aligned}
\tag{2.54}
$$

where

$$
\begin{aligned}
\delta_1 &= \delta^{\mu_1\mu_2}\Big\{\delta^{\nu_1\nu_2}(\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}+\delta^{\alpha_1\beta_2}\delta^{\beta_1\alpha_2})+\delta^{\nu_1\alpha_2}(\delta^{\alpha_1\nu_2}\delta^{\beta_1\beta_2}+\delta^{\alpha_1\beta_2}\delta^{\beta_1\nu_2}) \\
&\quad+\delta^{\nu_1\beta_2}(\delta^{\alpha_1\nu_2}\delta^{\beta_1\alpha_2}+\delta^{\alpha_1\alpha_2}\delta^{\beta_1\nu_2})\Big\}\ ,
\end{aligned}
\tag{2.55}
$$

and

$$
\begin{aligned}
\delta_2 &= \delta^{\mu_1\nu_2}\Big\{\delta^{\nu_1\mu_2}(\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}+\delta^{\alpha_1\beta_2}\delta^{\beta_1\alpha_2})+\delta^{\nu_1\alpha_2}(\delta^{\alpha_1\mu_2}\delta^{\beta_1\beta_2}+\delta^{\alpha_1\beta_2}\delta^{\beta_1\mu_2}) \\
&\quad+\delta^{\nu_1\beta_2}(\delta^{\alpha_1\mu_2}\delta^{\beta_1\alpha_2}+\delta^{\alpha_1\alpha_2}\delta^{\beta_1\mu_2})\Big\} \\
&\quad+\delta^{\mu_1\alpha_2}\Big\{\delta^{\nu_1\nu_2}(\delta^{\alpha_1\mu_2}\delta^{\beta_1\beta_2}+\delta^{\alpha_1\beta_2}\delta^{\beta_1\mu_2})+\delta^{\nu_1\mu_2}(\delta^{\alpha_1\nu_2}\delta^{\beta_1\beta_2}+\delta^{\alpha_1\beta_2}\delta^{\beta_1\nu_2}) \\
&\quad+\delta^{\nu_1\beta_2}(\delta^{\alpha_1\nu_2}\delta^{\beta_1\mu_2}+\delta^{\alpha_1\mu_2}\delta^{\beta_1\nu_2})\Big\} \\
&\quad+\delta^{\mu_1\beta_2}\Big\{\delta^{\nu_1\nu_2}(\delta^{\alpha_1\alpha_2}\delta^{\beta_1\mu_2}+\delta^{\alpha_1\mu_2}\delta^{\beta_1\alpha_2})+\delta^{\nu_1\alpha_2}(\delta^{\alpha_1\nu_2}\delta^{\beta_1\mu_2}+\delta^{\alpha_1\mu_2}\delta^{\beta_1\nu_2}) \\
&\quad+\delta^{\nu_1\mu_2}(\delta^{\alpha_1\nu_2}\delta^{\beta_1\alpha_2}+\delta^{\alpha_1\alpha_2}\delta^{\beta_1\nu_2})\Big\}\ ,
\end{aligned}
\tag{2.56}
$$

again by rescaling the τ variables as $\tau_i = Tu_i$ and using integration by parts, $\ddot{G}(\tau_1,\tau_2)$ can be eliminated as

$$\begin{aligned}\int_0^T d\tau_1 \int_0^T d\tau_2 \ddot{G}_B(\tau_1,\tau_2) G_B^3(\tau_1,\tau_2) &= T^4 \int_0^1 du \ddot{G}_B(u,0) G_B^3(u,0) \\ &= -3T^4 \int_0^1 du \dot{G}_B^2(u,0) G_B^2(u,0) \,.\end{aligned} \tag{2.57}$$

$\mathcal{M}'$ can be simplify as

$$\begin{aligned}\mathcal{M}' F_{\mu_1\nu_1,\alpha_1\beta_1} F_{\mu_2\nu_2,\alpha_2\beta_2} &= +3\dot{G}_B^2(u,0)G_B^2(u,0)\Big\{2F_{\mu_2\nu_2,\alpha_2\beta_2}(F_{\mu_2\nu_2,\alpha_2\beta_2} + F_{\mu_2\alpha_2,\nu_2\beta_2} + F_{\mu_2\beta_2,\nu_2\alpha_2})\Big\} \\ &\quad -\dot{G}_B^2(u,0)G_B^2(u,0)\Big\{2F_{\mu_2\nu_2,\alpha_2\beta_2}(-F_{\mu_2\nu_2,\alpha_2\beta_2} + F_{\alpha_2\mu_2,\nu_2\beta_2} + F_{\beta_2\mu_2,\nu_2\alpha_2})\Big\} \\ &= 8\dot{G}_B^2(u,0)G_B^2(u,0)\Big\{F_{\mu_2\nu_2,\alpha_2\beta_2}(F_{\mu_2\nu_2,\alpha_2\beta_2} + F_{\mu_2\alpha_2,\nu_2\beta_2} + F_{\mu_2\beta_2,\nu_2\alpha_2})\Big\} \,,\end{aligned} \tag{2.58}$$

where we used this property of $F_{\mu\nu,\alpha\beta}$ which is antisymmetric under $\mu \leftrightarrow \nu$ and symmetric under $\alpha \leftrightarrow \beta$.

By performing the u integral we get

$$T^2 \int_0^1 du \dot{G}_B^2(u,0) G_B^2(u,0) = T^4 \int_0^1 du\, (1-2u)^2\, u^2 (1-u)^2 = \frac{T^4}{210} \,. \tag{2.59}$$

Going back to the (2.53), it can be written as

$$\begin{aligned}\Gamma_{\rm scal}^{\partial\partial F \partial\partial F}[A] &= \frac{8}{128 \times 16\pi^2 \times 210} \int_0^\infty dT T e^{-m^2 T} \\ &\quad \times \int d^4x_0 \Big\{F_{\mu_2\nu_2,\alpha_2\beta_2}(F_{\mu_2\nu_2,\alpha_2\beta_2} + F_{\mu_2\alpha_2,\nu_2\beta_2} + F_{\mu_2\beta_2,\nu_2\alpha_2})\Big\} \,.\end{aligned} \tag{2.60}$$

T-integral leads to

$$\Gamma_{\rm scal}^{\partial\partial F \partial\partial F}[A] = \frac{1}{128 \times 420\pi^2 m^4} \int d^4x_0 \Big(T_1 + T_2 + T_3\Big) \,, \tag{2.61}$$

where

$$\begin{aligned}T_1 &= F_{\mu_2\nu_2,\alpha_2\beta_2} F_{\mu_2\nu_2,\alpha_2\beta_2} \,, \\ T_2 &= F_{\mu_2\nu_2,\alpha_2\beta_2} F_{\mu_2\alpha_2,\nu_2\beta_2} \,, \\ T_3 &= F_{\mu_2\nu_2,\alpha_2\beta_2} F_{\mu_2\beta_2,\nu_2\alpha_2} \,,\end{aligned} \tag{2.62}$$

but $T_3 = 0$ since the first factor is antisymmetric in $\mu \leftrightarrow \nu$ and the second factor is symmetric under this exchange. From the Bianchi identity

$$T_1 = 2T_2 \ . \tag{2.63}$$

Finally (2.53) can be written as

$$\begin{aligned}
\Gamma_{\rm scal}^{\partial\partial F\partial\partial F}[A] &= \frac{1}{128\times 420\pi^2 m^4}\int d^4x_0\Big(T_1+2T_2\Big) \\
&= \frac{2}{128\times 420\pi^2 m^4}\int d^4x_0 T_1 = \frac{1}{26880\pi^2 m^4}\int d^4x_0 F^2_{\mu_2\nu_2,\alpha_2\beta_2} \ .
\end{aligned} \tag{2.64}$$

The computation of the spinor case is analogous to the previous one, including the use of the replacement rule. Here we only quote the result:

$$\Gamma_{\rm scal}^{\partial\partial F\partial\partial F}[A] = \frac{1}{26880\pi^2 m^4}\int d^4x_0 F^2_{\mu_2\nu_2,\alpha_2\beta_2} \ , \tag{2.65}$$

$$\Gamma^{\partial\partial F\partial\partial F}[A] = \frac{1}{2240\pi^2 m^4}\int d^4x_0 F^2_{\mu_2\nu_2,\alpha_2\beta_2} \ . \tag{2.66}$$

At this same subleading mass dimension level, we also have the terms with four F's. Their contributions are contained in the effective Lagrangians for a constant field, due to Heisenberg and Euler [80] in the spinor QED case, and Weisskopf [114] for the scalar QED case.
Let us look at this part in details. By considering four times the first term of the (2.33) one gets

$$\begin{aligned}
\Gamma_{\rm scal}^{FFFF} &= -\frac{(-i)^4}{4!}\int_0^\infty \frac{dT}{T}e^{-m^2T}\int d^4x_0\int_0^T d\tau_1\int_0^T d\tau_2\int_0^T d\tau_3\int_0^T d\tau_4 \\
&\quad\times\int \mathcal{D}y\Big[\frac{1}{16}\dot y^{\mu_1}F_{\mu_1\nu_1}y^{\nu_1}\,\dot y^{\mu_2}F_{\mu_2\nu_2}y^{\nu_2}\,\dot y^{\mu_3}F_{\mu_3\nu_3}y^{\nu_3}\,\dot y^{\mu_4}F_{\mu_4\nu_4}y^{\nu_4}\Big]e^{-\int_0^T d\tau \dot y^2/4} \\
&= -\frac{1}{4!\times 16^2\pi^2}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\int d^4x_0\int_0^T d\tau_1\int_0^T d\tau_2\int_0^T d\tau_3\int_0^T d\tau_4 \\
&\quad\times\int \mathcal{D}y\, F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}\,\mathcal{M}'' \ ,
\end{aligned} \tag{2.67}$$

where

$$\mathcal{M}'' = \Big\langle \dot y^{\mu_1}(\tau_1)y^{\nu_1}(\tau_1)\,\dot y^{\mu_2}(\tau_2)y^{\nu_2}(\tau_2)\,\dot y^{\mu_3}(\tau_3)y^{\nu_3}(\tau_3)\,\dot y^{\mu_4}(\tau_4)y^{\nu_4}(\tau_4)\Big\rangle \ . \tag{2.68}$$

We have programed this contraction using Mathematica [115] which presents in Appendix C.

$$\frac{1}{4!\times 16}\int_0^T d\tau_1 d\tau_2 d\tau_3 \mathcal{M}'' F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4} = \frac{({\rm tr}F^2)^2}{288}+\frac{{\rm tr}(F^4)}{360} \ . \tag{2.69}$$

From those Lagrangians, one easily finds

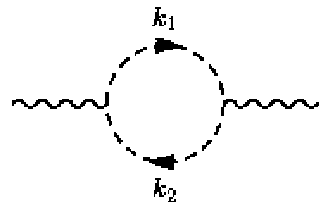


Figure 2.1: Vacuum polarization diagram.

$$\Gamma^{FFFF}_{\text{scal}}[A] = -\frac{1}{16\pi^2 m^4}\int d^4x_0\left[\frac{1}{360}\text{tr}(F^4)+\frac{1}{288}(\text{tr}F^2)^2\right], \tag{2.70}$$
$$\Gamma^{FFFF}[A] = -\frac{1}{16\pi^2 m^4}\int d^4x_0\left[\frac{7}{90}\text{tr}(F^4)-\frac{1}{36}(\text{tr}F^2)^2\right]. \tag{2.71}$$

Finally, we insert our background field, defined by (2.1) and (2.24) (with $\rho=\nu=1$) and expand in inverse powers of mass. In the limit when $\alpha\to 0$, the coefficients of the inverse squared and inverse quartic terms, up to cubic order in α, are respectively given by

$$\begin{aligned} c_{\text{scal},2} &= -\frac{2}{15}\alpha^3\left(\ln(2\alpha)+\gamma_E+\frac{19}{80}\right)-\frac{31\alpha^2}{600}+\frac{23\alpha}{1200}-\frac{1}{75}\,, \\ c_{\text{scal},4} &= \frac{203}{270}\alpha^3\left(\ln(4\alpha)+\gamma_E-\frac{60601}{59682}\right)+\frac{11}{60}\alpha^2\left(\ln(4\alpha)+\gamma_E+\frac{32663}{776160}\right)+\frac{2941\alpha}{66150}-\frac{107}{105840}\,, \end{aligned} \tag{2.72}$$

in the scalar case, whereas for the spinor case,

$$\begin{aligned} c_{\text{spin},2} &= 8\,c_{\text{scal},2}, \\ c_{\text{spin},4} &= \frac{232}{135}\alpha^3\left(\ln(4\alpha)+\gamma_E+\frac{137}{588}\right)+\frac{2}{15}\alpha^2\left(\ln(4\alpha)+\gamma_E+\frac{8819}{35280}\right)+\frac{3149\alpha}{33075}+\frac{683}{13230}\,, \end{aligned} \tag{2.73}$$

where $\gamma_E\simeq 0.57721$ is the Euler-Mascheroni constant. These expressions were obtained with MATHEMATICA [115], the detail of this program appears in Appendix D.

2.4.2 Two-point functions

We will now compute the two-point contributions to the scalar Fig. 2.1 and spinor effective actions in the background defined by (2.1) and (2.24). We will set $\nu=\rho=1$, but keep α, m and μ general, at least at first.
For the scalar case, to get the two-point contribution we start again from the worldline representation of the effective action (2.31), and expand out the interaction exponential to second order.

This yields

$$\Gamma^{(2)}_{\rm scal} = \frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T}\int_0^T d\tau_1 d\tau_2 \int \mathcal{D}x \prod_{i=1}^2 \dot{x}_i \cdot A(x_i) \exp\Bigl[-\int_0^T d\tau \frac{\dot{x}^2}{4}\Bigr] . \tag{2.74}$$

where we have put $x_i \equiv x(\tau_i)$. Fourier transforming $A_\mu(x)$,

$$A_\mu(x) = \int \frac{d^Dk}{(2\pi)^D} e^{ik\cdot x}\bar{A}_\mu(k) \; , \tag{2.75}$$

we find that

$$\bar{A}_\mu(k) = -i\eta^3_{\mu\nu}k^\nu \bar{b}(k^2,\alpha) \; , \tag{2.76}$$

where $\bar{b}(k^2,\alpha)$ can be written as

$$\bar{b}(k^2,\alpha) = \frac{\pi^2}{2}\, e^\alpha\, \Gamma(-2,\alpha; k^2/4) \; , \tag{2.77}$$

with Γ the *generalized incomplete gamma function*

$$\Gamma(a,x;b) = \int_x^\infty dz\, z^{a-1} e^{-z-bz^{-1}} \; . \tag{2.78}$$

Introducing (fictitious) polarization vectors by

$$\varepsilon_{i\mu} \quad := \quad \eta^3_{\mu\nu}k^\nu_i \quad , \quad i=1,2 \tag{2.79}$$

and the two-point function in momentum space,

$$\Gamma^{(2)}_{\rm scal}[k_1,\varepsilon_1;k_2,\varepsilon_2] \quad = \quad -\int_0^\infty \frac{dT}{T} e^{-m^2T}\int_0^T d\tau_1 d\tau_2 \int Dx \prod_{i=1}^2 \varepsilon_i \cdot \dot{x}_i\, e^{ik\cdot x} e^{-\int_0^T d\tau \frac{\dot{x}^2}{4}} \; , \tag{2.80}$$

we can then rewrite $\Gamma^{(2)}_{\rm scal}$ as

$$\Gamma^{(2)}_{\rm scal} = \frac{1}{2}\prod_{i=1}^2 \int \frac{d^Dk_i}{(2\pi)^D}\bar{b}(k_i^2,\alpha)\Gamma^{(2)}_{\rm scal}[k_1,\varepsilon_1;k_2,\varepsilon_2] \, . \tag{2.81}$$

The calculation of $\Gamma^{(2)}_{\rm scal}[k_1,\varepsilon_1;k_2,\varepsilon_2]$ is a standard textbook calculation, and we give the result here only. In the $\overline{\rm MS}$ scheme with mass scale μ, one finds

$$
\begin{aligned}
\Gamma^{(2)}_{\rm scal,ren}[k_1,\varepsilon_1;k_2,\varepsilon_2] &= (2\pi)^D\delta(k_1+k_2)(\varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2-\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1)\Pi_{\rm scal,ren}(k_1^2,m,\mu),\\
\Pi_{\rm scal,ren}(k^2,m,\mu) &= -\frac{1}{(4\pi)^2}\biggl\{\frac{1}{3}\ln\frac{m^2}{\mu^2}-\frac{8}{9}\Bigl(1+3\frac{m^2}{k^2}\Bigr)+\frac{1}{3}\Bigl(1+4\frac{m^2}{k^2}\Bigr)^{\frac{3}{2}}\\
&\qquad\times{\rm ArcTanh}\Biggl[\frac{\sqrt{1+4\frac{m^2}{k^2}}}{1+2\frac{m^2}{k^2}}\Biggr]\biggr\}.
\end{aligned}
\tag{2.82}
$$

Setting now $k=k_1=-k_2$ and using (2.79) together with $\eta^{3T}=-\eta^3$ and $(\eta^3)^2=-\mathbb{1}$, we compute

$$
(\varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2-\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1)=k^4\,. \tag{2.83}
$$

After using the δ - function in (2.82) to remove the k_2 - integral, our final renormalized result for the two-point function becomes

$$
\Gamma^{(2)}_{\rm scal,ren}[m,\mu]=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}k^4\bar{b}^2(k^2,\alpha)\Pi_{\rm scal}(k^2,m,\mu)\,. \tag{2.84}
$$

We will also need its massless limit. Setting now $\mu=1$ and taking the limit $m\to 0$, we find

$$
\Pi_{\rm scal,ren}(k^2,0,1)=\frac{1}{(4\pi)^2}\Bigl(\frac{8}{9}-\frac{1}{3}\ln k^2\Bigr)\,. \tag{2.85}
$$

Even in the massless case, it seems not to be possible to evaluate the triple integral in (2.84) in closed form. However, it is not difficult to determine its asymptotic behavior for $\alpha\to 0$. We find

$$
\Gamma^{(2)}_{\rm scal,ren}[0,1]=\frac{1}{48}(\ln\alpha)^2+\Bigl(-\frac{23}{288}+\frac{1}{8}\ln 2-\frac{1}{24}\gamma_E\Bigr)\ln\alpha+\text{finite}\,. \tag{2.86}
$$

Moving on to the spinor case, for $\Gamma^{(2)}_{\rm spin,ren}[m,\mu]$ we get the same formula (2.84) with $\Pi_{\rm scal,ren}$ replaced by the spinor QED vacuum polarization $\Pi_{\rm ren}$,

$$
\Pi_{\rm ren}(k^2,m,\mu) = -\frac{1}{(4\pi)^2}\biggl\{\frac{4}{3}\ln\frac{m^2}{\mu^2}-\frac{20}{9}\Bigl(1-\frac{12}{5}\frac{m^2}{k^2}\Bigr)+\frac{8}{3}\Bigl(1-2\frac{m^2}{k^2}\Bigr)\sqrt{1+4\frac{m^2}{k^2}}{\rm ArcCoth}\sqrt{1+4\frac{m^2}{k^2}}\biggr\}\,. \tag{2.87}
$$

In the massless limit, this yields

$$
\Pi_{\rm ren}(k^2,0,1)=\frac{1}{(4\pi)^2}\Bigl(\frac{20}{9}-\frac{4}{3}\ln k^2\Bigr)\,, \tag{2.88}
$$

and for the small α limit, we find

$$\Gamma^{(2)}_{\rm ren}[0,1] = \frac{1}{12}(\ln\alpha)^2 + \Big(-\frac{11}{72} + \frac{1}{2}\ln 2 - \frac{1}{6}\gamma_E\Big)\ln\alpha + \text{finite}\,. \tag{2.89}$$

We remark that the leading $(\ln\alpha)^2$ terms in (2.86), (2.89) come from the $\ln k^2$ terms in (2.85), (2.88), so that their coefficients are related to the $\frac{1}{\epsilon}$ poles of the two-point functions, and thus ultimately to the QED β - functions. The details of this calculation is presented in the Appendix E and also see [110].

2.4.3 Finiteness of the quartic and higher contributions for $m = \alpha = 0$

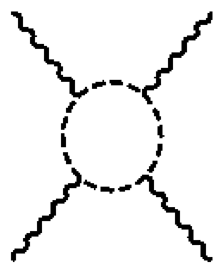


Figure 2.2: Four point contribution.

Next, we consider the quartic contribution of the one-loop effective action, Fig. 2.2. Contrary to the case of the two-point function treated in the previous part, at the four-point level a detailed calculation is out of the question, and our only goal is to demonstrate that the four-point function is finite in the double limit $m, \alpha \to 0$. Thus here we consider only the $\alpha = 0$ case, and wish to show that there are no divergences in the zero mass limit. Since the four-point contribution is already UV finite, contrary to the case of the two-point function here we can set $D = 4$ from the beginning.

We start with the scalar QED case. Expanding the worldline path integral (2.31) to quartic order, we can write this quartic contribution to the effective action as

$$\Gamma^{(4)}_{\rm scal}[A] \quad = \quad -\frac{1}{4!}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_0^T d\tau_1 d\tau_2 d\tau_3 d\tau_4 \int \mathcal{D}x \prod_{i=1}^{4} \dot{x}_i\cdot A(x_i)\exp\left[-\int_0^T d\tau \frac{\dot{x}^2}{4}\right]\,. \tag{2.90}$$

As in the two-point case above, we next Fourier transform $A_\mu(x)$, where due to our setting $\alpha = 0$ the Fourier transform can now be given more explicitly in terms of the modified Bessel function of the second kind $K_2(x)$:

$$\bar{A}_\mu(k) \quad = \quad -i\eta^3_{\mu\nu}k^\nu \bar{a}(k^2), \tag{2.91}$$

$$\bar{a}(k^2) \quad = \quad \bar{b}(k^2,0) = 4\pi^2 \frac{K_2(\sqrt{k^2})}{k^2}\,. \tag{2.92}$$

Later on, we will need the small and large k behavior of $\bar{a}(k^2)$,

$$\bar{a}(k^2) \quad = \quad \frac{8\pi^2}{k^4} - \frac{2\pi^2}{k^2} + \cdots \tag{2.93}$$

$$\bar{a}(k^2) \quad = \quad 2\sqrt{2}\pi^{5/2}\frac{1}{k^{5/2}}e^{-\sqrt{k^2}} + \cdots \tag{2.94}$$

Introducing polarization vectors as in (2.79), we can rewrite (2.90) as

$$\begin{aligned}\Gamma^{(4)}[A] \quad &= \quad -\frac{1}{4!}\prod_{i=1}^4 \int \frac{d^4k_i}{(2\pi)^4}\bar{a}(k_i^2)\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\int_0^T d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ &\times \int \mathcal{D}x \prod_{i=1}^4 \varepsilon_i \cdot \dot{x}_i\, e^{ik_i\cdot x_i} \exp\left[-\int_0^T d\tau \frac{\dot{x}^2}{4}\right]\,.\end{aligned} \tag{2.95}$$

We separate off the zero mode x_0 contained in the path integral,

$$\begin{aligned}\int \mathcal{D}x \quad &= \quad \int dx \int \mathcal{D}y\,, \\ x^\mu(\tau) \quad &= \quad x_0^\mu + y^\mu(\tau)\,, \\ \int_0^T d\tau y^\mu(\tau) \quad &= \quad 0\,,\end{aligned} \tag{2.96}$$

Its integral gives the usual δ - function for energy-momentum conservation. Thus we have

$$\Gamma^{(4)}[A] = -\frac{1}{4!}\prod_{i=1}^4 \int \frac{d^4k_i}{(2\pi)^4}\bar{a}(k_i^2)(2\pi)^4\delta^4(\sum k_i)\Gamma[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4]\,, \tag{2.97}$$

where Γ is the worldline representation of the off-shell Euclidean four-photon amplitude in momentum space:

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] \quad &= \quad -\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\int_0^T d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ &\times \int \mathcal{D}y \prod_{i=1}^4 \varepsilon_i \cdot \dot{y}_i\, e^{ik_i\cdot y_i} \exp\left[-\int_0^T d\tau \frac{\dot{y}^2}{4}\right]\,.\end{aligned} \tag{2.98}$$

Here after performing all Wick contractions of $\prod_{i=1}^{4}\varepsilon_i\cdot\dot{y}_i e^{ik_i\cdot y_i}$, one gets a polynomial function of $\ddot{G}_{Bij}$ and $\dot{G}_{Bij}$ which we call it $P_4(\ddot{G}_{Bij},\dot{G}_{Bij})$, then by suitable integrations by parts P_4 get replaced by another polynomial which is a function of only $\dot{G}_{Bij}$, $Q_4(\dot{G}_{Bij})$ (see [50] and Chapter 3 and 5 for the details).
After rescaling $\tau_i = Tu_i, i = 1,\dots,4$

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] &= -\frac{1}{(4\pi)^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}T^4\int_0^1 du_1du_2du_3du_4 \\ &\quad\times Q_4(\dot{G}_{B12},\cdots,\dot{G}_{B34})e^{\frac{T}{2}\sum_{i,j=1}^4 G_{Bij}k_i\cdot k_j}\ ,\end{aligned} \tag{2.99}$$

and after performing the T-integral

$$\Gamma[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] = -\frac{1}{(4\pi)^2}\int_0^1 du_1du_2du_3du_4\frac{Q_4(\dot{G}_{B12},\dots,\dot{G}_{B34})}{\left(m^2-\frac{1}{2}\sum_{i,j=1}^4 G_{Bij}k_i\cdot k_j\right)^2}\ . \tag{2.100}$$

Here, $G_{Bij}\equiv G_B(u_i,u_j) = |u_i-u_j|-(u_i-u_j)^2$ is the worldline Green's function and $\dot{G}_{Bij} = \mathrm{sign}(u_i-u_j)-2(u_i-u_j)$ its derivative. Q_4 is a polynomial in the various $\dot{G}_{Bij}$'s, as well as in the momenta and polarizations.
Now, the QED Ward identity implies that (2.100) is $O(k_i)$ in each of the four momenta, which can also be easily verified using properties of the numerator polynomial Q_4 given in [50]. Using this fact and (2.93) in (2.95), we see that there is no singularity at $k_i = 0$, and convergence at large k_i is assured by (2.94). Further, after specializing the u_i integrals to the standard ordering $u_1\geq u_2\geq u_3\geq u_4 = 0$ (all ordered sectors give the same here by permutation symmetry) and changing from the u_i variables to standard Schwinger parameters a_i, in the denominator, we find the standard off-shell four point expression

$$-\frac{1}{2}\sum_{i,j=1}^4 G_{Bij}k_i\cdot k_j = a_1a_3(k_1+k_2)^2+a_2a_4(k_2+k_3)^2+a_1a_2k_1^2+a_2a_3k_2^2+a_3a_4k_3^2+a_4a_1k_4^2\ . \tag{2.101}$$

Thus, for nonzero mass the denominator in the rhs of (2.100) is alway positive for our Euclidean momenta. Taking now the massless limit, any divergence of the effective action in this limit would have to come from a non-integrable singularity due to a zero of (2.101). However, it is easily seen that all such zeros require multiple pinches in the total $k_1,\dots,a_4$ space, whose measure factors render integrable the corresponding second-order pole.
This method can be easily extended to show that also all higher N - point functions are finite in the double limit $m,\alpha\to 0$.

2.5 Nonperturbative results

We now show our numerical results for the (Spinor) QED effective action based on [110] in the family of backgrounds (2.24). In our calculations we fixed $\rho=\nu=1$ throughout.
We calculated the physically renormalized effective action for the full mass range [110]. Fig. 2.3 shows the behavior of $\Gamma_{\mathrm{ren}}^{\mathrm{OS}}(m)$ for $\alpha = 1/100$. What we expect from effective action behavior is

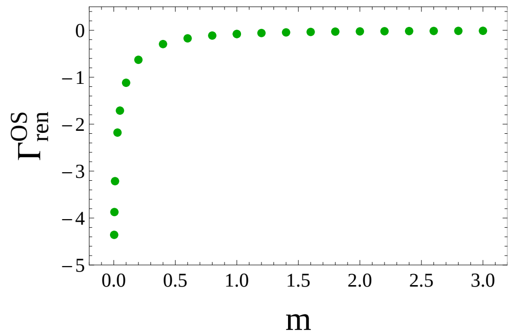


Figure 2.3: The effective action $\Gamma^{\mathrm{OS}}_{\mathrm{ren}}(m)$ for $\alpha = 1/100$.

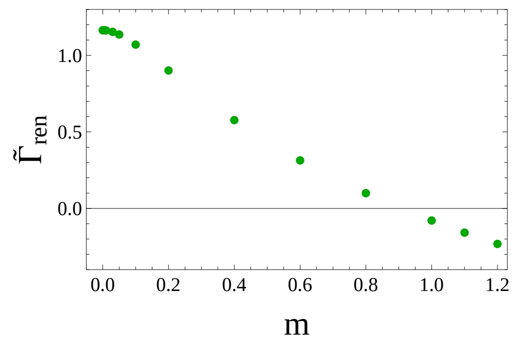


Figure 2.4: The effective action $\tilde{\Gamma}_{\mathrm{ren}}(m)$ for $\alpha = 1/100$.

the following:
The leading order should be proportional to $(\int F^2 dx)\ln(m)$. We define a new effective action $\tilde{\Gamma}_{\mathrm{ren}}(m)$ which is the previous one after removing the $(\int F^2 dx)\ln(m)$ term, in other words: $\tilde{\Gamma}_{\mathrm{ren}}(m) = \Gamma^{\mathrm{OS}}_{\mathrm{ren}} - \int F^2 dx \ln(m)$, which is finite as $m \to 0$. A plot is shown in Fig. 2.4.

The finiteness of $\tilde{\Gamma}_{\mathrm{ren}}(m)$ for $m \to 0$ was already shown in [98], and for the scalar QED case, a comparison was made with the leading term of the derivative expansion, finding good agreement. However, we go beyond the results of [98] here and we have also obtained extensive results for small masses and, in addition, we have calculated the effective action taking the mass to be exactly zero which was not known before. Results are shown if Fig. 2.5 for different vales of α [110].

Here we add another Fig. 2.5 which suggests that $\tilde{\Gamma}_{\mathrm{ren}}(m)$ at $m = 0$ diverges in the limit $\alpha \to 0$. From our perturbative results of the previous Section (2.4.2) we can confirm and interpret this fact that perturbatively all the N-point contributions to the effective action are finite in the double limit $m, \alpha \to 0$ except for the two-point function.
For the two-point function we have given the asymptotic small α behavior in (2.89), so we expect the full $\tilde{\Gamma}_{\mathrm{ren}}(m = 0)$ to have the same behavior as the two-point function for small α, means

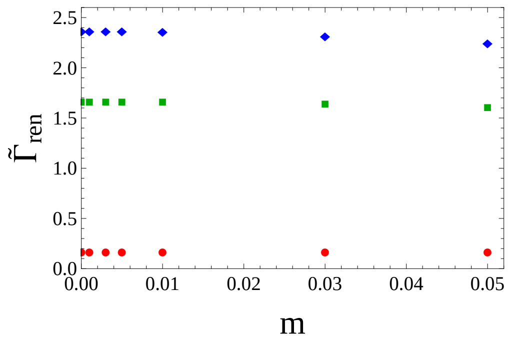


Figure 2.5: The effective action $\tilde{\Gamma}_{\text{ren}}(m)$ in the small-mass regime for different values of α. Dots correspond to $\alpha = 1/10$, squares to $\alpha = 1/200$ and diamonds to $\alpha = 1/450$.

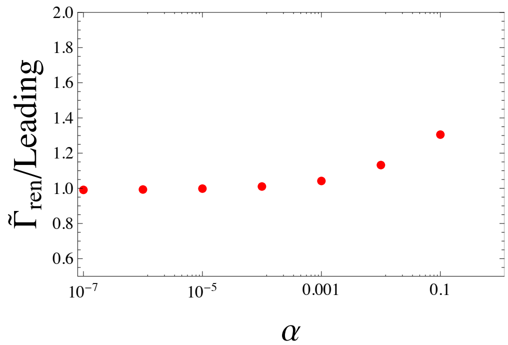


Figure 2.6: The ratio of $\tilde{\Gamma}_{\text{ren}}(m = 0)$ vs. the leading asymptotic behavior of its two-point contribution as a function of α.

$$\tilde{\Gamma}_{\text{ren}}(m = 0) \quad \overset{\alpha \to 0}{\sim} \quad \frac{1}{12}(\ln \alpha)^2 + \Big(-\frac{11}{72} + \frac{1}{2}\ln 2 - \frac{1}{6}\gamma_E \Big) \ln \alpha \,. \tag{2.102}$$

Fig. 2.6 shows a numerical plot of the ratio of the left hand and right hand side of (2.102) as a function of α which confirms our expectation for small α behavior of the $\tilde{\Gamma}_{\text{ren}}(m = 0)$.
We can also trace the origin of the divergence of $\tilde{\Gamma}_{\text{ren}}(0)$ for $\alpha \to 0$: it is easy to see that the k - integral in our final result for the two-point function (2.84) is, for $\alpha \to 0$, dominated by the region close to 0, which implies that this divergence is related to the divergence of the integral of the induced Maxwell term for $\alpha = 0$ (see (2.26)), of which a finite part is still contained in the two-point function (for our choice of the unphysical renormalization condition $\mu = 1$). This fact that the small α divergence comes purely from the perturbative two-point contribution can be checked also in a very different way: if the two-point contribution to the effective action becomes dominant over the higher-point ones for sufficiently small α, then also the dependence of the whole effective action on the normalization constant ν should become close to quadratic. In Fig.

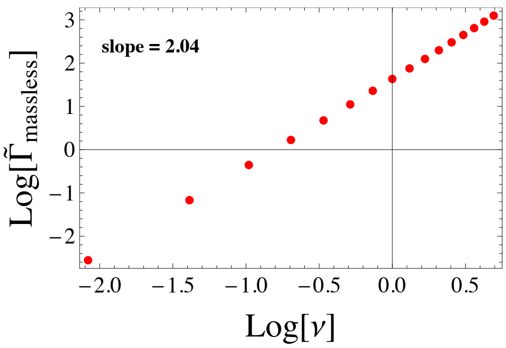


Figure 2.7: The log of $\tilde{\Gamma}_{\mathrm{ren}}(m = 0)$ as a function of $\log \nu$ for $\alpha = 1/5000$.

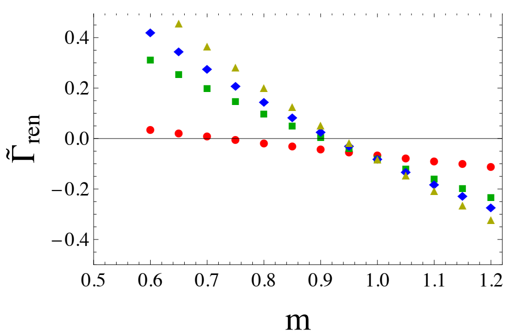


Figure 2.8: The effective action $\tilde{\Gamma}_{\mathrm{ren}}$ for different values of α. Dot correspond to $\alpha = 1/10$, squares to $\alpha = 1/100$ and diamonds to $\alpha = 1/200$ and triangles to $\alpha = 1/450$.

2.7 we show that indeed numerically the exponent of the ν - dependence of $\tilde{\Gamma}_{\mathrm{ren}}$ becomes 2.04 for zero mass and $\alpha = 1/5000$.
One advantage of being able to calculate the effective action for the full range of masses is that one can look for zeros. As seen from Fig. 5.2 for $\alpha = 1/100$ $\tilde{\Gamma}_{\mathrm{ren}}(m)$ vanishes close to $m = 1$. A more detailed study reveals that, remarkably, not only the existence but also the location of this zero seems to be rather stable under variation of α, as shown in Fig. 2.8. These zeros of mass of $\tilde{\Gamma}_{\mathrm{ren}}(m)$ are shown in Table 2.1 for different values of α.
In the large-mass regime, we compare our numerical calculation of the physically renormalized effective action with its inverse mass (= heat kernel) expansion, using the leading and subleading terms in this expansion:

$$\Gamma^{\mathrm{OS}}_{\mathrm{ren}}(m) = \frac{c_{\mathrm{spin},2}}{m^2} + \frac{c_{\mathrm{spin},4}}{m^4} + O\left(\frac{1}{m^6}\right) . \tag{2.103}$$

The coefficients $c_{\mathrm{spin},2}$ and $c_{\mathrm{spin},4}$ (which are still functions of α) were given in (2.73). As can be seen from Fig. 2.9, the leading order approximation fits the numerical results very well in the large mass region, and in an intermediate range of masses (between about $m = 1.5$ and $m = 2.0$)

α	Crossing
1/10	0.735540
1/100	0.907293
1/200	0.925169
1/450	0.939393

Table 2.1: Mass zero of the effective action as a function of α.

adding the subleading term leads to a better agreement with the numerical data (in interpreting these results it should be kept in mind that, in applications of the inverse mass expansion, typically any truncation to finite order breaks down completely at small enough masses, and adding a few terms more will lower this point of breakdown only slightly; see, e.g., [116]).

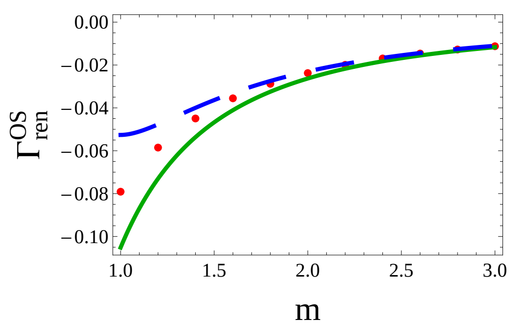


Figure 2.9: The effective action $\Gamma^{\mathrm{OS}}_{\mathrm{ren}}$ as a function of m in the large-mass limit for $\alpha = 1/100$. Dots represent the exact effective action, the solid curve is produced by fitting terms only up to $1/m^2$ whereas the dashed curve corresponds to the behavior (2.103).

2.6 Conclusions

The calculation of effective actions is a very important matter simply because the fermion determinant appears everywhere in the standard model (see, e.g., [117, 118] and refs. therein). It is an exact contribution to the gauge field measure in the functional integral. In this work, we have continued and extended the full mass range analysis of the spinor QED effective action for the $O(2) \times O(3)$ symmetric backgrounds (2.24), started in [98], by a more detailed numerical study of both its small and large mass behavior.

In [98], only the unphysically renormalized version $\tilde{\Gamma}_{\rm ren}(m)$ of this effective action was considered (corresponding to $\mu = 1$), which is appropriate for the small mass limit, but has a logarithmic divergence in m in the large m limit. This asymptotic logarithmic behavior was numerically well-reproduced in [98], but its presence prevented one from probing into the physical part of the large mass expansion, whose leading term is already $1/m^2$ - suppressed. Here, we have used the physically renormalized effective action $\Gamma^{\rm OS}_{\rm ren}(m)$ instead of $\tilde{\Gamma}_{\rm ren}(m)$ for the study of the large mass expansion. Going to large masses demands higher values of the cutoff L in the numerical calculations, and is computationally more challenging. However, we have matched our numerical results not only against the leading term, but also against the subleading $O(1/m^4)$ term in the expansion. We have also calculated the expansion coefficients analytically for these backgrounds. At the intermediate mass range, we have demonstrated the ability of the method to compute zeroes of the effective action.

Most of our effort here has, however, gone into the study of the small-mass limit of the effective action. Our study of the perturbative N - point functions in this background has shown that, with the exception of the two-point function, all of them are finite in the double limit $m, \alpha \to 0$ (the latter meaning the removal of the exponential IR suppression factor). The two-point function is, for $\alpha > 0$, made finite in the massless limit using the renormalization condition $\mu = 1$. Letting also $\alpha \to 0$ in it however produces an IR divergence whose α - dependence we have been able to calculate. In our numerical study of the small mass limit of the effective action, we have improved on [98] by obtaining good numerical results for $\tilde{\Gamma}_{\rm ren}(m)$ even at $m = 0$, showing continuity for $m \to 0$ for various values of α, and moreover verifying that the full effective action at zero mass has the same diverging asymptotic behavior for $\alpha \to 0$ as its two-point contribution.

Our results further provide strong support for M. Fry's conjecture [106, 107] according to which the effective action for this type of background should, after the subtraction of its two-point and four-point contributions, in the small-mass limit be dominated by a logarithmic divergence in the mass entirely due to the chiral anomaly term. This term exists for the backgrounds (2.24) only at $\alpha = 0$, which case is difficult to access with our method since, even after the subtraction of the true IR divergence contained in the two-point function, one would still have spurious IR divergences in $\Gamma^{(\pm)}_{\rm ren}$ which will cancel only in the sum of the low and high angular momentum contributions. This poses a formidable challenge for a numerical treatment. Nevertheless, our results show that, as long as $\alpha > 0$ and after the subtraction of the two-point function only, the effective action is already finite in the zero mass limit, both perturbatively and non-perturbatively. Given the finiteness of the double limit $m, \alpha \to 0$ for all the N - point functions but for the discarded two-point one, it is clear that the appearance at $\alpha = 0$ of some term singular in the massless limit other than the known chiral anomaly one would signal some new nonperturbative effect different from, but similar to the chiral anomaly, which is hardly to be expected in QED at the one-loop level.

We believe that the work presented here not only provides an impressive demonstration of the power of the "partial-wave-cutoff method", but also constitutes the most complete study performed so far of a one-loop QED effective action in a nontrivial background field [110].

Chapter 3

String-inspired representations of photon/gluon amplitudes

3.1 Introduction

[1] As we already mentioned in the introduction, Strassler has derived "Bern-Kosower rules" in a simpler way based on the representation of one-loop amplitudes in term of first-quantized path integrals. The central formula in the Bern-Kosower formalism is the following 'master formula':

$$\begin{aligned}\Gamma_{\rm scal}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i \\ &\times \exp\biggl\{\sum_{i,j=1}^N\Bigl[\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot G_{Bij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\Bigr]\biggr\}\,|_{\rm lin(\varepsilon_1,\dots,\varepsilon_N)} \quad .\end{aligned} \tag{3.1}$$

As it stands, this formula represents the one-loop N – photon amplitude in scalar QED, with photon momenta k_i and polarisation vectors ε_i. m denotes the mass, e the charge and T the total proper time of the scalar loop particle [2]. Each of the integrals $\int d\tau_i$ represents one photon leg moving around the loop. The integrand is written in terms of the bosonic worldline Green's function G_B and its derivatives.
As was mentioned before, dots generally denote a derivative acting on the first variable, $\dot G_B(\tau_1,\tau_2) \equiv \frac{\partial}{\partial\tau_1}G_B(\tau_1,\tau_2)$, and we abbreviate $G_{Bij} = G_B(\tau_i-\tau_j)$ etc.

From the Bern-Kosower rules one can obtain from the scalar QED integrand, the corresponding integrand for the photon amplitudes in fermion QED, as well as for the (on-shell) N-gluon amplitudes in QCD. However those rules do not apply to the master formula as it stands, for this purpose one needs to write the master formula as a polynomial function of G_{Bij} and $\dot G_{Bij}$, so it is clear that one has to remove all the $\ddot G_{Bij}$ to get an appropriate formula to apply the rules. By expanding the exponential in (3.1) one obtains an integrand

[1] This Chapter is based on [55].
[2] We work in the Euclidean throughout. With our conventions a Wick rotation $k_i^4 \to -ik_i^0, T \to is$ yields the N - photon amplitude in the conventions of [119].

with a certain polynomial P_N depending on the various $\dot{G}_{Bij}$, $\ddot{G}_{Bij}$ and the kinematic invariants. The resulting parameter integrals are directly related to the ones arising in a standard Feynman parameter calculation of this amplitude [28, 37, 120].The exponential factor in particular will, after performance of the global T-integration, turn into the standard one-loop N - point Feynman denominator polynomial. To arrive at the Bern-Kosower rules, one now has to remove all the $\ddot{G}_{Bij}$ appearing in P_N by suitable integration by part (IBP) in the variables τ_i. That this removal of all $\ddot{G}_B$'s is possible for any N was shown in Appendix B of [121]. The new integrand is written in terms of the G_{Bij}'s and $\dot{G}_{Bij}$'s alone, and serves as the input for the Bern - Kosower rules. Those allow one to classify the various contributions to the N – photon/gluon amplitude in terms of ϕ^3 – diagrams, and moreover lead to simple relations between the integrands for the scalar, spinor and gluon loop cases. For our present purposes, the most relevant part of the rules is that, up to a global factor of -2 correcting for the differences in degrees of freedom and statistics, the integrand for the spinor loop case can be obtained from the one for the scalar loop simply by replacing every closed cycle of $\dot{G}_B$'s appearing in Q_N according to the "replacement rule" eq. (1.118).

As was discussed already in in the Introduction 1, the IBP procedure is generally ambiguous [121], however this ambiguity does not affect the application of the Bern-Kosower rules since one only needs to write the polynomials just in term of the Green function and its first derivatives. For this reason in the calculation of gluon amplitudes in [27, 33] arbitrary way of IBP has been taken. A closer look at the IBP was taken by Strassler in [122], he noted that this procedure has an interesting relation with the gauge invariance. Each photon leg can be represented as a field strength tensor in momentum space as

$$f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu \ . \tag{3.2}$$

Remove all $\ddot{G}_{Bij}$'s and combine all terms contributing to a given 'τ - cycle' $\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}$. Then the sum of their Lorentz factors can be written as a 'Lorentz cycle' $Z_n(i_1i_2\dots i_n)$, defined by

$$Z_2(ij) \equiv \frac{1}{2}\mathrm{tr}\Big(F_iF_j\Big) = \varepsilon_i\cdot k_j\varepsilon_j\cdot k_i - \varepsilon_i\cdot\varepsilon_i k_i\cdot k_j \ , \tag{3.3}$$

$$Z_n(i_1i_2\dots i_n) \equiv \mathrm{tr}\Big(\prod_{j=1}^{n} F_{i_j}\Big) \quad (n\geq 3) \ . \tag{3.4}$$

Thus Z_n generalizes the familiar transversal projector. However, in [122] no systematic way was found to perform the partial integrations at arbitrary N, and also the absorption of polarisation vectors into field strength tensors (a process to be called "covariantization" in the following) worked only partially; after the IBP and the sorting of the resulting integrand in terms of "cycle content" some terms are just cycles or products of cycles, but, starting from the three-point case, there are also terms with left-overs, called "tails" in [122], and the polarisation vectors in them were not absorbed yet into field strength tensors. The IBP procedure was further studied by Schubert in [123], where a definite IBP has been introduced which works for any N, and preserves the full permutation symmetry even suitable for computerization. But still one interesting question remains whether some IBP algorithm exist which to an integrand where $a$ll polarization vectors would be absorbed in field strength tensors, thus making gauge invariance , i.e. transversality, manifest at the integrand level. The purpose of this Chapter is to address this question and to present an algorithm which indeed imposes the manifest transversality at the integrand level [55].

In detail, we will do the following: In Section 3.2 and 3.3 we find a surprisingly simple way of using IBPs to covariantize the Bern-Kosower master formula itself; the resulting representation will be called the R-representation. In 3.4 we summarize the "symmetric IBP procedure" of [123], leading to the Q - and Q' - representations. The next two Sections 3.5 and 3.6 define our new algorithm, which combines elements of both the R - and the Q' - representation, and results in what we will call the S-representation of the N photon/gluon amplitudes. As an aside, in Section 3.7 we present a further improvement of the "two-tail" which leads to a particularly compact integrand at the four-point level (but does not seem to generalize to the N - point case). In Section 3.8 we shortly comment on a direct treatment of the spinor QED case in the worldline super formalism. Section 3.9 is devoted to our main application, which is the calculation of the one-loop off-shell one-particle-irreducible N - gluon amplitudes (or "N-vertices"). While in the abelian case there are never any boundary terms in our IBPs, since all integrations run over the full loop and the integrand is written in terms of worldline Green's functions with the appropriate boundary conditions, in the nonabelian case the color ordering of the gluon legs leads to the restriction of the multiple parameter integrals to ordered sectors, and to the appearance of such boundary terms [37, 122]. Those generally can be combined into color commutators and, in x-space, would in principle allow one to achieve a complete nonabelian extension of the covariantization, namely to rewrite the final integrand in terms of full nonabelian field strength tensors, and to complete all derivatives to covariant ones. This is not possible in momentum space, but here instead the IBP procedure generates a natural form factor decomposition of the N-vertices, where the bulk terms are manifestly transversal and all non-transversality has been pushed into boundary terms. Finding such a decomposition by standard methods usually involves a tedious analysis of the nonabelian Ward identities, and so far has been completed only for the three-point case [124] which will be discussed in details in Chapter 4 and also four-point case in Chapter 5. In the conclusions Section 4.8,Chapter we summarize the properties of our new IBP algorithm, and discuss various applications, some of which have already been published or are actually in progress. We shortly comment on possible generalizations to gravity and string theory.

3.2 The P-representation

We will call "P-representation" the integrand obtained directly from the expansion of the Bern-Kosower master formula,

$$\begin{aligned}\Gamma_{\rm scal}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] \quad &= \quad (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\\ &\times\prod_{i=1}^N\int_0^T d\tau_i P_N(\dot G_{Bij},\ddot G_{Bij})\exp\biggl\{\sum_{i,j=1}^N\frac12 G_{Bij}k_i\cdot k_j\biggr\}\,.\end{aligned} \tag{3.5}$$

Explicitly, the polynomial P_N is given by

$$
\begin{aligned}
P_N \;=\;& \dot G_{B1i_1}\varepsilon_1\cdot k_{i_1}\dot G_{B2i_2}\varepsilon_2\cdot k_{i_2}\cdots \dot G_{BNi_N}\varepsilon_N\cdot k_{i_N}\\
&-\sum_{\substack{a,b=1\\ a<b}}^{N}\ddot G_{Bab}\varepsilon_a\cdot\varepsilon_b\dot G_{B1i_1}\varepsilon_1\cdot k_{i_1}\cdots\widehat{\dot G_{Bai_a}\varepsilon_a\cdot k_{i_a}}\cdots\widehat{\dot G_{Bbi_b}\varepsilon_b\cdot k_{i_b}}\cdots\dot G_{BNi_N}\varepsilon_N\cdot k_{i_N}\\
&+\sum_{\substack{a,b,c,d=1\\ a<b<c<d}}^{N}\big(\ddot G_{Bab}\varepsilon_a\cdot\varepsilon_b\ddot G_{Bcd}\varepsilon_c\cdot\varepsilon_d+\ddot G_{Bac}\varepsilon_a\cdot\varepsilon_c\ddot G_{Bbd}\varepsilon_b\cdot\varepsilon_d+\ddot G_{Bad}\varepsilon_a\cdot\varepsilon_d\ddot G_{Bbc}\varepsilon_b\cdot\varepsilon_c\big)\\
&\quad\times\dot G_{B1i_1}\varepsilon_1\cdot k_{i_1}\cdots\widehat{\dot G_{Bai_a}\varepsilon_a\cdot k_{i_a}}\cdots\widehat{\dot G_{Bbi_b}\varepsilon_b\cdot k_{i_b}}\cdots\widehat{\dot G_{Bci_c}\varepsilon_c\cdot k_{i_c}}\cdots\widehat{\dot G_{Bdi_d}\varepsilon_d\cdot k_{i_d}}\cdots\dot G_{BNi_N}\varepsilon_N\cdot k_{i_N}\\
&-\dots\;.
\end{aligned}
\tag{3.6}
$$

Here and in the following the dummy indices $i_1, i_2, \dots$ should be summed over from 1 to N, and a 'hat' denotes omission. Note that all terms in P_N are obtained from the first one by a simultaneous replacement of pairs of $\dot G_{Bri_r}\varepsilon_r\cdot k_{i_r}\dot G_{Bsi_s}\varepsilon_s\cdot k_{i_s}$ by $-\ddot G_{Brs}\varepsilon_r\cdot\varepsilon_s$, which has to be done in all possible ways. Note also that $\dot G_{Bii}=0$ by antisymmetry.
These P-representation integrals are still directly related to the ones arising in a standard Feynman or Schwinger parameter calculation of the N photon amplitude [28, 37]. The exponential factor will, after a multiple rescaling and performance of the global $T-$ integration, turn into the standard one-loop N - point Feynman denominator polynomial. The δ - function contained in $\ddot G_{Bij}$ will bring together the photons i and j, corresponding to a quartic vertex, and the contributions of such terms match the ones from the seagull vertex of scalar QED which was mentioned in the introduction.

3.3 The R-representation

Before coming to the "old" IBP procedure of [37, 123] and its intended improvement, it will be useful to solve a simpler problem, namely how to covariantize the Bern-Kosower master formula itself. In the following we will often abbreviate

$$e^{\frac12\sum G_{Bij}k_i\cdot k_j}\equiv e^{(\cdot)}\;. \tag{3.7}$$

Consider first the case of $N=2$, where

$$P_2=\dot G_{B12}\varepsilon_1\cdot k_2\dot G_{B21}\varepsilon_2\cdot k_1-\ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\;. \tag{3.8}$$

We choose two vectors r_1, r_2 that fulfill

$$r_i\cdot k_i\neq 0\;, \tag{3.9}$$

but are arbitrary otherwise. Adding to the integrand $P_2\,e^{(\cdot)}$ the following sum of total derivative terms,

$$
\begin{aligned}
& -\frac{r_1\cdot\varepsilon_1}{r_1\cdot k_1}\partial_1\Bigl(\dot G_{B21}\varepsilon_2\cdot k_1\,e^{(\cdot)}\Bigr)-\frac{r_2\cdot\varepsilon_2}{r_2\cdot k_2}\partial_2\Bigl(\dot G_{B12}\varepsilon_1\cdot k_2\,e^{(\cdot)}\Bigr)+\frac{r_1\cdot\varepsilon_1}{r_1\cdot k_1}\frac{r_2\cdot\varepsilon_2}{r_2\cdot k_2}\partial_1\partial_2\,e^{(\cdot)}\\
= & -\frac{r_1\cdot\varepsilon_1}{r_1\cdot k_1}\Bigl(-\ddot G_{B21}\varepsilon_2\cdot k_1+\dot G_{B21}\varepsilon_2\cdot k_1\dot G_{B12}k_1\cdot k_2\Bigr)\,e^{(\cdot)}\\
& -\frac{r_2\cdot\varepsilon_2}{r_2\cdot k_2}\Bigl(-\ddot G_{B12}\varepsilon_1\cdot k_2+\dot G_{B12}\varepsilon_1\cdot k_2\dot G_{B21}k_1\cdot k_2\Bigr)\,e^{(\cdot)}\\
& +\frac{r_1\cdot\varepsilon_1}{r_1\cdot k_1}\frac{r_2\cdot\varepsilon_2}{r_2\cdot k_2}\Bigl(-\ddot G_{B21}k_2\cdot k_1+\dot G_{B21}k_1\cdot k_2\dot G_{B12}k_1\cdot k_2\Bigr)\,e^{(\cdot)}\\
= & \ \ddot G_{B12}\Bigl(\frac{r_1\cdot\varepsilon_1\varepsilon_2\cdot k_1}{r_1\cdot k_1}+\frac{r_2\cdot\varepsilon_2\varepsilon_1\cdot k_2}{r_2\cdot k_2}-\frac{r_1\cdot\varepsilon_1 r_2\cdot\varepsilon_2 k_1\cdot k_2}{r_1\cdot k_1 r_2\cdot k_2}\Bigr)\,e^{(\cdot)}\\
& +\dot G_{B12}\dot G_{B21}\Bigl(\frac{r_1\cdot\varepsilon_1 r_2\cdot\varepsilon_2(k_1\cdot k_2)^2}{r_1\cdot k_1 r_2\cdot k_2}-\frac{r_1\cdot\varepsilon_1\varepsilon_2\cdot k_1 k_1\cdot k_2}{r_1\cdot k_1}-\frac{r_2\cdot\varepsilon_2\varepsilon_1\cdot k_2 k_1\cdot k_2}{r_2\cdot k_2}\Bigr)\,e^{(\cdot)}\ ,
\end{aligned}
\tag{3.10}
$$

($\partial_i \equiv \frac{\partial}{\partial\tau_i}$) the total result is a change of P_2 into R_2,

$$
R_2 := \dot G_{B12}\frac{r_1\cdot F_1\cdot k_2}{r_1\cdot k_1}\dot G_{B21}\frac{r_2\cdot F_2\cdot k_1}{r_2\cdot k_2}+\ddot G_{B12}\frac{r_1\cdot F_1\cdot F_2\cdot r_2}{r_1\cdot k_1 r_2\cdot k_2}\ . \tag{3.11}
$$

Thus we have managed to absorb the polarization vectors into field strength tensors. And this procedure can be immediately generalized to the N-point case: let us abbreviate

$$
\rho_i := \frac{r_i\cdot\varepsilon_i}{r_i\cdot k_i}\ , \tag{3.12}
$$

$$
T_r(i) := \sum_j \dot G_{Bij}\frac{r_i\cdot F_i\cdot k_j}{r_i\cdot k_j}\ , \tag{3.13}
$$

$$
W_r(ij) := \ddot G_{Bij}\frac{r_i\cdot F_i\cdot F_j\cdot r_j}{r_i\cdot k_i k_j\cdot r_j}\ , \tag{3.14}
$$

and choose vectors $r_1,\dots,r_N$ fulfilling (3.9). Then it is a matter of simple combinatorics to verify that

$$
P_N\,e^{(\cdot)}+\Bigl[\prod_{a=1}^{N}(1-\rho_a\partial_a\Delta_a)-1\Bigr]\Bigl[P_N\,e^{(\cdot)}\Bigr]=P_N\Bigl(\dot G_{Bai_a}\varepsilon_a\cdot k_{i_a}\to T_r(a),-\ddot G_{Bab}\varepsilon_a\cdot\varepsilon_b\to W_r(ab)\Bigr)\,e^{(\cdot)}\ , \tag{3.15}
$$

where the operator Δ_a is defined as follows: each term in $P_N\,e^{(\cdot)}$ either involves the index a in a second derivative factor $\ddot G_B$, or it carries a factor of $\dot G_{Bai_a}\varepsilon_a\cdot k_{i_a}$. In the former case the term will be annihilated by Δ_a, in the latter case the action of Δ_a is to replace the factor of $\dot G_{Bai_a}\varepsilon_a\cdot k_{i_a}$ by 1.

We can then reexponentiate the new integrand, and arrive at the following covariantized version of the Bern-Kosower master formula (3.1):

$$
\begin{aligned}
\Gamma_{\rm scal}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i \\
&\times\exp\Big\{\sum_{i,j=1}^N\Big[\frac12 G_{Bij}k_i\cdot k_j - i\dot G_{Bij}\frac{r_i\cdot F_i\cdot k_j}{r_i\cdot k_i}-\frac12\ddot G_{Bij}\frac{r_i\cdot F_i\cdot F_j\cdot r_j}{r_i\cdot k_i\, r_j\cdot k_j}\Big]\Big\}\Big|_{{\rm lin}(F_1,\dots,F_N)}\,.
\end{aligned}
\tag{3.16}
$$

Thus, we have achieved manifest gauge invariance at the integrand level, with a large freedom of choosing the vectors $r_1,\dots,r_N$. We will call this the "R-representation" of the N - photon amplitudes. Note that it reduces to the original master formula (3.1) if $r_i\cdot\varepsilon_i=0$ for all i.

3.4 The Q and Q$'$ -representations

Next, we review the IBP procedure motivated by the Bern-Kosower rules, whose primary purpose is to get rid of all second derivatives $\ddot G_B$ [121–123].
For the two-point case P_2 has been written down already in (3.8). After an IBP of the second term in either τ_1 or τ_2, and using $\dot G_{B12}=-\dot G_{B21}$, it turns into

$$
Q_2 = \dot G_{B12}\dot G_{B21}\big(\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1-\varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2\big)=\dot G_{B12}\dot G_{B21}Z_2(12)\,. \tag{3.17}
$$

Proceeding to the three-point case, here (3.6) becomes

$$
\begin{aligned}
P_3 &= \dot G_{B1i}\varepsilon_1\cdot k_i\dot G_{B2j}\varepsilon_2\cdot k_j\dot G_{B3k}\varepsilon_3\cdot k_k\\
&\quad -\Big[\ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B3i}\varepsilon_3\cdot k_i\,+\,(1\to2\to3)\,+\,(1\to3\to2)\Big]\,.
\end{aligned}
\tag{3.18}
$$

In this three-point case it is still possible to remove all $\ddot G_B$'s in a single step. To remove, e. g., the term involving $\ddot G_{B12}\dot G_{B31}$ in the second term of P_3, we can add the total derivative term

$$
-\partial_2\Big(\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B31}\varepsilon_3\cdot k_1e^{(\cdot)}\Big)=\Big(\ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B31}\varepsilon_3\cdot k_1-\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B2i}k_2\cdot k_i\dot G_{B31}\varepsilon_3\cdot k_1\Big)\,e^{(\cdot)}\,, \tag{3.19}
$$

where $i=1,2,3$, so

$$
-\ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B31}\varepsilon_3\cdot k_1\,e^{(\cdot)}-\partial_2\Big(\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B31}\varepsilon_3\cdot k_1e^{(\cdot)}\Big)=-\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B2i}k_2\cdot k_i\dot G_{B31}\varepsilon_3\cdot k_1\,e^{(\cdot)}\,. \tag{3.20}
$$

This term together with five similar ones removes all the $\ddot{G}_B$'s. Here are the other terms:

$$
\begin{aligned}
-\ddot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B32}\varepsilon_3\cdot k_2\,e^{(\cdot)}+\partial_1\Big(\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B32}\varepsilon_3\cdot k_2e^{(\cdot)}\Big) &= \dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B1i}k_1\cdot k_i\dot{G}_{B32}\varepsilon_3\cdot k_2\,e^{(\cdot)}\,,\\
-\ddot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B21}\varepsilon_2\cdot k_1\,e^{(\cdot)}-\partial_3\Big(\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B21}\varepsilon_2\cdot k_1e^{(\cdot)}\Big) &= -\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B3i}k_3\cdot k_i\dot{G}_{B21}\varepsilon_2\cdot k_1\,e^{(\cdot)}\,,\\
-\ddot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B23}\varepsilon_2\cdot k_3\,e^{(\cdot)}+\partial_1\Big(\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B23}\varepsilon_2\cdot k_3e^{(\cdot)}\Big) &= \dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B1i}k_1\cdot k_i\dot{G}_{B23}\varepsilon_2\cdot k_3\,e^{(\cdot)}\,,\\
-\ddot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\dot{G}_{B12}\varepsilon_1\cdot k_2\,e^{(\cdot)}-\partial_3\Big(\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\dot{G}_{B12}\varepsilon_1\cdot k_2e^{(\cdot)}\Big) &= -\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\dot{G}_{B3i}k_3\cdot k_i\dot{G}_{B12}\varepsilon_1\cdot k_2\,e^{(\cdot)}\,,\\
-\ddot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\dot{G}_{B13}\varepsilon_1\cdot k_3\,e^{(\cdot)}+\partial_2\Big(\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\dot{G}_{B13}\varepsilon_1\cdot k_3e^{(\cdot)}\Big) &= \dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\dot{G}_{B2i}k_2\cdot k_i\dot{G}_{B13}\varepsilon_1\cdot k_3\,e^{(\cdot)}\,.
\end{aligned}
\tag{3.21}
$$

Decomposing the new integrand according to its "cycle content", P_3 gets replaced by $Q_3 = Q_3^3 + Q_3^2$, where

$$
\begin{aligned}
Q_3^3 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\,,\\
Q_3^2 &= \dot{G}_{B12}\dot{G}_{B21}Z_2(12)T(3)+\dot{G}_{B13}\dot{G}_{B31}Z_2(13)T(2)+\dot{G}_{B23}\dot{G}_{B32}Z_2(23)T(1)\,.
\end{aligned}
\tag{3.22}
$$

Note that Q_3^3 contains a cycle of length three and Q_3^2 a cycle of length two, as indicated by the upper indices, and that each τ-cycle appears together with the corresponding "Lorentz-cycle". This motivates the further definition of a "bicycle" as the product of the two:

$$
\dot{G}(i_1i_2\cdots i_n) := \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}Z_n(i_1i_2\cdots i_n)\,. \tag{3.23}
$$

But the terms of Q_3^2 have, apart from the cycle, also a "one-tail", defined by

$$
T(a) := \dot{G}_{Bai}\varepsilon_a\cdot k_i\,. \tag{3.24}
$$

This tail still has a polarization vector that is not absorbed into a field strength tensor. Now in this three-point case there are already various chains of IBP that can be used to remove the $\ddot{G}_B$'s, but if one assumes that the corresponding total derivative terms are added with constant coefficients (i.e., they involve no functions of momentum or polarization other than the ones already present in same Q_3 in (3.22)), then it is easy to convince oneself that they all lead to the same Q_3 of (3.22). It is easy to see that, nonetheless, each term in Q_3^2 is individually gauge invariant; if one replaces in, e.g., the term

$$
\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\dot{G}_{B3k}\varepsilon_3\cdot k_k\,e^{(\cdot)}\,,
$$

ε_3 by k_3, then it becomes proportional to

$$
\partial_3\Big(\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\,e^{(\cdot)}\Big)\,.
$$

However, our aim here is to make gauge invariance manifest even at the integrand level. Now in the three-point case there are already various chains of IBP that can be used to remove all the $\ddot{G}_B$'s, but if one assumes that the corresponding total derivative terms are added with constant coefficients (i.e., they involve no dependences on momentum or polarization other than the ones already present in the term which one wishes to modify), then it is easy to convince oneself that they all lead to the same Q_3 of (4.25). Thus we have to look for a more general type of IBP. We will now essentially apply the procedure of the previous Section to the tails. Consider again the first term in Q_3^2 above, eq. (4.25). Choose a momentum vector r_3 such that $r_3 \cdot k_3 \neq 0$, and add the total derivative

$$-\frac{r_3 \cdot \varepsilon_3}{r_3 \cdot k_3} Z_2(12) \partial_3 \Big(\dot{G}_{B12} \dot{G}_{B21} e^{(\cdot)} \Big) \,. \tag{3.25}$$

The addition of this term to the first term in Q_3^2, and of similar terms to the second and third one, transforms Q_3^2 into

$$\begin{aligned} R_3^2 \quad := \quad & \dot{G}_{B12} \dot{G}_{B21} Z_2(12) \dot{G}_{B3k} \frac{r_3 \cdot F_3 \cdot k_k}{r_3 \cdot k_3} + \dot{G}_{B13} \dot{G}_{B31} Z_2(13) \dot{G}_{B2j} \frac{r_2 \cdot F_2 \cdot k_j}{r_2 \cdot k_2} \\ & + \dot{G}_{B23} \dot{G}_{B32} Z_2(23) \dot{G}_{B1i} \frac{r_1 \cdot F_1 \cdot k_i}{r_1 \cdot k_1} \,. \end{aligned} \tag{3.26}$$

Thus we have completed the covariantization of the integrand.
In the abelian case the 3-point amplitude must, of course, vanish, which we can see by noting that the integrand is odd under the orientation-reversing transformation of variables $\tau_i = T - \tau_i'$, $i = 1, 2, 3$. Note that in the three-point case, Q_3 is still unique, all possible ways of performing the partial integrations lead to the same result. The same is not true anymore in four-point case when the result of the partial integration procedure turns out to depend on the specific chain of partial integrations chosen. This ambiguity was discussed in [122] but no systematic way was found to perform the partial integrations at arbitrary N. This ambiguity was considered again in [123] and a systematic way and an algorithm had introduced based on preserving the manifest permutation symmetry of the N-photon amplitude. Here is the algorithm:

1. In every step, partially integrate away *all* the second derivative factors $\ddot{G}_{Bij}$'s appearing in the term under inspection simultaneously. This is possible since different $\ddot{G}_{Bij}$'s never share variables.

2. In the first step, for every factor of $\ddot{G}_{Bij}$ present use both τ_i and τ_j for the IBP, and take the mean of the results.

3. At every following step, any $\ddot{G}_{Bij}$ appearing must have been created in the previous step. Therefore either both variables τ_i and τ_j were used in the previous step, or just one of them. If both were used, then both should be used again in the actual IBP step, and the mean of the results be taken. If only one of the variables was used in the previous step, then the other variable should be used in the actual step.

The P-representation of $N = 4$ would be

$$
\begin{aligned}
P_4(\dot G_{Bij}, \ddot G_{Bij}) = \quad & + \quad \ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\, \ddot G_{B34}\varepsilon_3\cdot\varepsilon_4 + \ddot G_{B13}\varepsilon_1\cdot\varepsilon_3\, \ddot G_{B24}\varepsilon_2\cdot\varepsilon_4 + \ddot G_{B14}\varepsilon_1\cdot\varepsilon_4\, \ddot G_{B23}\varepsilon_2\cdot\varepsilon_3 \\
& - \quad \dot G_{B1i}\varepsilon_1\cdot k_i\, \dot G_{B2j}\varepsilon_2\cdot k_j\, \ddot G_{B34}\varepsilon_3\cdot\varepsilon_4 - \dot G_{B1i}\varepsilon_1\cdot k_i\, \dot G_{B3j}\varepsilon_3\cdot k_j\, \ddot G_{B24}\varepsilon_2\cdot\varepsilon_4 \\
& - \quad \dot G_{B1i}\varepsilon_1\cdot k_i\, \dot G_{B4j}\varepsilon_4\cdot k_j\, \ddot G_{B23}\varepsilon_2\cdot\varepsilon_3 - \dot G_{B2i}\varepsilon_2\cdot k_i\, \dot G_{B3j}\varepsilon_3\cdot k_j\, \ddot G_{B14}\varepsilon_1\cdot\varepsilon_4 \\
& - \quad \dot G_{B2i}\varepsilon_2\cdot k_i\, \dot G_{B4j}\varepsilon_4\cdot k_j\, \ddot G_{B13}\varepsilon_1\cdot\varepsilon_3 - \dot G_{B3i}\varepsilon_3\cdot k_i\, \dot G_{B4j}\varepsilon_4\cdot k_j\, \ddot G_{B12}\varepsilon_1\cdot\varepsilon_2 \\
& + \quad \dot G_{B1i}\varepsilon_1\cdot k_i\, \dot G_{B2j}\varepsilon_2\cdot k_j\, \dot G_{3k}\varepsilon_3\cdot k_k\, \dot G_{4l}\varepsilon_4\cdot k_l\ .
\end{aligned}
\tag{3.27}
$$

Let us apply this algorithm for $\ddot G_{B12}\ddot G_{B34}$ appearing in the first term of P_4:

$$
\begin{aligned}
\ddot G_{B12}\ddot G_{B34}\, e^{(\cdot)} \quad \to \quad & \frac{1}{4}\dot G_{B12}\dot G_{B34}\Big\{\Big[\dot G_{B1i}k_1\cdot k_i - \dot G_{B2i}k_2\cdot k_i\Big]\Big[\dot G_{B3j}k_3\cdot k_j - \dot G_{4j}k_4\cdot k_j\Big] \\
& -\ddot G_{B13}k_1\cdot k_3 + \ddot G_{B14}k_1\cdot k_4 + \ddot G_{B23}k_2\cdot k_3 - \ddot G_{B24}k_2\cdot k_4\Big\}\, e^{(\cdot)}\ .
\end{aligned}
\tag{3.28}
$$

The terms in the second line contain some new $\ddot G$ which have been created in the first step so they have to further processed. Let us consider just the first term which contains $\ddot G_{B13}$, since both variables were active in the first step, both also must be used in the second step. This yields

$$
\begin{aligned}
-\frac{1}{4}\dot G_{B12}\dot G_{B34}\ddot G_{B13}\, e^{(\cdot)} \quad \to \quad & \frac{1}{8}\dot G_{B12}\dot G_{B34}\dot G_{B13}\Big[\dot G_{B1i}k_1\cdot k_i - \dot G_{B3i}k_3\cdot k_i\Big]\, e^{(\cdot)} \\
& +\frac{1}{8}\dot G_{B13}\Big[\ddot G_{B12}\dot G_{B34} - \dot G_{B12}\ddot G_{B34}\Big]\, e^{(\cdot)}\ .
\end{aligned}
\tag{3.29}
$$

Considering again the first term in the second line which contains $\ddot G_{B12}$ needs to one more step, here since τ_1 was active in the previous step therefor only τ_2 must be used now, and the third step is the final one for this especial case

$$
\frac{1}{8}\dot G_{B13}\ddot G_{B12}\dot G_{B34}\, e^{(\cdot)} \to \frac{1}{8}\dot G_{B13}\dot G_{B12}\dot G_{B34}\dot G_{B2i}k_2\cdot k_i\, e^{(\cdot)}\ . \tag{3.30}
$$

Since this algorithm treats all variable at the same footing, therefore it leads to the permutation symmetric results. The nontrivial fact is that this process terminates after a finite number of steps and does not become cyclic (as would be the case if, for example, one would always treat the indices in a $\ddot G_{Bij}$ symmetrically). This algorithm transforms P_4 into Q_4 [123] which is

$$
\begin{aligned}
Q_4 \;=\;& \dot G_{B1i}\varepsilon_1\cdot k_i \dot G_{B2j}\varepsilon_2\cdot k_j \dot G_{B3k}\varepsilon_3\cdot k_k \dot G_{B4l}\varepsilon_4\cdot k_l\\
&+\biggl\{\frac12 \dot G_{B12}\varepsilon_1\cdot\varepsilon_2\Bigl\{\dot G_{B3i}\varepsilon_3\cdot k_i \dot G_{4j}\varepsilon_4\cdot k_j\Bigl[\dot G_{B1k}k_1\cdot k_k-\dot G_{B2k}k_2\cdot k_k\Bigr]\\
&+\Bigl[\dot G_{B3i}\varepsilon_3\cdot k_i(\dot G_{B41}\varepsilon_4\cdot k_1-\dot G_{B42}\varepsilon_4\cdot k_2)\dot G_{B4k}k_4\cdot k_k+(3\leftrightarrow 4)\Bigr]\\
&+\Bigl[(\dot G_{B31}\varepsilon_3\cdot k_1-\dot G_{B32}\varepsilon_3\cdot k_2)\dot G_{B43}\varepsilon_4\cdot k_3\dot G_{B4k}k_4\cdot k_k+(3\leftrightarrow 4)\Bigr]+5\text{ permutations}\biggr\}\\
&+\biggl\{\frac14 \dot G_{B12}\dot G_{B34}\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4\Bigl\{\Bigl[\dot G_{B1i}k_1\cdot k_i-\dot G_{B2i}k_2\cdot k_i\Bigr]\Bigl[\dot G_{B3j}k_3\cdot k_j-\dot G_{B4j}k_4\cdot k_j\Bigr]\\
&+\frac12\Bigl[\dot G_{B13}k_1\cdot k_3-\dot G_{B23}k_2\cdot k_3-\dot G_{B14}k_1\cdot k_4+\dot G_{B24}k_2\cdot k_4\Bigr]\\
&\times\Bigl[\dot G_{B1i}k_1\cdot k_i+\dot G_{B2i}k_2\cdot k_i-\dot G_{B3i}k_3\cdot k_i-\dot G_{B4i}k_4\cdot k_i\Bigr]\Bigr\}+2\text{ permutations}\biggr\}\,, \qquad (3.31)
\end{aligned}
$$

which can be rewritten more compactly as

$$
Q_4 \;=\; Q_4^4+Q_4^3+Q_4^2-Q_4^{22}\,, \qquad (3.32)
$$

where

$$
\begin{aligned}
Q_4^4 &= \dot G(1234)+\dot G(1243)+\dot G(1324)\,,\\
Q_4^3 &= \dot G(123)T(4)+\dot G(234)T(1)+\dot G(341)T(2)+\dot G(412)T(3)\,,\\
Q_4^2 &= \dot G(12)T(34)+\dot G(13)T(24)+\dot G(14)T(23)+\dot G(23)T(14)+\dot G(24)T(13)+\dot G(34)T(12)\,,\\
Q_4^{22} &= \dot G(12)\dot G(34)+\dot G(13)\dot G(24)+\dot G(14)\dot G(23)\,.
\end{aligned} \qquad (3.33)
$$

Here we have now further introduced the two-tail,

$$
T(ij) \;:=\; \sum_{r,s}\Bigl\{\dot G_{Bir}\varepsilon_i\cdot k_r \dot G_{Bjs}\varepsilon_j\cdot k_s+\frac12 \dot G_{Bij}\varepsilon_i\cdot\varepsilon_j\Bigl[\dot G_{Bir}k_i\cdot k_r-\dot G_{Bjs}k_j\cdot k_s\Bigr]\Bigr\}\,. \qquad (3.34)
$$

Note that in rhs of (3.32) the the product of two cycles Q_4^{22} appears with a minus sign, the reason is that we corrected for an over-counting here.
Thus the final representation of the (still off-shell) four-photon amplitude in scalar QED becomes

$$
\begin{aligned}
\Gamma_{\rm scal}[k_1,\varepsilon_1;\dots;k_4,\varepsilon_4] \;=\;& \frac{e^4}{(4\pi)^{\frac D2}}(2\pi)^D\delta(\sum k_i)\int_0^\infty\frac{dT}{T}T^{-\frac D2}e^{-m^2T}\\
&\times\int_0^T d\tau_1\cdots d\tau_4\,Q_4(\dot G_{Bij})\exp\Bigl\{\frac T2\sum_{i,j=1}^4 G_{Bij}k_i\cdot k_j\Bigr\}\,.
\end{aligned} \qquad (3.35)
$$

We shortly summarize the advantages of this representation compared to a standard Feynman/Schwinger parameter integral representation (see [50] for details):
First, the rhs of (3.35) represents already the complete amplitude, with no need to add "crossed" terms. The summation over "crossed" diagrams which would have to be done in a standard field theory calculation here is implicit in the integration over the various ordered sectors.
Second, the IBP procedure has homogenized the integrand; every term in Q_N has N factors of $\dot{G}_{Bij}$ and N factors of external momentum. In the four-point case this has the additional advantage of making the UV finiteness of the photon-photon scattering amplitude manifest before integration. While the original numerator P_4 contains terms involving products of two $\ddot{G}_{Bij}$'s which lead to spurious divergences in the T - integration, after the IBP the integrand is finite term by term.
Third, it allows one to obtain the corresponding spinor QED amplitude by the application of the replacement rule (4.13). In applying the rules it must be observed, though, that the form of the integrand given above still contains, apart from the explicit cycle factors, additional cycles from the tail factors for certain values of the dummy indices. In the four – point case this occurs for Q_4^2 only: The two – tails contained in Q_4^2 as given in (3.33) above each contain a two-cycle, since the content of the braces on the rhs of (3.34) for $r = j, s = i$ turns into

$$\dot{G}_{Bij}\dot{G}_{Bji}\Big(\varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_j\Big) = \dot{G}(ij) \; . \tag{3.36}$$

For the application of the "replacement rules" it is therefore convenient to decompose Q_4 in a slightly different way [123]. Namely, note that

$$Q_4^2 \quad = \quad Q_4^{'2} + 2Q_4^{22} \; , \tag{3.37}$$

where $Q_4^{'2}$ is obtained from Q_4^2 by eliminating the term with $r = j, s = i$ from the sum over dummy indices. With this definition, and setting $Q_4^{'(\cdot)} = Q_4^{(\cdot)}$ for the remaining components, we can write

$$Q_4 = Q_4^{'4} + Q_4^{'3} + Q_4^{'2} + Q_4^{'22} \; . \tag{3.38}$$

In this form all cycle factors are explicit, and moreover all the coefficients in the decomposition turn out to be unity. Generally, we will denote by $T'(i_1 \dots i_n)$ a tail whose cycles have been removed by the appropriate restrictions on the multiple dummy sums appearing in it.
Thus the four-photon amplitude for the spinor loop case can now be obtained from the scalar loop formula (3.35) simply by multiplying with a global factor of -2 from spin and statistics, and by replacing, simultaneously, each bicycle $\dot{G}(i_1 \dots i_n)$ by the corresponding "super-bicycle"

$$\dot{G}_S(i_1 \dots i_n) := (\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1})Z_n(i_1\cdots i_n) \; , \tag{3.39}$$

(the notation refers to the worldline supersymmetry underlying the replacement rule (4.13), see [50]). This "symmetric partial integration" procedure has been worked out explicitly for up to the six-photon case; see [50, 123] for the explicit formulas.

3.5 The QR representation

So far we have established two seemingly unrelated IBP procedures, the first one leading to the manifestly gauge invariant R-representation, the second one to the Q' - representation that is suitable for the application of the Bern-Kosower rules, but manifestly gauge invariant only in the cycle factors, not in the tails. We will now combine the two IBP strategies, using the following three simple observations:

First, it had been noted in the appendix C of [50] that the Q - representation is recursive, in the following sense: Each term in the cycle decomposition of Q_N contains at least one cycle [50], so that any tail appearing in the N - point amplitude has at most $N-2$ arguments. And a tail of length, say, M, is related to the (undecomposed) lower-order Q_M simply by writing Q_M in the tail variables, and then extending the range of all dummy variables occurring in it to run over the full set of indices $1,\ldots,N$. For example, writing out (3.34) for the two-tail $T(12)$ in the last term of Q_4^2 in (3.33) gives

$$T(12) = \sum_{r,s=1}^{4} \dot G_{B1r}\varepsilon_1\cdot k_r \dot G_{B2s}\varepsilon_2\cdot k_s + \frac{1}{2}\dot G_{B12}\varepsilon_1\cdot\varepsilon_2 \sum_{r=1}^{4}\Big[\dot G_{B1r}k_1\cdot k_r - \dot G_{B2r}k_2\cdot k_r\Big] \,, \tag{3.40}$$

which can also be obtained by writing Q_2, defined in (4.18), as

$$Q_2(12) \equiv \dot G_{B12}\varepsilon_1\cdot k_2 \dot G_{B21}\varepsilon_2\cdot k_1 + \frac{1}{2}\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\Big[\dot G_{B12}k_1\cdot k_2 - \dot G_{B21}k_2\cdot k_1\Big] \,, \tag{3.41}$$

and introducing appropriate dummy index summations. That this property holds in general can be easily seen by considering those terms in the cycle factors of the decomposition of Q_M that do not contain factors of $\varepsilon_i\cdot\varepsilon_j$, and thus can have involved IBPs only in the tail and not in the cycle variables; see the appendix C of [50] for more details.

Second, for each N the symmetric IBP procedure defines a unique Q_N and thus a total derivative term

$$S_N\, e^{(\cdot)} \equiv Q_N\, e^{(\cdot)} - P_N\, e^{(\cdot)} \,. \tag{3.42}$$

Consider now an arbitrary term in the cycle decomposition of Q_N. It will have the form $C(i_1\ldots i_L)T(j_1\ldots j_M)$, where $M\le N-2$, $C(i_1\ldots i_L)$ is a bicycle or product of bicycles involving the variables $\tau_{i_1},\ldots,\tau_{i_L}$, and $T(j_1\ldots j_M)$ is the unique (in the symmetric IBP scheme) tail of M variables, written in the remaining variables $\tau_{j_1},\ldots,\tau_{j_M}$. Consider $C(i_1\ldots i_L)S_M(j_1\ldots j_M)\, e^{(\cdot)}$, where possible dummy variable summations in S_M are extended to run over the full range of variables $\tau_1,\ldots,\tau_N$ as above. This is still a total derivative term (involving only derivatives in the tail variables), and the above simple relation between the M - tail and Q_M implies, that

$$C(i_1\ldots i_L)T(j_1\ldots j_M)\, e^{(\cdot)} - C(i_1\ldots i_L)S_M(j_1\ldots j_M)\, e^{(\cdot)} = C(i_1\ldots i_L)T_p(j_1\ldots j_M)\, e^{(\cdot)} \,, \tag{3.43}$$

with a new version T_p of the M - tail which relates to P_M in the same way as the standard tail T to Q_M, i.e. by an extension of the dummy index sums.

Continuing with our example above, here we have

$$\begin{aligned} S_2\, e^{G_{B12}k_1\cdot k_2} &\equiv (Q_2 - P_2)\, e^{G_{B12}k_1\cdot k_2} \\ &= \Big\{\frac{1}{2}\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\Big[\dot G_{B12}k_1\cdot k_2 - \dot G_{B21}k_2\cdot k_1\Big] + \ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\Big\}\, e^{G_{B12}k_1\cdot k_2} \\ &= \frac{1}{2}(\partial_1 - \partial_2)\Big(\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\, e^{G_{B12}k_1\cdot k_2}\Big)\ , \end{aligned} \tag{3.44}$$

and (3.43) becomes

$$\dot G(34)T(12)\, e^{(\cdot)} - \frac{1}{2}(\partial_1 - \partial_2)\Big(\dot G(34)\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\, e^{(\cdot)}\Big) = \dot G(34)T_p(12)\, e^{(\cdot)}\ , \tag{3.45}$$

where $T(12)$ was given in (3.40) and $T_p(12)$ is given by

$$T_p(12) = \sum_{r,s=1}^{4} \dot G_{B1r}\varepsilon_1\cdot k_r \dot G_{B2s}\varepsilon_2\cdot k_s - \ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\ . \tag{3.46}$$

Finally, the new tail $T_p(\cdots)$ can now be covariantized by a simple extension of (3.15):

$$\prod_{a=1}^{M}(1-\rho_{j_a}\partial_{j_a}\Delta_{j_a})\Big(C(\cdot)T_p(j_1\cdots j_M)\, e^{(\cdot)}\Big) = C(\cdot)T_p\Big(\dot G_{Bai_a}\varepsilon_a\cdot k_{i_a}\to T_r(a), -\ddot G_{Bab}\varepsilon_a\cdot\varepsilon_b \to W_r(ab)\Big)\, e^{(\cdot)}\ . \tag{3.47}$$

In our two-tail example, this lead to the following result:

$$T_r(12) = \dot G_{B1r}\frac{r_1\cdot F_1\cdot k_r}{r_1\cdot k_1}\dot G_{B2s}\frac{r_2\cdot F_2\cdot k_s}{r_2\cdot k_2} + \ddot G_{B12}\frac{r_1\cdot F_1\cdot F_2\cdot r_2}{r_1\cdot k_1 k_2\cdot r_2}\ , \tag{3.48}$$

where the sums over r and s run from 1 to 4. This should be compared with R_2 of (3.11).

3.6 The S representation

Thus in the QR - representation we have the usual decomposition into cycle and tail factors, with the tails already covariantized, and in a form that generalizes the (lower order) R - representation by an extension of the dummy index sums to run over all N variables, including those belonging to the cycle factors of the term under consideration. To finish our quest for an integrand that would be both covariant and suitable for the application of the Bern-Kosower rules, two more steps are needed: First, we need to remove the remaining $\ddot G_B$'s; this can be done by reapplying the symmetric IBP procedure of Section 3.4, without any modifications. And finally all cycles still contained in the tails have to be separated out. We will call this final, in some sense optimized result for the integrand of the N photon/gluon amplitudes, the "S - representation", and denote the corresponding tails by $T'_s(\cdot)$.
Continuing with our example of the two-tail in Q_4^2, the first step transforms $T_r(12)$ of (3.48) into

$$
\begin{aligned}
T_{\rm s}(12) \quad &\equiv \quad \sum_{r,s=1}^{4} \dot G_{B1r}\frac{r_1\cdot F_1\cdot k_r}{r_1\cdot k_1}\dot G_{B2s}\frac{r_2\cdot F_2\cdot k_s}{r_2\cdot k_2} \\
&-\frac{1}{2}\dot G_{B12}\Bigl(\sum_{r=1}^{4}\dot G_{B1r}k_1\cdot k_r-\sum_{s=1}^{4}\dot G_{B2s}k_2\cdot k_s\Bigr)\frac{r_1\cdot F_1\cdot F_2\cdot r_2}{r_1\cdot k_1 k_2\cdot r_2}\,.
\end{aligned}
\tag{3.49}
$$

This form of the two-tail, like the original two-tail $T(12)$ of (3.34), still contains a cycle - the terms on the rhs with $r=2$ and $s=1$ combine to form a $\dot G(12)$, as in (3.36). Eliminating these terms from the tail one arrives at the final form,

$$
\begin{aligned}
T'_{\rm s}(12) \quad &\equiv \quad \sum_{\substack{r,s=1\\ r,s\neq(2,1)}}^{4} \dot G_{B1r}\frac{r_1\cdot F_1\cdot k_r}{r_1\cdot k_1}\dot G_{B2s}\frac{r_2\cdot F_2\cdot k_s}{r_2\cdot k_2} \\
&-\frac{1}{2}\dot G_{B12}\Bigl(\sum_{\substack{r=1\\ r\neq 2}}^{4}\dot G_{B1r}k_1\cdot k_r-\sum_{\substack{s=2\\ s\neq 1}}^{4}\dot G_{B2s}k_1\cdot k_s\Bigr)\frac{r_1\cdot F_1\cdot F_2\cdot r_2}{r_1\cdot k_1 k_2\cdot r_2}\,.
\end{aligned}
\tag{3.50}
$$

For the one-tail there is no difference between $T_r(i)$, $T_{\rm s}(i)$ and $T'_{\rm s}(i)$, being all given by (3.13).

We can now write down a covariantized version of the four photon amplitude, simply by taking over (3.33), (3.35), and (3.38) and replacing all one-tails by (3.13) and all two-tails by (3.50). Explicit formulas for higher point amplitudes will be given elsewhere.

3.7 Alternative version of the two-tail

For the two-tail, there is actually yet another form which is covariant, free of $\ddot G_B$'s and at the same time more compact than (3.49). Starting again with $T_p(12)$ of (3.46) we add the following total derivative term to $T_p(12)$ (omitting now the inert cycle factors $C(\cdot)$)

$$
\begin{aligned}
&\frac{1}{(k_1\cdot k_2)^2}{\rm tr}\,(F_1F_2)\,\partial_1\partial_2\,e^{(\cdot)}+\frac{1}{k_1\cdot k_2}\Bigl[\varepsilon_1\cdot\varepsilon_2\,\partial_1\partial_2\,e^{(\cdot)}-\varepsilon_1\cdot k_2\varepsilon_2\cdot k_j\partial_1\Bigl(\dot G_{B2j}\,e^{(\cdot)}\Bigr) \\
&\qquad\qquad -\varepsilon_2\cdot k_1\varepsilon_1\cdot k_i\partial_2\Bigl(\dot G_{B1i}\,e^{(\cdot)}\Bigr)\Bigr]\,.
\end{aligned}
\tag{3.51}
$$

One obtains the new two-tail

$$\begin{aligned}
T_H(12) &= \ddot{G}_{B12}\Big[2\frac{\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1 - k_1\cdot k_2\varepsilon_1\cdot\varepsilon_2}{k_1\cdot k_2} - \frac{2\mathrm{tr}(F_1F_2)}{k_1\cdot k_2}\Big] \\
&\quad -\frac{\dot{G}_{Bi1}\dot{G}_{B2j}}{(k_1\cdot k_2)^2}\Big[\mathrm{tr}(F_1F_2)k_1\cdot k_i k_2\cdot k_j + \varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2 k_2\cdot k_j k_1\cdot k_i - \varepsilon_1\cdot k_2 k_1\cdot k_2 k_1\cdot k_i\varepsilon_2\cdot k_j \\
&\quad -\varepsilon_2\cdot k_1 k_1\cdot k_2 k_2\cdot k_j\varepsilon_1\cdot k_i + \varepsilon_1\cdot k_i\varepsilon_2\cdot k_j(k_1\cdot k_2)^2\Big] \\
&= -\dot{G}_{Bi1}\dot{G}_{B2j}\frac{\mathrm{tr}(F_1F_2)k_1\cdot k_i k_2\cdot k_j}{(k_1\cdot k_2)^2} + \frac{\dot{G}_{Bi1}\dot{G}_{B2j}(k_1\cdot k_2)}{(k_1\cdot k_2)^2}\Big[-\varepsilon_1\cdot\varepsilon_2 k_2\cdot k_j k_1\cdot k_i \\
&\qquad +\varepsilon_1\cdot k_2 k_1\cdot k_i\varepsilon_2\cdot k_j + \varepsilon_2\cdot k_1 k_2\cdot k_j\varepsilon_1\cdot k_i - \varepsilon_1\cdot k_i\varepsilon_2\cdot k_j k_1\cdot k_2\Big] ,
\end{aligned} \tag{3.52}$$

which in a compact form would be

$$T_H(12) \equiv \dot{G}_{Bi1}\dot{G}_{B2j}k_i\cdot H_{12}\cdot k_j , \tag{3.53}$$

where we have introduced the tensor

$$H_{12}^{\mu\nu} \equiv \frac{(F_1F_2)^{\mu\nu}k_1\cdot k_2 - k_1^\mu k_2^\nu \mathrm{tr}\,(F_1F_2)}{(k_1\cdot k_2)^2} . \tag{3.54}$$

Note that $\mathrm{tr}\,H_{12} = 0$ and $H_{12}^T = H_{21}$. Note also that the term with $i = 2, j = 1$ in (3.53) as before produces a $\dot{G}(12)$, since

$$k_2\cdot H_{12}\cdot k_1 = \frac{1}{2}\mathrm{tr}\,(F_1F_2) - \mathrm{tr}\,(F_1F_2) = -\frac{1}{2}\mathrm{tr}\,(F_1F_2) = -Z_2(12) . \tag{3.55}$$

Thus $T_H(ij)$ can be used as well as $T(ij)$ and $T_{\mathrm{s}}(ij)$ in the construction of the four-point amplitudes, including the application of the replacement rule (4.13) and the simple sign change in passing from (3.33) to (3.38). However, contrary to $T_{\mathrm{s}}(ij)$ it appears that $T_H(ij)$ has no natural generalization to the higher-point tails.

3.8 The case of spinor QED

One of the main purposes of the IBP procedure is to trivialize the transition to the spinor QED case though the replacement rule (4.13). Still, it is interesting to note (and of possible practical relevance for the generalization to the case of open fermion lines, where the IBP is less attractive due to the existence of boundary terms) that in the worldline formalism there is also a more direct treatment of the spin $\frac{1}{2}$ case using an approach based on explicit worldline supersymmetry [50,62,65,125,126]. It allows one to write down a master formula for N – photon scattering [65] which is formally analogous to the one for the scalar loop, eq.(3.1):

$$\begin{aligned}
\Gamma_{\mathrm{spin}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -2(-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T} \\
\times\prod_{i=1}^N\int_0^T d\tau_i\int d\theta_i\exp&\Big\{\sum_{i,j=1}^N\Big[\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j + iD_i\hat{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}D_iD_j\hat{G}_{ij}\varepsilon_i\cdot\varepsilon_j\Big]\Big\}\Big|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)} .
\end{aligned} \tag{3.56}$$

Here we have further introduced integrals over the Grassmann variables $\theta_1,\dots,\theta_N$, such that $\int d\theta_i\theta_i = 1$, and the super derivative

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau} \,. \tag{3.57}$$

The two worldline Green's functions $G_{B,F}$ now appear combined in the super Green's function

$$\hat{G}(\tau_1,\theta_1;\tau_2,\theta_2) \equiv G_B(\tau_1,\tau_2) + \theta_1\theta_2 G_F(\tau_1,\tau_2) \,. \tag{3.58}$$

Now, also the polarization vectors $\varepsilon_1,\dots,\varepsilon_N$ are to be treated as Grassmann variables. The overall sign of the master formula refers to the standard ordering of the polarization vectors $\varepsilon_1\varepsilon_2\dots\varepsilon_N$.

Starting with this master formula, all the manipulations which we have applied in the previous Chapters to the scalar loop integrands can, *mutatis mutandis* [3], also be used in the spinor loop case starting from (4.37). Here we will be satisfied with pointing out that the covariantized Bern-Kosower master formula (3.16) generalizes to the spinor QED case as follows:

$$\begin{aligned}\Gamma_{\rm spin}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -2(-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\\ \times\prod_{i=1}^N\int_0^T d\tau_i\int d\theta_i\exp\Biggl\{\sum_{i,j=1}^N\biggl[\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j &+ iD_i\hat{G}_{ij}\frac{r_i\cdot F_i\cdot k_j}{r_i\cdot k_i} - \frac{1}{2}D_iD_j\hat{G}_{ij}\frac{r_i\cdot F_i\cdot F_j\cdot r_j}{r_i\cdot k_i\, r_j\cdot k_j}\biggr]\Biggr\}\Bigg|_{\mathrm{lin}(F_1,\dots,F_N)}\,,\end{aligned} \tag{3.59}$$

where now $F_1,\dots,F_N$ have to be treated as Grassmann variables.

3.9 The nonabelian case

As far as concerns the calculation of the on-shell QED photon amplitudes, or of the on-shell gluon S-matrix elements via the Bern-Kosower rules, the availability of a representation that is manifestly transversal at the integrand level is satisfying, but the significance of this fact for practical calculations is not obvious. To the contrary, it is easy to recognize the advantages of such a representation when it comes to the gluon amplitudes off-shell. Here the natural objects to consider in QCD are the "N-vertices", that is the one-particle-irreducible N - point functions, and for applications of those it is often essential to decompose them into a basis of transversal and longitudinal tensor structures (see, e.g., [124,127–129]). Such a "transversality-based form factor decomposition" in the present approach emerges essentially automatically in the IBP procedure through the appearance of field strength tensors. We have seen how this happens for the abelian case, but it is true also for the nonabelian case; here in principle one would like to see the full nonabelian field strength tensor emerging,

$$F_{\mu\nu} \equiv F^a_{\mu\nu}T^a = F^0_{\mu\nu} + ig[A^b_\mu T^b, A^c_\nu T^c] \,, \tag{3.60}$$

[3] *Mutatis Mutandis* is a Latin phrase meaning "changing [only] those things which need to be changed" or more simply "[only] the necessary changes having been made".

where by

$$F^0_{\mu\nu} \equiv (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T^a \,, \tag{3.61}$$

we now denote its "abelian part"; and indeed Strassler demonstrated already for some simple cases how this happens [37, 122]: When the external particles are gluons, the various ordered sectors of the integral $\int d\tau_1 \cdots d\tau_N$ need to be considered separately, since they carry different color factors. Therefore boundary terms now arise in the IBP procedure, and the commutator terms are generated as differences of boundary terms between adjacent sectors that in the abelian case would cancel, but cannot do so any more in the presence of color since two of the color matrices appear in different orders. In an x-space calculation of the effective action, those commutator terms could then be combined with the "abelian" parts of the field strength tensor, but this is not possible in a momentum space calculation of the N - point function at fixed N, since any term in the nonabelian effective action after Fourier transformation contributes to amplitudes with various numbers of external particles; e.g., the term $\mathrm{tr}\,(D_\mu F_{\alpha\beta} D^\mu F^{\alpha\beta})$ will contribute to the N - point functions with N between two and six. Generally, each term in the nonabelian effective Lagrangian has a "core" term, which has a counterpart already in the abelian case (in the example this would be $\partial_\mu F^0_{\alpha\beta}\partial^\mu F^{0\alpha\beta}$) and a number of "covariantizing" terms that all involve commutators, and belong to amplitudes with more legs than the core term. In this Section, we will explain the essentials of how to calculate the scalar, spinor and gluon loop contributions to the one-loop N - gluon vertex. The details and a full recalculation of the three-gluon vertex and also its extension to the four-gluon case will be discussed in Chapter 4 and Chapter 5.

Starting with the scalar loop case, here the master formula (3.1) generalizes to the nonabelian case simply by supplying a global color factor, and keeping the gluons in a fixed order:

$$\begin{aligned}\Gamma^{a_1\dots a_N}_{\mathrm{1PI,scal}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] \;&=\; (-ig)^N \mathrm{tr}(T^{a_1}\dots T^{a_N})(2\pi)^D i\delta(\sum k_i)\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\ \times\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\dots\int_0^{\tau_{N-2}} d\tau_{N-1}&\exp\left\{\sum_{i,j=1}^N\left[\frac12 G_{Bij}k_i\cdot k_j - i\dot G_{Bij}\varepsilon_i\cdot k_j + \frac12\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Bigg|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)}.\end{aligned} \tag{3.62}$$

Here the T^a are the generators of the gauge group in the representation of the loop scalar. This treatment of color correponds to a "color-ordered" representation (although not necessarily in the usual sense, where the T^a would be in the fundamental representation, see, e.g., [130, 131]). Note that we have not only fixed the ordering of the gluons along the loop but also used the translation invariance in τ to set $\tau_N = 0$. Summation over all $(N-1)!$ inequivalent orderings of the N vertex operators is implied. Also, Γ has been given an index "1PI" to indicate that the rhs gives only the one-particle-irreducible part of the N - gluon amplitude, not including the reducible part that now also exists, differently from the abelian case. Starting from (3.62) one can apply the IBP procedure leading from the P - representation to the S - representation as before, the only novelty being the boundary terms. Since for the bulk terms all the polarization vectors ε_i get absorbed into the transversal structures (3.60) (which now, however, represent only the "abelian part" of the field strength tensor), in the final representation the non-transversal part of the N - vertex must be entirely in the boundary terms, given by lower-point integrals. For those one still has to choose the vectors r_i, preferably in a way that is consistent with the

cyclic invariance of the nonabelian amplitudes. In the three-point case a convenient cyclic choice is $r_1 = k_2 - k_3$, $r_2 = k_3 - k_1$, $r_3 = k_1 - k_2$, and indeed it turns out [132] that, with this choice, the resulting form factor decomposition matches precisely the standard Ball-Chiu decomposition [124] of the three-gluon-vertex.
Coming to the spinor loop case, here the only issue is whether the replacement rule (4.13) can be applied also to all the boundary terms now arising in the IBP procedure. This is indeed the case, as we can see as follows: it suffices to show the corresponding statement for the effective action, rather than the momentum space Green's functions. Now, the effective Lagrangian can in principle be written as an infinite series of terms that are Lorentz scalars formed using any number of field strength tensors and covariant derivatives. As was already mentioned, each such term has a core term, whose calculation is not different from the abelian case for either the scalar or spinor loop, such that the replacement rule applies to it. All covariantizing terms of a core term must share its coefficient, and low-order calculations show that, as one would expect, the way this works is that they all involve the same parameter integral [122,132]. And for the whole structure to continue to be gauge invariant for the spinor loop case it is necessary that the same replacement rule applies to all the covariantizing terms as well as for the core term.
Similarly one can convince oneself that also the replacement rules that connect the scalar with the gluon loop cases [33,37,50,133,134] can be extended from the core terms to the ones involving boundary contributions [132]. An additional issue with the gluon loop contribution to the N - vertex is that one has to choose a gauge for the gluon propagator. The application of the gluonic replacement rules gives the N - vertex corresponding to the use of the background field method with Feynman gauge for the quantum part [37, 134], which is also known to coincide with the result of the application of the pinch technique [135–137]. This version of the gluonic vertex is also the one that leads to SUSY sum rules [138].

3.10 Conclusions

We have continued here the systematic investigation of the Bern-Kosower partial integration procedure, initiated in [122] and continued in [123]. We have presented an IBP algorithm that unambiguously leads to a form of the integrand of the one-loop N photon amplitude in scalar QED which is manifestly gauge invariant (transversal) at the integrand level, and suitable for an application of the Bern-Kosower rules. We have worked out this integrand explicitly at the four-point (two-tail) level. The N - point integrand contains N vectors $r_1, \dots, r_N$ which are constrained only by the condition (3.9). Further study will be needed to find out what is the significance of this ambiguity, and how to make the best use of it. As far as concerns the on-shell N - photon/gluon amplitudes, one would surmise that the dependence of the integrand on the vectors r_i is related to the usual dependence on the reference vectors $q_1, \dots, q_N$ which one would normally have in the application of the spinor helicity formalism, but does not exist any more once all polarization vectors are absorbed into field strength tensors. And indeed, there is clearly a relation: for example, consider the case of the the on-shell N-photon amplitudes with all helicities positive. When using the P representation together with the standard spinor helicity formalism (see, e.g., [130, 139]), one can remove all terms involving a $\ddot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j$ by choosing, in the spinor helicity formalism, the same reference vector $q_i = q$ for all legs, since then $\varepsilon_i^+ \cdot \varepsilon_j^+ = 0$. Similarly, in the S representation one could make disappear all the factors $r_i \cdot F_i \cdot F_j \cdot r_j$ by choosing all $r_i = r$ equal, and r on-shell, since then

$$r \cdot F_i \cdot F_j \cdot r = \frac{1}{2} r \cdot \{F_i, F_j\} \cdot r = -\frac{1}{4}[ij]^2 r^2 = 0 \tag{3.63}$$

on account of the identity [140]

$$\{F_i^+, F_j^+\}^{\mu\nu} = -\frac{1}{2}[ij]^2\eta^{\mu\nu} \tag{3.64}$$

However, it is clear that this match between the freedom of choosing the r_i's and the q_i's cannot be a perfect one, since the r_i's need not be chosen as on-shell.

All our representations are valid off-shell. This makes them relevant for state-of-the-art calculations already at the four-point level, since neither the four-photon nor the four-gluon amplitudes are presently available in the literature fully off-shell (for any spin in the loop). This fact is particularly conspicuous in the case of spinor QED, where the on-shell four-photon scattering amplitude was obtained already in 1951 by Karplus and Neumann [141], and the extension to the case of two off-shell legs in 1971 by Costantini et al. [142]. Our integral representations for the QED four-photon amplitudes are, with any of the various definitions of the tails, manifestly finite term by term and thus suitable for a numerical evaluation as they stand. For analytical purposes one would still like to reduce the various parameter integrals appearing in them to scalar box, triangle and bubble integrals. This could be done using existing tensor reduction algorithms (see [143] and refs. therein), but in a companion paper [144] we will rather perform this tensor reduction in a way that is specifically adapted to the structure of the worldline integrals.

Due to this validity off-shell our representations can also be used to construct, by sewing, all higher-loop N - photon amplitudes. From the calculation of the two-loop QED β function [62] it is clear that when calculating those multiloop amplitudes in the worldline formalism significative simplifications can be expected from a judicious application of IBP.

The manifest transversality is probably a more significant issue in the nonabelian case. Here we have in mind not so much the calculation of on-shell gluon amplitudes, for which other extremely powerful methods have been developed in recent years (see, e.g., [145, 146] and refs. therein), but rather of the one-particle-irreducible off-shell N - gluon vertices, for which there is presently still a dearth of efficient methods. For the use of these amplitudes, e.g. in Schwinger-Dyson equations, it is usually important to have them in a form that separates them into transversal and non-transversal parts, which normally requires a tedious analysis of the nonabelian Ward identities. This is one of the reasons why, using standard methods, the explicit calculation of these off-shell vertices has been completed so far only for the $N = 3$ case [124, 138, 147]. In another companion paper [132] we use the S representation to recalculate the three-gluon vertex, for the scalar, spinor, and gluon loops, achieving a drastic reduction in computational effort compared to earlier attempts (a summary of this calculation was given in [148]). The main advantages of our approach are that the gluon and spinor loop cases can be effortlessly obtained from the scalar loop one through the (off-shell extended) Bern-Kosower replacement rules, and that there is no necessity to solve the Ward identities, rather the decomposition into transversal and non-transversal pieces emerges automatically in the IBP procedure. In relation with the latter point it must also be mentioned that the boundary terms in the IBP procedure applied to the N-vertex always involve color-commutators, and are always connected to some lower-point term, even appearing with the same integral. Thus the algorithm makes it also easy to separate the genuinely new structures appearing in the N-vertex from those that, in terms of the effective action, only serve the completion of lower-point expressions to fully gauge-invariant ones; see [132].

It should be possible, and very interesting, to generalize our approach to the inclusion of gravity. On-shell, the gauge theory Bern-Kosower rules were generalized to the construction of string-based representations for the one-loop N-graviton amplitudes in [149], and these gravity rules

were then successfully applied at the four-point level in [150]. They also involve an IBP and "replacement rules" connecting the amplitudes with different spins in the loop. Worldline path integral representations of the one-loop effective actions for gravity have so far been constructed for spin zero [51], spin half [74, 151] and spin one [152] in the loop, and can be used to obtain parameter integral representations of the corresponding off-shell one-particle irreducible N-graviton amplitudes that are closely related to those string-based representations. They are written in terms of the same worldline Green's functions as the ones for gauge theory, and the challenge is again to find an IBP algorithm that would allow one to apply the replacement rules and at the same time make covariance manifest, where the latter now means the emergence of full Riemann tensors in the IBP. This algorithm can, however, not be a simple extension of the one which we have presented here, as one can see from the fact that in gravity there are no boundary terms in the IBP, so that the nonlinear terms in the Riemann tensor now have to be created by δ-functions; therefore a complete removal of all $\ddot{G}_B$'s in the IBP is not called for, and in fact also not possible, as one can easily see already from the case of the graviton propagator.
Finally, in open string theory the one-loop gauge boson amplitudes can be written in terms of a master formula that is analogous to the master formula (3.1), only that the variables τ_i parametrize positions along the boundary of the string worldsheet, and the Green's functions are worldsheet Green's functions (see, e.g., [153]). Since in all of our manipulations we have, apart from the translation invariance in proper-time, not used any specific properties of the worldline Green's functions, our IBP procedure could as well be applied at the string level to achieve a form factor decomposition based on gauge invariance.

Chapter 4

A covariant representation of the three-gluon vertex (Ball-Chiu vertex)

4.1 Introduction

[1] In theoretical physics, quantum chromodynamics (QCD) is a theory of strong interaction, a fundamental force describing the interaction between QCD ingredients as quarks and gluons which make up hadron such as the proton, neutron and pion. QCD is a nonabelian theory which has a $SU(3)$ symmetry group which is free in ultraviolet (UV) and confining in infrared (IR). The idea that the IR singularities of QCD provides the mechanism for quark and color confinement is widely accepted, in spite of the limited understanding of these singularities. In comparing the IR behavior of QCD with that of QED, which is well understood, we observe that QCD has two complications. First, the field quanta carry a charge (color in QCD) and hence are self-coupled; and second the quanta are coupled to massless particles (quarks and gluons). Both of these features are required in QCD. In this and next Chapter we study the analytic properties of the off-shell three and four-gluon vertex function. As noted by Ball and Chiu [154] there are three reasons for studying vertex function: First, in massless theories, functions of a single momentum such as the propagator have singularities in the dimensionless variable q^2/Λ^2 where Λ^2 is the UV cutoff. Thus the IR singularities at the one loop level are trivially related to the UV behavior. In contrast to the propagator the off-shell vertex function depends on three scalars let say k^2, k'^2 and q^2. Hence, even in the one-loop result, logarithmic singularities in k^2/k'^2 or q^2/k'^2 can occur which are not related to the UV behavior. Second, the vertex function in a gauge theory satisfies a Ward identity, which relates the longitudinal components of a vertex to simpler function. The fact that all of the IR singularities in the vertex function for massive QED occur in this longitudinal component allows a simple and direct investigation of the general IR properties. It has been suggested [155] that a similar procedure might be followed for QCD. And the essential step as described in [155] is to construct a vertex function which automatically satisfies the Ward identity. The longitudinal part of the vertex can be represented explicitly in terms of simpler scalar functions that appear in the Ward identity.

[1] This Chapter is based on [132].

During the last decade, the IR behavior of Yang Mills Green functions has been studied extensively. Especially studying the propagators is of interest because they are linked to the confinement problem via two exciting scenarios, Kugo-Ojima [156] and Gribov-Zwanziger [157,158]. Both of these scenarios predict that in Landau gauge the IR enhanced ghost propagator that is responsible for the appearance of long-ranged forces, whereas the gluon propagator should vanish (or be finite for the former). Indeed such behavior was found by functional methods as Schwinger-Dyson equations (SDEs) and Renormalization Group (RG) equations (see, e.g., [159] and references therein).
The one-particle-irreducible ('1PI') (any diagram that cannot be split in two by removing a single line) off-shell three-gluon Green's function (in the following simply called "three-vertex" or"vertex") is a basic object of interest in nonabelian gauge theory and quantum chromodynamics. As we already mentioned it contains important information on IR and it is a main ingredient of the SDEs. In perturbation theory three-gluon vertex can be computed explicitly to a certain loop order and be used a s building block for higher-loop calculations. Historically, the first calculation of off-shell three-gluon vertex has been considered by Celmaster and Gonsalves in 1979 [160] where they computed the dimensionally regularized the two and three-point functions at the symmetric point $k_2^2 = k_2^2 = k_3^2$ in arbitrary covariant gauge in one-loop order. Later in 1980 Pascual and Tarrach studied the Slavnov-Taylor identity in Weinberg's renormalization scheme [161]. In 1980 first by Kim and Baker [128] the most general form of the three-gluon vertex consistent with requirement of gauge invariance was determined in both axial and covariant gauge. Later Ball and Chiu [124] constructed from first principle the general form of the three-gluon vertex but restricted to Feynman gauge which satisfies the Ward identity which nowadays is know as Ball-Chiu form factor decomposition. In 1989 Cornwall and Papavassiliou [129] constructed the gauge invariant three-gluon vertex through the *pinch technique*. In 1996 Davydychev et al [162] treated the gluon loop for three-gluon vertex in arbitrary covariant gauge also the massless fermion loop case and in other paper in 2001 they studied the massive fermion loop case [147]. By using the BFM in 2006, Binger and Brodsky [138] studied the scalar, fermion and gluon loop cases in various dimensions, their result is equivalent to the pinch technique for N-gluon amplitude but they obtained an explicit form for three-gluon vertex. At two loop, the three-gluon vertex has been obtained, so far only for some very special momentum configurations [163–165].

Multi-gluon amplitude in perturbative QCD pose two very different computational challenges. On one hand, there is the calculation of on-shell matrix elements, a field that has been tremendous activity and progress during the last decades, particularly for the massless and/or SUSY cases. A whole host of new concepts and techniques have emerged, such as unitarity based method [166, 167], twistors [168], BCFW recursion [169, 170], and Grassmannians [171, 172]; see [139] and [173] for recent reviews. In this Chapter and Chapter 5 we show how to use he Bern-Kosower master formula, originally a generating function for on-shell matrix element, to derive well-organaized form factor decomposition of the off-shell one particle-irreducible three and four gluon vertex.

Now this question may arise, why studying the three-gluon vertex is important?

We already mentioned that one of the original reasons for studying this object is the believe that its IR properties might shed light on the mechanism of confinement. The three gluon vertex is perhaps the most obvious manifestation of the nonabelian aspect of QCD [174] (for a review see [175]). In real physical calculations, the perturbative corrections to gluonic vertex are also very

important such as multijet production at the hadron colliders [176].
In this Chapter we recalculate, in a simple and unified way, the scalar, spinor and gluon loop contribution to the one-loop three-vertex. In Fig. 4.1 for definiteness we show the scalar loop contribution.

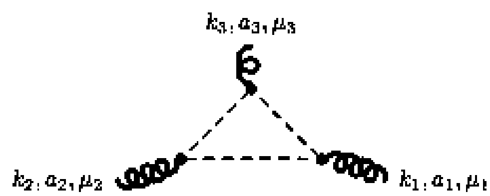


Figure 4.1: Three-gluon vertex for scalar loop particle.

A priori, six tensor structures (and their permutations) are needed to decompose the three-gluon vertex [124]. The Yang-Mills term of the QCD Lagrangian yields the following well-known expression for the lowest-order three-gluon vertex:

$$-igf^{a_1a_2a_3}[g_{\mu_1\mu_2}(k_1-k_2)_{\mu_3}+g_{\mu_1\mu_3}(k_1-k_3)_{\mu_2}+g_{\mu_2\mu_3}(k_2-k_3)_{\mu_1}]\ , \tag{4.1}$$

where k_1, k_2 and k_3 are the are the momenta of the gluons, all of which are ingoing, $k_1+k_2+k_3=0$, we are following the notation of [147, 162] and the adjoint representation is given by $(T^a)^{bc}=-if^{abc}$.
In (4.1) the $f^{a_1a_2a_3}$ are the totally antisymmetric color structure constants in adjoint representation of the gauge group. So one can define the general three gluon vertex as

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)=-igf^{a_1a_2a_3}\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)\ . \tag{4.2}$$

Since gluons are bosons then $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$ must be symmetric under interchange of a pair gluon momenta and the corresponding Lorentz indices. Since $f^{a_1a_2a_3}$ are antisymmetric hence $\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$ must also be antisymmetric under any of these changes.
For Fig. 4.1 we have two diagrams differing by the two inequivalent orderings of the external gluons along the loop (or equivalently by a change of the scalar line orientation). Those diagrams add to produce a factor of two. We recall that the lowest-order gluon propagator is

$$\delta^{a_1a_2}\frac{1}{k^2}\Big(g_{\mu_1\mu_2}-\xi\frac{k_{\mu_1}k_{\mu_2}}{k^2}\Big)\ , \tag{4.3}$$

where ξ is the gauge parameter corresponding to a general covariant gauge, defined such $\xi=0$ is the Feynman gauge. When one calculates the radiative corrections (here one-loop as depicted in Fig. 4.1) in addition to the lowest order (4.1) other tensor structures arise, and the general form of these tensor structure should be considered. So we want to construct a general tensor decomposition consistent with Bose symmetry out of three Lorentz indices and two linearly independent four vectors (energy momentum conservation leaves only two independent four momentum vectors). It is easy to construct tensors which are odd under interchange of $(k_1,\mu_1)\leftrightarrow(k_2,\mu_2)$ and making $\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$ invariant under cyclic permutations. $\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$ must satisfy the Ward identity so it is more convenient to construct tensors which are transverse, i.e. , orthogonal to $k_{1\mu_1}$, $k_{2\mu_2}$ and $k_{3\mu_3}$. In term of these quantities, the transverse part of the vertex is given by four (which actually two of them are independent because of cyclic permutations) and there are 10 (in reality five) tensor structures which make the longitudinal part of the vertex. Ball and Chiu [124] in 1980 found a form factor decomposition of the three-gluon vertex which is valid at any order in perturbation theory, and also independent of whether the particle in the loop is scalar, fermion or gluon, (in Feynman gauge). This tensor decomposition as we discussed above can be written in term of really six independent tensor structure, as

$$\begin{aligned}\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \;=\; & -A(k_1^2,k_2^2;k_3^2)g_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} - B(k_1^2,k_2^2;k_3^2)g_{\mu_1\mu_2}(k_1+k_2)_{\mu_3} \\ & +C(k_1^2,k_2^2;k_3^2)\Big[k_{1\mu_2}k_{2\mu_1} - k_1\cdot k_2\, g_{\mu_1\mu_2}\Big](k_1-k_2)_{\mu_3} \\ & +\frac{1}{3}S(k_1^2,k_2^2,k_3^2)\Big[k_{1\mu_3}k_{2\mu_1}k_{3\mu_2} + k_{1\mu_2}k_{2\mu_3}k_{3\mu_1}\Big] \\ & +F(k_1^2,k_2^2;k_3^2)\Big[k_{1\mu_2}k_{2\mu_1} \; - k_1\cdot k_2\, g_{\mu_1\mu_2}\Big]\Big[k_1\cdot k_3\, k_{2\mu_3} - k_2\cdot k_3\, k_{1\mu_3}\Big] \\ & +H(k_1^2,k_2^2,k_3^2)\Big\{-g_{\mu_1\mu_2}\Big[k_{1\mu_3}k_2\cdot k_3 - k_{2\mu_3}k_1\cdot k_3\Big] \\ & \qquad\qquad +\frac{1}{3}\Big[k_{1\mu_3}k_{2\mu_1}k_{3\mu_2} - k_{1\mu_2}k_{2\mu_3}k_{3\mu_1}\Big]\Big\} \\ & +\Big\{\text{cyclic permutations of } (k_1,\mu_1),(k_2,\mu_2),(k_3,\mu_3)\Big\}\,.\end{aligned} \tag{4.4}$$

Here the functions A, C and F are symmetric in the first two arguments, the function B is antisymmetric in the first two arguments, H is totally symmetric and S totally antisymmetric with respect to interchange of any pair of arguments. F and H functions belong to the transverse part of the vertex and A, B, C and S to the longitudinal part.
In the following we wish to study off-shell three-gluon vertex, with our calculation method it will be convenient to contract it with polarization vectors $\varepsilon_{1,2,3}$ which are arbitrary and serve book-keeping purposes only. So for (4.4)

$$\begin{aligned}\varepsilon_1^{\mu_1}\varepsilon_2^{\mu_2}\varepsilon_3^{\mu_3}\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \;\equiv\; & A(k_1^2,k_2^2;k_3^2)T_A + B(k_1^2,k_2^2;k_3^2)T_B + C(k_1^2,k_2^2;k_3^2)T_C \\ + \; & S(k_1^2,k_2^2,k_3^2)T_S + F(k_1^2,k_2^2;k_3^2)T_F + H(k_1^2,k_2^2,k_3^2)T_H\,.\end{aligned} \tag{4.5}$$

Note that the tensors T_F and T_H are totally transversal, i.e., they give zero when any ε_i is replaced by k_i.
For the gluon loop in the Feynman gauge and at the one-loop level, Ball and Chiu also calculated the coefficient functions A to H, up to their constant terms in the ϵ-expansion; here it turn out that S actually vanishes. In other words, one-loop correction to three-gluon vertex has only five independent form factor. Later Davydychev, Osland and Tarasov [162] computed this gluon loop contrition vertex more generally for an arbitrary covariant gauge, and in arbitrary space-time dimension. Their interest for doing the calculations in arbitrary gauge and dimension was the following [162]:

i. one can explicitly keep track of gauge invariance for physical quantities by knowing the results in an arbitrary gauge.

ii. if one is interested in the two-loop calculation of the three-gluon coupling, one should know one-loop contributions in more detail.

iii. results in arbitrary dimension make it possible to consider all on-shell limits (when some $k_i = 0$) directly from these expressions, this is impossible if one only has the results valid around four dimensions.

iv. QCD is also a theory of interest in three and two dimensions (see e.g. [177] and a review [178]).

As we have mentioned, the quark loop contribution to the vertex was first calculated for massless quarks and in the symmetric limit $k_1^2 = k_2^2 = k_3^2$ by Celmaster and Gonsalves [160] and Pascual and Tarrach [161]. But latter in [162] the massless quark contribution was obtained for general off-shell momenta and in [147] for massive quark loop again in arbitrary dimension.
However, this is not the whole story, since for the gluon loop contribution to the vertex there are subtle issues with gauge dependence. In the standard formalism using any covariant gauge for gluon loop, it satisfies rather complicated Slavnov-Taylor identities involving not only the gluon propagator, but also the ghost propagator and the gluon-gluon-ghost vertex (see, e.g., [124, 138, 175]). In contrast to the gluon loop, the scalar and fermion loop contributions, satisfy the simple QED like Ward identity,

$$\begin{aligned}\Gamma_{\text{scalar,spinor}}(\varepsilon_3 \to k_3) \quad &= \quad (k_1^2 \varepsilon_1 \cdot \varepsilon_2 - k_1 \cdot \varepsilon_1 k_1 \cdot \varepsilon_2)\Big(1 - \Pi_{\text{scalar,spinor}}(k_1^2)\Big) \\ &\quad -(k_2^2 \varepsilon_1 \cdot \varepsilon_2 - k_2 \cdot \varepsilon_1 k_2 \cdot \varepsilon_2)\Big(1 - \Pi_{\text{scalar,spinor}}(k_2^2)\Big)\,.\end{aligned} \tag{4.6}$$

where $\Pi(k^2)$ is the corresponding vacuum polarization function, for example for the spinor case is

$$\begin{aligned}\Pi^{\mu\nu}(k^2) \quad &= \quad (k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi(k^2)\,, \\ \Pi_2(k^2) \quad &= \quad -(-ie)^2 \int \frac{d^D q}{(2\pi)^D} \text{tr}\Big[\gamma^\mu \frac{(-\not{q} + m)}{q^2 + m^2 - i\epsilon} \gamma^\nu \frac{(-\not{q} - \not{k} + m)}{(q+k)^2 + m^2 - i\epsilon}\Big]\,,\end{aligned} \tag{4.7}$$

where $\Pi_2(k^2)$ is the second order (in e) contribution to $\Pi(k^2)$.

Having the same simple Ward identity for the gluon loop case is possible, but requires more sophisticated techniques; it can be achieved using either the BFM [179–181] with Feynman gauge for the quantum field, or the pinch technique [127, 129]. Although very different, those two methods turn out to lead to precisely the same Green's functions [135, 182]. The corresponding three-gluon vertex, also called the "gauge-invariant vertex", was studied first by Freedman et al. [183]. Somewhat surprisingly it was found to be conformally invariant, despite of the gauge fixing. Binger and Brodsky [138] calculated the scalar loop contribution, which enabled them to find certain sum rules between the massless scalar, fermion and gluon loop contributions. In particular in the adjoint representation for all particles massless they obtained the following identity

$$3\Gamma_0 + \frac{1}{2}\Gamma_{\frac{1}{2}} + \Gamma_1 = 0\,, \tag{4.8}$$

(here we introduce the convention that a subscript $s = 0$ refers to a scalar loop, $s = \frac{1}{2}$ to spinor loop, and $s = 1$ to a gluon including the ghost loop contribution). This identity is related to supersymmetry, and implies that the off-shell three-gluon amplitude in $\mathcal{N} = 4$ Super-Yang-Mills theory vanishes. It generalizes the well-known vanishing of the gluon self-energy in that theory,

but contrary to that fact does not obviously relate to the finiteness of the theory.
In this Chapter, we will recalculate the scalar, spinor and gluon loop contributions to this "gauge-invariant" three-gluon vertex using the "string-inspired" formalism the line of [30,33,37,84,121–123] (for a review, see [50]). Our starting point is the "Bern-Kosower master formula" [30,33,121]:

$$
\begin{aligned}
&\Gamma^{a_1\dots a_N}_{\text{scalar}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N]\\
&= (-ig)^N \mathrm{tr}(T^{a_1}\dots T^{a_N})(2\pi)^D i\delta(\sum k_i)\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\
&\times \int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\dots\int_0^{\tau_{N-2}}d\tau_{N-1}\\
&\times \exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot G_{Bij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Bigg|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)}.
\end{aligned}
\tag{4.9}
$$

As it stands, this formula represents the color-ordered contribution to the 1PI N - gluon amplitude due to a (complex) scalar loop of mass m, calculated in D spacetime dimensions. The ith gluon carries the momentum k_i, polarization ε_i and a gauge group generator T^{a_i}. T is the total proper-time length of the loop, and τ_i is the position in proper-time along the loop of gluon i. One integration is redundant and has been eliminated by setting $\tau_N = 0$. The derivation of this formula involved a formal exponentiation, which still needs to be undone by expanding out the exponential factor and keeping only the terms linear in each of the N polarization vectors. The color-ordering means that one still has to sum over all $(N-1)!$ inequivalent orderings of the gluons along the loop to get the full amplitude. $G_{Bij} \equiv G_B(\tau_i,\tau_j)$ denotes the "bosonic" worldline Green's function, defined by

$$
G_B(\tau_1,\tau_2) = \mid \tau_1-\tau_2\mid - \frac{(\tau_1-\tau_2)^2}{T}\,, \tag{4.10}
$$

and dots generally denote the derivative acting on the first variable, here we need first and second derivatives of the "bosonic" worldline Green's function

$$
\begin{aligned}
\dot G_B(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}\,,\\
\ddot G_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2) - \frac{2}{T}\,.
\end{aligned}
\tag{4.11}
$$

The main motivation of Bern and Kosower to derive the (4.9) from string theory [30,33,121] was a representation of the N-gluon amplitude for the heterotic string and analyzing its field theory limit. They were interested in on-shell calculation of the full N-gluon amplitude rather than the 1PI amplitude off-shell. Thus on one hand on-shell conditions were used from the beginning, already at the string level; on the other hand the fact that the distinction between reducible and irreducible diagrams emerges only in the field theory made it possible to establish certain formal rules that allow one to reconstruct, from the formula for the 1PI amplitude (4.9), also all the missing reducible contributions to the full on-shell matrix element. Bern and Kosower were moreover able to derive simple "loop replacement rules", based on worldsheet supersymmetry, that allow one to obtain from (4.9) also integral representations for the spinor and gluon loop contributions to the full on-shell N-gluon amplitudes [30, 33, 121]. We need not discuss these

"Bern-Kosower rules" here in full, but it is important to note that they all involve integration-by-parts ('IBP') in an essential way. We have discussed this issue in Chapter 3 in detail, see also [55].
Namely, performing the expansion of the exponential factor in (4.9) will yield an integrand $\sim P_N\, e^{(\cdot)}$, where we abbreviated

$$e^{(\cdot)} := \exp\biggl\{\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij}p_i\cdot p_j\biggr\}, \tag{4.12}$$

where as we already discussed in Chapter 3, P_N is a polynomial in $\dot G_{Bij}$, $\ddot G_{Bij}$ and the kinematic invariants. One can remove all the $\ddot G_{Bij}$s by suitable integration-by-part, leading to a new integrand $\sim Q_N e^{(\cdot)}$ which is the real starting point for the application of Bern-Kosower rules. We recall from Chapter 3, the rule for passing from the scalar to the spinor is easy to state in general: Q_N contains "τ-cycles", that is, product of $\dot G_{Bij}$'s whose indices form a closed chain which can be written as $\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots \dot G_{Bi_ni_1}$. Then, apart from a global factor of -2 correcting for degrees of freedom and statistics, the integrand for the spinor loop case can be obtained from the one for the scalar case simply by simultaneously replacing every τ-cycle appearing in Q_N by

$$\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots \dot G_{Bi_ni_1} \to \dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots \dot G_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\,.$$

where $G_{Fij}\equiv \mathrm{sign}(\tau_i-\tau_j)$ denotes the fermionic worldline Green's function. The rule for passing from the scalar to the gluon loop case is similar but more complicated, and will be dealt in Section (4.4). This formalism was used for the first calculation of the one-loop one-shell QCD five-gluon amplitudes [27], but due to emergence of other extremely powerful methods for computation of one-loop on-shell amplitudes such as generalized unitarity [146] no further employed for such on-shell gluon amplitude calculations. In this Chapter, we will start an effort to exploit the Bern-Kosower formulation as a tool for the calculation of the N-gluon vertex in one loop. In the original string-based formalism (which is on-shell) going off-shell is highly nontrivial, but not impossible [184, 185]. In [186] Bern and Kosower present the derivation of the rules and apply them to the computation of two-to-two gluon scattering in one-loop which previously was difficult enough to challenge the most expert calculators [187]. Later Bern and Dunbar [188] showed how to map the Bern-Kosower rules onto Feynman diagrams and demonstrated that the BFM plays and important rule. In [184, 185] a formalism was developed that, in principle, allows one to obtain the one-loop off-shell gluon amplitudes in pure Yang-Mills theory from the open string, and it was shown correctly reproduce the renormalization constant for the two-three and four point vertices. In our calculations we use however the simpler approach to the Bern-Kosower formation due to Strassler [37, 122], which uses string theory only as a guiding principle. In his approach the effective action is written as a one-dimensional path integral, which can be calculated to any order in a gauge coupling, evaluation leads to Feynman parameter integral directly, bypassing the usual algebra required from Feynman diagrams, and leading to compact and organized expression. This formalism is valid off-shell and is explicitly gauge invariant. The starting point of this "string-inspired worldline formalism" is the following path integral representation of the nonabelian one-loop effective action due to a scalar loop [37] (this generalizes Feynman's famous 1950 formula for scalar QED, see the Introduction 1),

$$\Gamma_0[A] = \int_0^\infty \frac{dT}{T}\, e^{-m^2T}\mathrm{tr}\int \mathcal{D}x\,\exp\Bigl[-\int_0^T d\tau\Bigl(\frac{1}{4}\dot x^2 + ig\dot x\cdot A\Bigr)\Bigr]\,, \tag{4.13}$$

where the integral $\int \mathcal{D}x$ is over the space of all closed trajectories in spacetime with periodicity

T in proper-time, $x^\mu(T) = x^\mu(0)$ and subscript 0 indicates the scalar loop particle. In [122], Strassler started a systematic investigation of the IBP procedure, and discovered that it bears an interesting connection to gauge invariance, which we have discussed in Chapter 3. However in [122] no systematic way was found to perform the partial integration at arbitrary N, nor how to preserve the permutation symmetry. This issue was taken up again by Schubert in [123], where he found a computerizable IBP algorithm which works for any N and preserve the full permutation symmetry. This algorithm is still not satisfactory from point of view of gauge invariance. This issue has been discussed in proceeding Chapter. Very recently an extension of the algorithm of [123] was found which, for any N and preserving the permutation invariance, achieves this "covariantization" for all the polarization vectors [55].
This in some sense completes the investigation started in [122]. It also suggests that, with this optimized IBP at hand, the string-inspired formalism might become a powerful tool for the computation of the N-vertex. This is for three reasons:

i. The covariantization means that the bulk integrand after the IBP is manifestly transversal, so that any nontransversality must come from boundary terms. Thus the IBP procedure itself should generate a transversality-based form factor decomposition similar to the Ball-Chiu one (4.4).

ii. Like the Ball-Chiu one, this decomposition will respect the cyclic invariance (which is the remnant of the permutation invariance after the color ordering).

iii. The work of [37] also suggests that the "loop-replacement" part of the Bern-Kosower rules may hold off-shell for the 1PI amplitudes, which would reduce the calculational effort very significantly.

Here we will recalculate the three-gluon vertex along the above lines, and find these expectation will be fully justified. The organization of this Chapter is as follows:
In Section 4.2 we will start the scalar loop contribution to the vertex (scalar particle running in the loop). With perform the IBP using the old algorithm of [123] as well as the improved one of [55]. With both choices we obtained a very compact integral representation of the vertex, however the new algorithm has the advantage that the nontransversality is pushed into the boundary (two-point) terms.
In Sections 5.4 and 4.4 we show that the "loop replacement rules" indeed hold for the three-gluon vertex. It is known that the free relativistic spin-$\frac{N}{2}$ particle can be described by a first quantized action with N-extended worldline supersymmetry, then the Dirac fermion possesses an $N = 1$ supersymmetric worldline path integral representation [38, 58, 59, 61, 63, 64, 189], and the gluon an $N = 2$ supersymmetric one [37, 38, 59, 190, 191]. Analogously to the original string-based derivation of those rules [30, 33, 121], where worldsheet SUSY was identified as the underlying symmetry, in the worldline approach the same rules can be related to this worldline SUSY [37,50].
In Section 4.5 we summarize and unify our results for the scalar, spinor and gluon loop. In Section 4.6 we establish their exact relation to the Ball-Chiu decomposition (4.4), and also explicitly verify the Ward identities (4.6) and the Binger-Brodsky relation (4.8).
The 1PI vertices hold the same information as the effective action. Nevertheless, contrary to the QED case where there is no essential difference between the calculation of the effective action and of the off-shell N-photon amplitudes, in the nonabelian case the effective action is mathematically an intrinsically more natural object. This is because it can be written in terms of full field strength tensors

$$F_{\mu\nu} \equiv F^a_{\mu\nu} T^a = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T^a + ig[A^b_\mu T^b, A^c_\nu T^c] \,, \tag{4.14}$$

whereas upon Fourier transformation those inevitably get split up into their "abelian parts" $f^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ and the commutator terms. This suggests that the analysis of the structure of the 1PI vertices should benefit from a comparison with the low energy expansion of the effective action, and indeed we will show in Section 4.7 for the three-point case that in the present formalism, due to the systematic generation of "abelian" field strength tensors and commutator terms by the IBP, it is possible to keep the relation between the effective action and the vertex very transparent.
Our conclusions are given in Section 4.8. In particular, we give there a general argument showing that the off-shell validity of the loop replacement rules extends to the N-vertex.

4.2 The scalar loop case

Before coming to the calculation of the (off-shell, 1PI) three-gluon amplitude for a scalar loop, let us first consider the two-point (vacuum polarization) case. This will not only be useful as a warm-up, but also for the verification of the Ward identitiy (4.6) later on.
For $N = 2$ we get from the master formula (4.9), after expanding out the exponential (in the following we generally omit the global factor $(2\pi)^4 i\delta(\sum k_i)$ for energy-momentum conservation),

$$\begin{aligned}\Gamma_0^{a_1a_2}[k_1,\varepsilon_1;k_2,\varepsilon_2] &= (-ig)^2\mathrm{tr}(T^{a_1}T^{a_2})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\ &\quad\times\int_0^T d\tau_1\,(-i)^2P_2\,e^{G_{B12}k_1\cdot k_2}\,,\end{aligned}\tag{4.15}$$

where

$$P_2 = \dot G_{B12}\varepsilon_1\cdot k_2\dot G_{B21}\varepsilon_2\cdot k_1 - \ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\,.\tag{4.16}$$

In this stage we need to remove the $\ddot G_{B12}$ by adding a following total derivative

$$\partial_1\Big(\dot G_{B12}\varepsilon_1\cdot\varepsilon_2 e^{G_{B12}k_1\cdot k_2}\Big) = \Big(\ddot G_{B12}\varepsilon_1\cdot\varepsilon_2 - \dot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B12}k_1\cdot k_2\Big)e^{G_{B12}k_1\cdot k_2}\,.\tag{4.17}$$

Now P_2 transforms to Q_2,

$$\begin{aligned}Q_2 &= \dot G_{B12}\dot G_{B21}\big(\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1 - \varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2\big)\\ &= \frac{1}{2}\dot G_{B12}\dot G_{B21}\mathrm{tr}\,(f_1f_2) = \dot G_{B12}\dot G_{B21}Z_2(12)\,,\end{aligned}\tag{4.18}$$

where we used $\dot G_{Bij} = -\dot G_{Bji}$.
Here one can see that the IBP algorithm allows us to absorb the polarization vectors into the "abelian" field strength tensors f_i which defined as

$$f_i^{\mu\nu} \equiv k_i^\mu\varepsilon_i^\nu - k_i^\nu\varepsilon_i^\mu\,,\tag{4.19}$$

thereby making the transversality of the two point function manifest.
By using the energy-momentum conservation we set $k_1 \equiv -k_2 \equiv k$ and using $\mathrm{tr}(T^{a_1}T^{a_2}) = C(r)\delta^{a_1a_2}$, we can write

$$\begin{aligned}\Gamma_0^{a_1a_2}[k_1,\varepsilon_1;k_2,\varepsilon_2] &= \varepsilon_1^\mu\Pi^{a_1a_2}_{0\mu\nu}(k)\varepsilon_2^\nu\,,\\ \Pi^{a_1a_2}_{0\mu\nu}(k) &= \delta^{a_1a_2}(\eta_{\mu\nu}k^2 - k_\mu k_\nu)\Pi_0(k^2)\,,\end{aligned}\tag{4.20}$$

where

$$\Pi_0(k^2) \quad = \quad C(r)\frac{g^2}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1 \dot{G}_{B12}\dot{G}_{B21}\, e^{-G_{B12}k^2} \, . \tag{4.21}$$

Moving on to the three-point level, here the expansion of (4.9) yields

$$\begin{aligned}\Gamma_0^{a_1a_2a_3}[k_1,\varepsilon_1;k_2,\varepsilon_2;k_3,\varepsilon_3] \quad &= \quad (-ig)^3\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T} \\ &\quad \times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2\,(-i)^3 P_3\, e^{(\cdot)} \, , \end{aligned}\tag{4.22}$$

where

$$\begin{aligned} P_3 \quad = \quad & \dot{G}_{B1i}\varepsilon_1\cdot k_i \dot{G}_{B2j}\varepsilon_2\cdot k_j \dot{G}_{B3k}\varepsilon_3\cdot k_k - \ddot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B3k}\varepsilon_3\cdot k_k \\ & -\ddot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B2j}\varepsilon_2\cdot k_j - \ddot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\dot{G}_{B1i}\varepsilon_1\cdot k_i \, , \end{aligned}\tag{4.23}$$

and we have introduced the convention that repeated indices $i, j, k, \ldots$ are to be summed from 1 to $N = 3$. As the two point case we want to remove all the $\ddot{G}_B$'s appear here in P_3 to get a new polynomial as Q_3. So let us look at one of these terms and add a total derivative, for example consider the $\ddot{G}_{B12}\dot{G}_{B31}$ in the second term of P_3, we add the following total derivative

$$-\frac{\partial}{\partial\tau_2}\Bigl(\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B31}\varepsilon_3\cdot k_1 e^{(G_{B12}k_1\cdot k_2+G_{B13}k_1\cdot k_3+G_{23}k_2\cdot k_3)}\Bigr)\, . \tag{4.24}$$

Adding five more similar total derivative terms (as we have discussed in Chapter 3) removes all the $\ddot{G}_B$'s. Our new $Q_3 = Q_3^3 + Q_3^2$ polynomial contains two different "cycle content",

$$\begin{aligned} Q_3^3 \quad &= \quad \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\, , \\ Q_3^2 \quad &= \quad \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\dot{G}_{B3k}\varepsilon_3\cdot k_k + \dot{G}_{B13}\dot{G}_{B31}Z_2(13)\dot{G}_{B2j}\varepsilon_2\cdot k_j \\ & \quad +\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\dot{G}_{B1i}\varepsilon_1\cdot k_i \, , \end{aligned}\tag{4.25}$$

where Q_3^3 contains a τ-cycle of length three and Q_3^2 of length two, as indicated upper indices, and each τ-cycle appears together with the corresponding "Lorentz-cycle", as advertised in the introduction and in Chapter 3. The term of Q_3^2 have, apart from the cycle, also a "one-tail", defied by [50]

$$T_i(a) := \varepsilon_a\cdot k_i\dot{G}_{Bai}\, . \tag{4.26}$$

Although the form of the integrand reached in (4.25) is already suitable for the application of the Bern-Kosower rules, it is natural to ask whether the polarization vectors appearing in the tails can also somehow be completed to field strength tensors. A more general type of IBPs is called for is one wishes to achieve this "covariantization of the tails" which has been discussed in Chapter 3 according to [55].

Choose a momentum vector r_3 such that $r_3 \cdot k_3 \neq 0$, and add the total derivative

$$-\frac{r_3 \cdot \varepsilon_3}{r_3 \cdot k_3} Z_2(12) \frac{\partial}{\partial \tau_3} \Big(\dot{G}_{B12} \dot{G}_{B21} e^{(\cdot)} \Big) . \tag{4.27}$$

The addition of this term to the first term in Q_3^2, and of similar terms to the second and third one, transforms Q_3^2 into

$$\begin{aligned} S_3^2 \quad := \quad & \dot{G}_{B12} \dot{G}_{B21} Z_2(12) \dot{G}_{B3k} \frac{r_3 \cdot f_3 \cdot k_k}{r_3 \cdot k_3} + \dot{G}_{B13} \dot{G}_{B31} Z_2(13) \dot{G}_{B2j} \frac{r_2 \cdot f_2 \cdot k_j}{r_2 \cdot k_2} \\ & \dot{G}_{B23} \dot{G}_{B32} Z_2(23) \dot{G}_{B1i} \frac{r_1 \cdot f_1 \cdot k_i}{r_1 \cdot k_1} . \end{aligned} \tag{4.28}$$

Note that $S_3^3 \equiv Q_3^3$.
Thus now all polarization vectors have been absorbed into tensors f_i, leading to manifest transversality. This IBP procedure can be systematized to obtain closed-form integral representations of the Scalar and Spinor QED N - photon amplitudes that are manifestly gauge invariant at the integrand level [55].
Here, however, we are in nonabelian case, where the color-induced restriction of the parameter integrations to ordered sectors leads to appearance of boundary terms in the IBP [37, 122] (in the abelian case they would be zero).
Let us look again at our total derivative term (4.24). In the abelian case it would be integrated over the whole circle, and the result would be zero, since the worldline Green's function $G_B(\tau_1, \tau_2)$ has the appropriate periodicity properties to make the two boundary terms cancel. Here instead we find a nonzero result:

$$-\dot{G}_{B12} \varepsilon_1 \cdot \varepsilon_2 \dot{G}_{B31} \varepsilon_3 \cdot k_1 e^{(\cdot)} \Big|_{\tau_2 = \tau_3}^{\tau_2 = \tau_1} = 0 + \dot{G}_{B13} \varepsilon_1 \cdot \varepsilon_2 \dot{G}_{B31} \varepsilon_3 \cdot k_1 e^{G_{B13} k_1 \cdot (k_2 + k_3)} . \tag{4.29}$$

In the three point case there are already two inequivalent orderings for the external legs, say $(123 \equiv \tau_1 > \tau_2 > \tau_3)$ and (132), (4.29) give us the boundary terms for the (123) orderings so we need to consider the boundary terms for other ordering which simply is

$$-\dot{G}_{B12} \varepsilon_1 \cdot \varepsilon_2 \dot{G}_{B31} \varepsilon_3 \cdot k_1 e^{(\cdot)} \Big|_{\tau_2 = \tau_1}^{\tau_2 = \tau_3} = -\dot{G}_{B13} \varepsilon_1 \cdot \varepsilon_2 \dot{G}_{B31} \varepsilon_3 \cdot k_1 e^{G_{B13} k_1 \cdot (k_2 + k_3)} - 0 . \tag{4.30}$$

These two boundary terms would cancel each other in the abelian case, but now they would combine to produce a color commutator $\mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}])$. As we mentioned above we need five more total derivatives to convert P_3 into Q_3 and there is on that differs from (4.24) only by the interchange $2 \leftrightarrow 3$, as depicted in Fig. 4.2. With some relabeling of integration variables, we can combine the two boundary terms generated by that term with the two above to the structure

$$\mathrm{tr}\, (T^{a_1} [T^{a_2}, T^{a_3}]) \varepsilon_3 \cdot f_1 \cdot \varepsilon_2 \dot{G}_{B12} \dot{G}_{B21}\, e^{G_{B12} k_1 \cdot (k_2 + k_3)} . \tag{4.31}$$

Comparing with (4.21), we note that this term yields a parameter integral identical to the one of the two-gluon amplitude, except for the replacement of k_2 by $k_2 + k_3$. In terms of the effective

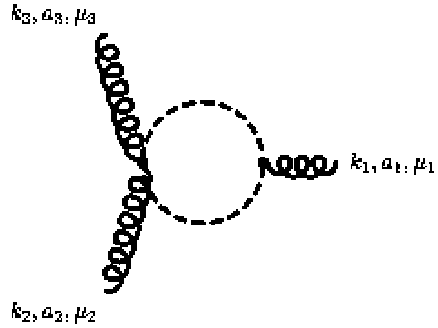


Figure 4.2: Two-gluon vertex of scalar loop particle from the boundary terms corresponds to $\Gamma_0^{\rm bt}$ when $2 \leftrightarrow 3$.

action, from (4.14) and (4.31) its role is evidently to provide a piece needed to extend the "abelian" Maxwell term $\mathrm{tr}\,(f_{\mu\nu}f^{\mu\nu})$ to the full nonabelian one $\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$. We will discuss this in more detail in Section 4.7 below.
To summarize so far, we can decompose the three-point amplitude for the scalar loop in Q-representation (as we discussed in Chapter 3) according to [55] as (here and in the following we will often suppress the superscript "$a_1a_2a_3$")

$$\Gamma_0 \quad = \quad \frac{g^3}{(4\pi)^{\frac{D}{2}}}(\Gamma_0^3 + \Gamma_0^2 + \Gamma_0^{\rm bt})\,, \tag{4.32}$$

where

$$\begin{aligned}
\Gamma_0^3 &= -\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\, Q_3^3\, e^{(\cdot)}\\
&\quad -\mathrm{tr}(T^{a_1}T^{a_3}T^{a_2})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_3\, Q_3^3\, e^{(\cdot)}\,,\\
\Gamma_0^2 &= \Gamma_0^3(Q_3^3 \to Q_3^2)\,,\\
\Gamma_0^{\rm bt} &= -\mathrm{tr}(T^{a_1}[T^{a_2},T^{a_3}])\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1 \dot G_{B12}\dot G_{B21}\\
&\quad \times\Big[\varepsilon_3\cdot f_1\cdot\varepsilon_2\, e^{G_{B12}k_1\cdot(k_2+k_3)} + \varepsilon_1\cdot f_2\cdot\varepsilon_3\, e^{G_{B12}k_2\cdot(k_1+k_3)}\\
&\qquad +\varepsilon_2\cdot f_3\cdot\varepsilon_1\, e^{G_{B12}k_3\cdot(k_1+k_2)}\Big]\,,
\end{aligned} \tag{4.33}$$

(here and in the following it is understood that always the last integration is eliminated by setting its integration variable equal to zero; e.g., for the ordering $\tau_1 > \tau_3 > \tau_2$ we set $\tau_2 = 0$). Alternatively, we can replace Q_3^2 by S_3^2 which gives us the S-representation of the scalar case [55], in the Γ_0^2 part, but then we have to also add to $\Gamma_0^{\rm bt}$ a term $\tilde\Gamma_0^{\rm bt}$ containing the further boundary contributions coming from the total derivative terms of the type (4.27). Collecting those, one finds

$$
\begin{aligned}
\tilde{\Gamma}_0^{\rm bt} &= \frac{1}{2}\mathrm{tr}(T^{a_1}[T^{a_2},T^{a_3}])\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1 \dot{G}_{B12}\dot{G}_{B21} \\
&\times\Big\{\Big[\mathrm{tr}(f_1f_2)\rho_3-\mathrm{tr}(f_3f_1)\rho_2\Big]\,e^{G_{B12}k_1\cdot(k_2+k_3)}+\Big[\mathrm{tr}(f_2f_3)\rho_1-\mathrm{tr}(f_1f_2)\rho_3\Big]\,e^{G_{B12}k_2\cdot(k_1+k_3)} \\
&\quad+\Big[\mathrm{tr}(f_3f_1)\rho_2-\mathrm{tr}(f_2f_3)\rho_1\Big]\,e^{G_{B12}k_3\cdot(k_1+k_2)}\Big\}\,,
\end{aligned}
\tag{4.34}
$$

where we have now abbreviated $\rho_i := r_i\cdot\varepsilon_i/r_i\cdot k_i$.

4.3 The spinor loop case

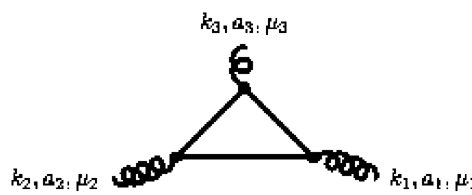


Figure 4.3: Three-gluon vertex for spinor loop particle.

For the spinor loop case, Fig. 4.3, it will be convenient to use the worldline super formalism [38, 62–65, 134, 192]. In worldline super formalism, one defines for each gluon leg a Grassmann variable θ_i where $\theta_i^2=0$, and also considers the polarization vectors ε_i as being Grassmann. Thus all ε_j^ν, θ_k and $d\theta_l$ anti commute with each other. One further introduces the super derivative as

$$
D=\frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial\tau}\,, \tag{4.35}
$$

and the super proper-time distance

$$
\hat{\tau}_{ij}:=\tau_i-\tau_j+\theta_i\theta_j\,. \tag{4.36}
$$

Then the Bern-Kosower master formula can be generalized to the case of a Dirac fermion loop case as follows [50, 65], as was discussed in the introduction:

$$
\begin{aligned}
\Gamma_{\frac{1}{2}}^{a_1\dots a_N}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -2(-ig)^N\mathrm{tr}(T^{a_1}\dots T^{a_N})\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T} \\
&\quad\times\prod_{k=1}^N\int_0^T d\tau_k\int d\theta_k\,\delta\Big(\frac{\tau_N}{T}\Big)\vartheta(\hat{\tau}_{1N})\prod_{l=1}^{N-1}\vartheta(\hat{\tau}_{l(l+1)}) \\
&\times\exp\Big\{\sum_{i,j=1}^N\Big[\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j+iD_i\hat{G}_{ij}\varepsilon_i\cdot k_j+\frac{1}{2}D_iD_j\hat{G}_{ij}\varepsilon_i\cdot\varepsilon_j\Big]\Big\}\Big|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)}\,.
\end{aligned}
\tag{4.37}
$$

Here ϑ is the Heaviside step function and the Green's functions

$$\hat{G}(\tau_i,\theta_i;\tau_j,\theta_j) \equiv G_B(\tau_i,\tau_j) + \theta_i\theta_j G_F(\tau_i,\tau_j)\,. \tag{4.38}$$

The overall sign of (4.37) refers to the standard ordering of the polarization vectors $\varepsilon_1\varepsilon_2\cdots\varepsilon_N$. Next note that

$$\vartheta(\hat{\tau}_{ij}) = \vartheta(\tau_i-\tau_j) + \theta_i\theta_j\delta(\tau_i-\tau_j)\,. \tag{4.39}$$

The terms arising in the expansion of the spinor loop master formula (4.37) can be divided into three types:

i. Terms they were there already for the scalar loop.

ii. New terms not involving any of the delta functions appearing in (4.39).

iii. Terms that do involve such delta functions (only single delta functions can appear up to three-point level).

Concerning the type (ii) terms, those are known already from the abelian case, and it was shown in [50] by a direct combinatorial argument, starting from the abelian version of (4.37), that they can be taken into account correctly by the "loop replacement rule" eq. (4.13). Terms of type (iii) are specific to the nonabelian case. They would cancel between adjacent ordered sectors in the abelian case, but now produce color commutators, so that it is natural to think of them as a fermionic counterpart to the boundary terms encountered in the scalar loop calculation.
In the two point case the loop replacement rules applies and its applications appears simply by replacing the $\dot{G}_{B12}\dot{G}_{B21}$ in Q_2 by $\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}$. Note that at the two point level, a terms of type (iii) appears but it gives an integrand proportional to

$$\delta(\tau_1-\tau_2)G_F(\tau_1,\tau_2)\ \mathrm{e}^{G_B(\tau_1,\tau_2)}\,, \tag{4.40}$$

that vanishes since $G_F(\tau,\tau)=0$ by antisymmetry. Therefore the vacuum polarization function for the spinor case (by taking the global normalization into account) becomes

$$\Pi_{\frac{1}{2}}(k^2) \quad = \quad -2C(r)\frac{g^2}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\big(\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\big)\,e^{-G_{B12}k^2}\,. \tag{4.41}$$

In the three point case, the effect of the type (ii) terms is to change each "bulk terms" $\Gamma_0^{2,3}$ of (4.33) to a corresponding $\Gamma_{\frac{1}{2}}^{2,3}$ differing from its scalar loop counterpart by a change of $Q_3^{2,3}$ to $\hat{Q}_3^{2,3}$, where

$$\begin{aligned}
\hat{Q}_3^3 \quad &= \quad \big(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} - G_{F12}G_{F23}G_{F31}\big)Z_3(123)\,,\\
\hat{Q}_3^2 \quad &= \quad \big(\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\big)Z_2(12)\dot{G}_{B3k}\varepsilon_3\cdot k_k + \text{ two permutations}\,.
\end{aligned} \tag{4.42}$$

But now the terms of type (iii) will really contribute. As we mentioned above, those are similar to the boundary terms (that come from the nonabelian characteristic of the theory and the total derivatives) of the scalar loop case, which is shown in Fig. 4.4 for spinor loop case, and a simple

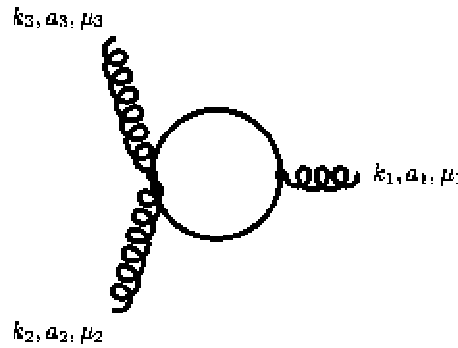


Figure 4.4: Two-gluon vertex of spinor loop particle from the boundary terms corresponds to $\Gamma^{\rm bt}_{\frac{1}{2}}$ when $2 \leftrightarrow 3$.

calculation shows that their role is precisely to make the "loop replacement rule" work for the boundary terms, too. Thus for the spinor loop contribution we find the Q-representation of the spinor case as

$$\Gamma_{\frac{1}{2}} = -2\frac{g^3}{(4\pi)^{\frac{D}{2}}}(\Gamma^3_{\frac{1}{2}} + \Gamma^2_{\frac{1}{2}} + \Gamma^{\rm bt}_{\frac{1}{2}}) \,, \tag{4.43}$$

where

$$\begin{aligned}
\Gamma^3_{\frac{1}{2}} &= -\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\, \hat{Q}^3_3|_{\tau_3=0}\, e^{(\cdot)} \\
&\quad -\mathrm{tr}(T^{a_1}T^{a_3}T^{a_2})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_3\, \hat{Q}^3_3|_{\tau_2=0}\, e^{(\cdot)} \,, \\
\Gamma^2_{\frac{1}{2}} &= \Gamma^3_{\frac{1}{2}}(\hat{Q}^3_3 \to \hat{Q}^2_3) \,, \\
\Gamma^{\rm bt}_{\frac{1}{2}} &= -\mathrm{tr}(T^{a_1}[T^{a_2},T^{a_3}])\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\big(\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\big) \\
&\quad \times\Big[\varepsilon_3\cdot f_1\cdot\varepsilon_2\, e^{G_{B12}k_1\cdot(k_2+k_3)} + \varepsilon_1\cdot f_2\cdot\varepsilon_3\, e^{G_{B12}k_2\cdot(k_1+k_3)} + \varepsilon_2\cdot f_3\cdot\varepsilon_1\, e^{G_{B12}k_3\cdot(k_1+k_2)}\Big] \,.
\end{aligned} \tag{4.44}$$

The alternative form of the scalar loop result which is S-representation, involving S^2_3 instead of Q^2_3 and the additional boundary contribution $\tilde{\Gamma}^{\rm bt}_0$, can similarly be generalized to the spinor loop case by an application of the replacement rule (4.13).

4.4 The gluon loop case

As was already mentioned, the case of the gluon loop is intrinsically more subtle, because here one has the issue of gauge (in)dependence not only for the background field but also for the loop particle. The preferred way of fixing the corresponding ambiguity for the three-vertex leads to

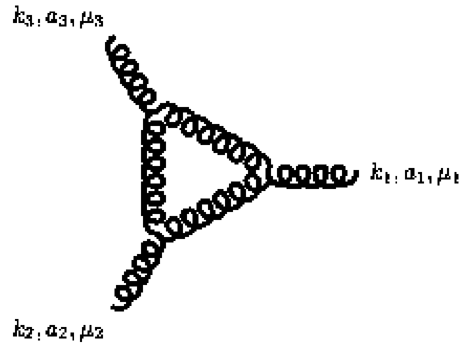


Figure 4.5: Three-gluon vertex for gluon loop particle.

the "gauge invariant vertex", which obeys the simple Ward identity (4.6). This version of the vertex is generated by the BFM with Feynman gauge for the quantum part, and it so happens that the only generalization of the worldline path integral representation (4.13) of the effective action to the gluon-loop case presently known is just based on the BMF with quantum Feynman gauge, and thus the right starting point for a calculation of the "gauge invariant vertex". This representation was developed in [37, 134] in component fields, and reformulated in terms of worldline superfields in [192]. Although from the point of view of string theory it is a "poor man's version" of the Polyakov path integral in the infinite string tension limit, it is perfectly adequate as far as the 1PI amplitudes are concerned. It is also consistent with full string theory in the sense that in the approach of [184,185], too, the field theory limit of the off-shell continued string gluon amplitudes naturally leads to the Green's functions corresponding to the BMF with Feynman gauge.

We will continue to take a user's approach here and proceed directly to the relevant master formula; the interested reader may consult [50] or the introduction for more details. This master formula for the (color-ordered) contribution to the off-shell 1PI N-gluon amplitude due to a gluon loop reads

$$\begin{aligned}\Gamma^{a_1\dots a_N}_{\rm gluon}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -\frac{(-ig)^N}{4}\mathrm{tr}(T^{a_1}\dots T^{a_N})\lim_{C\to\infty}\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-CT}\\ &\times\prod_{k=1}^N\int_0^T d\tau_k\int d\theta_k\,\delta\Bigl(\frac{\tau_N}{T}\Bigr)\vartheta(\hat\tau_{1N})\prod_{l=1}^{N-1}\vartheta(\hat\tau_{l(l+1)})\sum_{p=P,A}\sigma_p Z_p\\ &\times\exp\biggl\{\sum_{i,j=1}^N\biggl[\frac12\hat G^C_{p,ij}k_j\cdot k_j+iD_i\hat G^C_{p,ij}\varepsilon_i\cdot k_j+\frac12 D_iD_j\hat G^C_{p,ij}\varepsilon_i\cdot\varepsilon_j\biggr]\biggr\}\bigg|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)}.\end{aligned}\tag{4.45}$$

Here the generators T^a are now fixed to be in the adjoint representation. We have defined $\sigma_P = 1$, $\sigma_A = -1$ (corresponding to periodic ($p = P$) and antiperiodic ($p = A$) boundary conditions in the original path integral), and

$$
\begin{aligned}
Z_A &= (2\cosh[CT/2])^4\,, \\
Z_P &= (2\sinh[CT/2])^4\,,
\end{aligned}
\tag{4.46}
$$

$$
\hat{G}^C_{P,A}(\tau_1,\theta_1;\tau_2,\theta_2) = G_B(\tau_1,\tau_2) + \theta_1\theta_2 G^C_{P,A}(\tau_1,\tau_2)\,,
\tag{4.47}
$$

where

$$
\begin{aligned}
G^C_P(\tau_1,\tau_2) &= 2\mathrm{sign}(\tau_1-\tau_2)\frac{\sinh[C(\frac{T}{2}-|\tau_1-\tau_2|)]}{\sinh[CT/2]}\,, \\
G^C_A(\tau_1,\tau_2) &= 2\mathrm{sign}(\tau_1-\tau_2)\frac{\cosh[C(\frac{T}{2}-|\tau_1-\tau_2|)]}{\cosh[CT/2]}\,.
\end{aligned}
\tag{4.48}
$$

As one can see from (4.45) the limit $C\to\infty$ and the sum $\sum_{p=P,A}$ serve the purpose to remove unwanted degrees of freedom circulating in the loop. Now, at fixed C, p the gluon loop master formula (4.45) is isomorphic to the spinor loop case (4.37). For two point case, this formal analogy allows us to reuse (4.41) and write the vacuum polarization function due to a gluon loop in the form

$$
\begin{aligned}
\Pi_{\mathrm{gluon}}(k^2) &= -\frac{1}{4}C(r)\frac{g^2}{(4\pi)^{D/2}}\mathrm{lim}_{C\to\infty}\sum_{p=P,A}\sigma_p\int_0^\infty\frac{dT}{T^{\frac{D}{2}}}e^{-CT}Z_p \\
&\quad\times\int_0^T d\tau_1\left(\dot{G}_{B12}\dot{G}_{B21}-G^C_{p12}G^C_{p21}\right)e^{-G_{B12}k^2}\,.
\end{aligned}
\tag{4.49}
$$

Similarly, in the three-point case the isomorphism implies that we can generalize the decomposition (4.43) to

$$
\Gamma_{\mathrm{gluon}} = -\frac{1}{4}\frac{g^3}{(4\pi)^{D/2}}\lim_{C\to\infty}\sum_{p=P,A}\sigma_p\left(\Gamma^3_{\mathrm{gluon}}(C,p)+\Gamma^2_{\mathrm{gluon}}(C,p)+\Gamma^{\mathrm{bt}}_{\mathrm{gluon}}(C,p)\right)\,,
\tag{4.50}
$$

where $\Gamma^{(\cdot)}_{\mathrm{gluon}}(C,p)$ differs from the corresponding $\Gamma^{(\cdot)}_{\frac{1}{2}}$ in (4.44) only by a replacement of m^2 by C, G_{Fij} by $G^C_{p,ij}$, and the insertion of Z_p under the T integral.
It remains to analyze the limit $C\to\infty$ and the sum over boundary conditions; however, this has already been done in complete generality in [37,50,134]. For the integrands appearing in the two-point and three-point cases, the general rules found there give

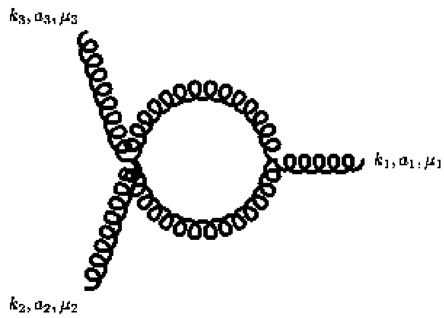


Figure 4.6: Two gluon vertex of gluon loop particle corresponds to $\Gamma_1^{\rm bt}$ when $2 \leftrightarrow 3$.

$$\begin{aligned}
\lim_{C\to\infty} \; \mathrm{e}^{-CT} \sum_{p=P,A} \sigma_p Z_p &= -8\,, \\
\lim_{C\to\infty} \; \mathrm{e}^{-CT} \sum_{p=P,A} \sigma_p Z_p G^C_{p,ij} G^C_{p,ji} &= 16\,, \\
\lim_{C\to\infty} \; \mathrm{e}^{-CT} \sum_{p=P,A} \sigma_p Z_p G^C_{p,12} G^C_{p,23} G^C_{p,31} &= 16\,.
\end{aligned} \tag{4.51}$$

For the two-point case, this results in

$$\Pi_{\rm gluon}(k^2) \;=\; 2C(r)\frac{g^2}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}\int_0^T d\tau_1 \big(\dot G_{B12}\dot G_{B21} + 2\big)\, e^{-G_{B12}k^2}\;. \tag{4.52}$$

For the Q-representation of the three-point case for the gluon loop, we can write, from (4.44), (4.50) and (5.53),

$$\Gamma_{\rm gluon} \;=\; 2\frac{g^3}{(4\pi)^{D/2}}(\Gamma^3_{\rm gluon} + \Gamma^2_{\rm gluon} + \Gamma^{\rm bt}_{\rm gluon})\,, \tag{4.53}$$

where, in terms of the spinor loop results of (4.44),

$$\begin{aligned}
\Gamma^3_{\rm gluon} &= \Gamma^3_{\frac12}(G_{F12}G_{F23}G_{F31} \to -2)\,, \\
\Gamma^2_{\rm gluon} &= \Gamma^2_{\frac12}(G_{F12}G_{F21} \to -2)\,, \\
\Gamma^{\rm bt}_{\rm gluon} &= \Gamma^{\rm bt}_{\frac12}(G_{F12}G_{F21} \to -2)\,,
\end{aligned} \tag{4.54}$$

where $\Gamma^{\rm bt}_{\rm gluon}$ corresponds to the boundary terms in the case of gluon loop particle which is shown in Fig. 4.6, when $2 \leftrightarrow 3$.

However, we must not forget the ghost loop contribution, which is necessary for the subtraction of the unphysical degrees of freedom of the gluon in the loop, and not contained in (4.45). This one is equal to the scalar loop contribution, but has to be taken with the opposite sign. Thus e.g. for the two-point case we get the total spin-one contribution

$$
\begin{aligned}
\Pi_1(k^2) &\equiv \Pi_{\rm gluon}(k^2) + \Pi_{\rm ghost}(k^2) \\
&= 2C(r)\frac{g^2}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}\int_0^T d\tau_1 \big(\dot G_{B12}\dot G_{B21}+2\big)\,e^{-G_{B12}k^2} \\
&\quad -C(r)\frac{g^2}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}\int_0^T d\tau_1 \dot G_{B12}\dot G_{B21}\,e^{-G_{B12}k^2} \\
&= C(r)\frac{g^2}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}\int_0^T d\tau_1 \Big(\dot G_{B12}\dot G_{B21}+4\Big)\,e^{-G_{B12}k^2}\,.
\end{aligned}
\tag{4.55}
$$

Finally, the S-representation of the gluon loop case, too, we have the option of using S_3^2 instead of Q_3^2 in the three-point vertex, with an additional boundary contribution $\tilde\Gamma^{\rm bt}_{\rm gluon}$, and it is easy to check that this form of the result still relates to the corresponding one for the spinor loop result by (4.54).

4.5 Summary

We will now summarize our results for the scalar, spinor, and gluon loop cases. For easy comparison, here we also rewrite the multiple τ_i - integrals in terms of the more standard Feynman/Schwinger parameters α_i. First, as usual we rescale $\tau_i = Tu_i$, after which the T integral can be done trivially. Then in the two-point case we set $u_2 = 0$ and change from u_1 to α, and in the three-point case we set $u_3 = 0$ and change from u_1, u_2 to $\alpha_1, \alpha_2, \alpha_3$ via

$$
\begin{aligned}
u_1 &= \alpha_2 + \alpha_3\,, \\
u_2 &= \alpha_3\,,
\end{aligned}
\tag{4.56}
$$

with $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Only six different parameter integrals appear:

Figure 4.7: Vacuum polarization diagrams for different particles running in the loop.

$$
\begin{aligned}
I^D_{\rm 2pt,B}(k^2) &= \int_0^1 d\alpha \frac{(1-2\alpha)^2}{\big(m^2+\alpha(1-\alpha)k^2\big)^{2-\frac{D}{2}}}\,,\\
I^D_{{\rm 2pt},F}(k^2) &= \int_0^1 d\alpha \frac{1}{\big(m^2+\alpha(1-\alpha)k^2\big)^{2-\frac{D}{2}}}\,,\\
I^D_{3,B}(k_1^2,k_2^2,k_3^2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3)\frac{(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)}{\Big(m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_1\alpha_3k_3^2\Big)^{3-\frac{D}{2}}}\,,\\
I^D_{3,F}(k_1^2,k_2^2,k_3^2) &= -\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3)\frac{1}{\Big(m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_1\alpha_3k_3^2\Big)^{3-\frac{D}{2}}}\,,\\
I^D_{2,B}(k_1^2,k_2^2,k_3^2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3)\frac{(1-2\alpha_2)^2(1-2\alpha_1)}{\Big(m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_1\alpha_3k_3^2\Big)^{3-\frac{D}{2}}}\,,\\
I^D_{2,F}(k_1^2,k_2^2,k_3^2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3)\frac{1-2\alpha_1}{\Big(m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_1\alpha_3k_3^2\Big)^{3-\frac{D}{2}}}\,.
\end{aligned}
\tag{4.57}
$$

Now we summarize our results for the vacuum polarization function for different particles running in the loop (4.21), (4.41), (4.55) as (see Fig. 4.7)

$$
\begin{aligned}
\Pi_0(k^2) &= -C(r)\frac{g^2}{(4\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right) I^D_{{\rm 2pt},B}(k^2)\,,\\
\Pi_{\frac12}(k^2) &= -2\Pi_0(k^2)\Big(I^D_{{\rm 2pt},B}\to I^D_{{\rm 2pt},B}-I^D_{{\rm 2pt},F}\Big)\,,\\
\Pi_1(k^2) &= \Pi_0(k^2)\Big(I^D_{{\rm 2pt},B}\to I^D_{{\rm 2pt},B}-4I^D_{{\rm 2pt},F}\Big)\,.
\end{aligned}
\tag{4.58}
$$

Here it is understood that the formula for Π_1 refers to the adjoint representation and to the massless case.
For the three point case, too, we can unify our results as follows:

$$
\Gamma_s = d_s\frac{g^3}{(4\pi)^{\frac{D}{2}}}(\Gamma^3_{\rm s}+\Gamma^2_{\rm s}+\Gamma^{\rm bt}_{\rm s})\,. \tag{4.59}
$$

Here $d_0 = d_1 = 1, d_{\frac{1}{2}} = -2$ and

$$\begin{aligned}
\Gamma_0^3 &= -\Gamma\left(3-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\mathrm{tr}\,(f_1f_2f_3)I^D_{3,B}(k_1^2,k_2^2,k_3^2) + (a_2\leftrightarrow a_3, f_2\leftrightarrow f_3, k_2\leftrightarrow k_3)\,,\\
\Gamma_0^2 &= \frac{1}{2}\Gamma\left(3-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\Big[\mathrm{tr}\,(f_1f_2)\big(\varepsilon_3\cdot k_1 I^D_{2,B}(k_1^2,k_2^2,k_3^2) - \varepsilon_3\cdot k_2 I^D_{2,B}(k_2^2,k_1^2,k_3^2)\big)\\
&\qquad +\mathrm{tr}\,(f_2f_3)\big(\varepsilon_1\cdot k_2 I^D_{2,B}(k_2^2,k_3^2,k_1^2) - \varepsilon_1\cdot k_3 I^D_{2,B}(k_3^2,k_2^2,k_1^2)\big)\\
&\qquad +\mathrm{tr}\,(f_3f_1)\big(\varepsilon_2\cdot k_3 I^D_{2,B}(k_3^2,k_1^2,k_2^2) - \varepsilon_2\cdot k_1 I^D_{2,B}(k_1^2,k_3^2,k_2^2)\big)\Big]\\
&\quad +(a_2\leftrightarrow a_3, f_2\leftrightarrow f_3, \varepsilon_2\leftrightarrow\varepsilon_3, k_2\leftrightarrow k_3)\,,\\
\Gamma_0^{\mathrm{bt}} &= \Gamma\left(2-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}[T^{a_2},T^{a_3}])\Big[\varepsilon_3\cdot f_1\cdot\varepsilon_2 I^D_{2\mathrm{pt},B}(k_1^2) + \varepsilon_1\cdot f_2\cdot\varepsilon_3 I^D_{2\mathrm{pt},B}(k_2^2)\\
&\qquad +\varepsilon_2\cdot f_3\cdot\varepsilon_1 I^D_{2\mathrm{pt},B}(k_3^2)\Big]\,,
\end{aligned}\tag{4.60}$$

$$\begin{aligned}
\Gamma_{\frac{1}{2}}^3 &= -\Gamma\left(3-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\mathrm{tr}\,(f_1f_2f_3)\big(I^D_{3,B}(k_1^2,k_2^2,k_3^2) - I^D_{3,F}(k_1^2,k_2^2,k_3^2)\big) + (2\leftrightarrow 3)\,,\\
\Gamma_{\frac{1}{2}}^2 &= \frac{1}{2}\Gamma\left(3-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\Big[\mathrm{tr}\,(f_1f_2)\Big(\varepsilon_3\cdot k_1\big(I^D_{2,B}(k_1^2,k_2^2,k_3^2) - I^D_{2,F}(k_1^2,k_2^2,k_3^2)\big)\\
&\qquad -\varepsilon_3\cdot k_2\big(I^D_{2,B}(k_2^2,k_1^2,k_3^2) - I^D_{2,F}(k_2^2,k_1^2,k_3^2)\big)\Big) + \text{ 2 perm.}\Big] + (2\leftrightarrow 3)\,,\\
\Gamma_{\frac{1}{2}}^{\mathrm{bt}} &= \Gamma\left(2-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}[T^{a_2},T^{a_3}])\Big[\varepsilon_3\cdot f_1\cdot\varepsilon_2\big(I^D_{2\mathrm{pt},B}(k_1^2) - I^D_{2\mathrm{pt},F}(k_1^2)\big) + \text{ 2 perm.}\Big]\,,
\end{aligned}\tag{4.61}$$

$$\begin{aligned}
\Gamma_1^3 &= -\Gamma\left(3-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\mathrm{tr}\,(f_1f_2f_3)\big(I^D_{3,B}(k_1^2,k_2^2,k_3^2) - 4I^D_{3,F}(k_1^2,k_2^2,k_3^2)\big) + (2\leftrightarrow 3)\,,\\
\Gamma_1^2 &= \frac{1}{2}\Gamma\left(3-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\Big[\mathrm{tr}\,(f_1f_2)\Big(\varepsilon_3\cdot k_1\big(I^D_{2,B}(k_1^2,k_2^2,k_3^2) - 4I^D_{2,F}(k_1^2,k_2^2,k_3^2)\big)\\
&\qquad -\varepsilon_3\cdot k_2\big(I^D_{2,B}(k_2^2,k_1^2,k_3^2) - 4I^D_{2,F}(k_2^2,k_1^2,k_3^2)\big)\Big) + \text{ 2 perm.}\Big] + (2\leftrightarrow 3)\,,\\
\Gamma_1^{\mathrm{bt}} &= \Gamma\left(2-\frac{D}{2}\right)\mathrm{tr}(T^{a_1}[T^{a_2},T^{a_3}])\Big[\varepsilon_3\cdot f_1\cdot\varepsilon_2\big(I^D_{2\mathrm{pt},B}(k_1^2) - 4I^D_{2\mathrm{pt},F}(k_1^2)\big) + \text{ 2 perm.}\Big]\,.
\end{aligned}\tag{4.62}$$

The parameter integrals at the three-point level are already highly nontrivial, and we refer the reader to [147], and refs. therein, for methods for their evaluation.
Further, note that the "loop replacement rules" now have, for both the two-point and three-point cases, assumed the form

$$\Gamma_{\frac{1}{2}}^{(\cdot)} = \Gamma_0^{(\cdot)}\Big(I^D_{(\cdot),B}\to I^D_{(\cdot),B} - I^D_{(\cdot),F}\Big)\,, \tag{4.63}$$

$$\Gamma_1^{(\cdot)} = \Gamma_0^{(\cdot)}\Big(I^D_{(\cdot),B}\to I^D_{(\cdot),B} - 4I^D_{(\cdot),F}\Big)\,. \tag{4.64}$$

Further, it can be easily checked that, for the bulk terms $\Gamma^{3,2}_{(\cdot)}$, the terms with the interchange $(2 \leftrightarrow 3)$ just provide the other half of a color commutator $[T^{a_2}, T^{a_3}]$, so that for them, as for the boundary terms, the color structure factors out in a $\mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}])$. Therefore we can now use

$$\mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}]) = iC(r) f^{a_1 a_2 a_3} \,, \tag{4.65}$$

to get the expected proportionality to $f^{a_1 a_2 a_3}$. Thus we can write

$$\begin{aligned} \Gamma_s^{a_1 a_2 a_3} &= d_s \frac{g^3}{(4\pi)^{\frac{D}{2}}} \mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}])(\gamma_{\mathrm{s}}^3 + \gamma_{\mathrm{s}}^2 + \gamma_{\mathrm{s}}^{\mathrm{bt}}) \\ &= i f^{a_1 a_2 a_3} C(r) d_s \frac{g^3}{(4\pi)^{\frac{D}{2}}} (\gamma_{\mathrm{s}}^3 + \gamma_{\mathrm{s}}^2 + \gamma_{\mathrm{s}}^{\mathrm{bt}}) \,, \end{aligned} \tag{4.66}$$

with

$$\begin{aligned} \gamma_0^3 &= -\Gamma\left(3 - \frac{D}{2}\right) \mathrm{tr}\,(f_1 f_2 f_3) I^D_{3,B}(k_1^2, k_2^2, k_3^2) \,, \\ \gamma_0^2 &= \frac{1}{2}\Gamma\left(3 - \frac{D}{2}\right) \Big[\mathrm{tr}\,(f_1 f_2)\Big(\varepsilon_3 \cdot k_1 I^D_{2,B}(k_1^2, k_2^2, k_3^2) - \varepsilon_3 \cdot k_2 I^D_{2,B}(k_2^2, k_1^2, k_3^2)\Big) \\ &\qquad + \mathrm{tr}\,(f_2 f_3)\Big(\varepsilon_1 \cdot k_2 I^D_{2,B}(k_2^2, k_3^2, k_1^2) - \varepsilon_1 \cdot k_3 I^D_{2,B}(k_3^2, k_2^2, k_1^2)\Big) \\ &\qquad + \mathrm{tr}\,(f_3 f_1)\Big(\varepsilon_2 \cdot k_3 I^D_{2,B}(k_3^2, k_1^2, k_2^2) - \varepsilon_2 \cdot k_1 I^D_{2,B}(k_1^2, k_3^2, k_2^2)\Big)\Big] \,, \\ \gamma_0^{\mathrm{bt}} &= \Gamma\left(2 - \frac{D}{2}\right) \Big[\varepsilon_3 \cdot f_1 \cdot \varepsilon_2 I^D_{2\mathrm{pt},B}(k_1^2) + \varepsilon_1 \cdot f_2 \cdot \varepsilon_3 I^D_{2\mathrm{pt},B}(k_2^2) + \varepsilon_2 \cdot f_3 \cdot \varepsilon_1 I^D_{2\mathrm{pt},B}(k_3^2)\Big] \,, \end{aligned} \tag{4.67}$$

and the $\gamma^{(\cdot)}_{\frac{1}{2},1}$'s obtained from the $\gamma_0^{(\cdot)}$'s by the rule (4.63) resp. (4.64).
In the S-representation, γ_0^2 gets replaced by (there is no change for γ_0^3 because $S_3^3 \equiv Q_3^3$)

$$\tilde\gamma_0^2 = \frac{1}{2}\Gamma\left(3 - \frac{D}{2}\right) \Big\{\mathrm{tr}\,(f_1 f_2)\Big[\frac{r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} I^D_{2,B}(k_1^2, k_2^2, k_3^2) - \frac{r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3} I^D_{2,B}(k_2^2, k_1^2, k_3^2)\Big] + 2\,\mathrm{perm.}\Big\} \,, \tag{4.68}$$

and one has the additional boundary contribution

$$\tilde\gamma_0^{\mathrm{bt}} = -\frac{1}{2}\Gamma\left(2 - \frac{D}{2}\right) \Big\{\Big[\mathrm{tr}(f_1 f_2)\rho_3 - \mathrm{tr}(f_3 f_1)\rho_2\Big] I^D_{2\mathrm{pt},B}(k_1^2) + 2\,\mathrm{perm.}\Big\} \,. \tag{4.69}$$

The rules (4.63) and (4.64) continue to hold.

4.6 Comparison with previous results

In this section we study the connection between our results for the three-gluon vertex and previous work. Since our treatment of the gluon loop case is equivalent to the use of the BFM with quantum Feynman gauge, we expect the Binger-Brodsky relation (4.8) to hold; and indeed this relation here follows immediately from the replacement rules (4.63) and (4.64). (Similarly we can use (4.58) to verify the vanishing of the gluon propagator in $\mathcal{N}=4$ SYM theory.)
For the same reason, the QED-like Ward identity (4.6) should be fulfilled not only for the scalar and spinor, but also for the gluon loop case. Here it is advantageous to use the S-representation instead of the Q-representation. Since S_3^2 is transversal, the Ward identity then involves only the boundary terms $\gamma_0^{\rm bt}, \tilde{\gamma}_0^{\rm bt}$, and can be easily verified using (4.58), (4.67),(4.69).
Next, we proceed to the less straightforward task of relating our representation to the Ball-Chiu decomposition. As usual we start with the scalar case. Comparing our final result (4.66),(4.67) with (4.4), (4.5) we first note that $T_H = {\rm tr}(f_1 f_2 f_3)$. Thus we must identify

$$H(k_1^2,k_2^2,k_3^2) \quad = \quad C(r)\frac{d_0 g^2}{(4\pi)^{D/2}}\Gamma\left(3-\frac{D}{2}\right) I^D_{3,B}(k_1^2,k_2^2,k_3^2)\ , \tag{4.70}$$

which is indeed totally symmetric in its arguments. The normalization of the generators is ${\rm tr}(T^aT^b) = C(r)\delta^{ab}$, where for $SU(N)$ one has $C(N)=\frac{1}{2}$ for the fundamental and $C(G)=N$ for the adjoint representation.
Further, it is also easy to recognize the functions A and B functions as symmetric and antisymmetric combinations of the functions contained in $\gamma_0^{\rm bt}$:

$$\begin{aligned} A(k_1^2,k_2^2;k_3^2) &= -C(r)\frac{d_0 g^2}{2(4\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\left[I^D_{2{\rm pt},B}(k_1^2)+I^D_{2{\rm pt},B}(k_2^2)\right],\\ T_A &= \varepsilon_1\cdot\varepsilon_2(k_1\cdot\varepsilon_3-k_2\cdot\varepsilon_3)\ , \end{aligned} \tag{4.71}$$

and

$$\begin{aligned} B(k_1^2,k_2^2;k_3^2) &= -C(r)\frac{d_0 g^2}{2(4\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\left[I^D_{2{\rm pt},B}(k_1^2)-I^D_{2{\rm pt},B}(k_2^2)\right],\\ T_B &= \varepsilon_1\cdot\varepsilon_2(k_1\cdot\varepsilon_3+k_2\cdot\varepsilon_3)\,. \end{aligned} \tag{4.72}$$

Coming to the structure F, the fact that T_F is transversal suggests that we should again use the transversal structure $\tilde{\gamma}_0^2$ rather than γ_0^2; the question is, how to choose the still undetermined vectors r_i? By inspection one finds that, with the cyclic choice $r_1=k_2-k_3, r_2=k_3-k_1, r_3=k_1-k_2$, and using the antisymmetry of f_i, e.g. the first term in braces in (4.68) turns into

$${\rm tr}(f_1f_2)\frac{k_1\cdot f_3\cdot k_2}{(k_1-k_2)\cdot k_3}\left[I^D_{2,B}(k_1^2,k_2^2,k_3^2)-I^D_{2,B}(k_2^2,k_1^2,k_3^2)\right]. \tag{4.73}$$

Noting that

$$T_F = \frac{1}{2}{\rm tr}(f_1f_2)k_1\cdot f_3\cdot k_2\ , \tag{4.74}$$

we are led to set

$$F(k_1^2,k_2^2;k_3^2) \quad = \quad C(r)\frac{d_0 g^2}{(4\pi)^{D/2}}\Gamma\left(3-\frac{D}{2}\right)\frac{I^D_{2,B}(k_1^2,k_2^2,k_3^2)-I^D_{2,B}(k_2^2,k_1^2,k_3^2)}{k_1^2-k_2^2}\,, \tag{4.75}$$

where we have also used momentum conservation to rewrite

$$(k_1-k_2)\cdot k_3 = k_2^2-k_1^2\,. \tag{4.76}$$

Thus the remaining structure C must match $\tilde{\gamma}_0^{\rm bt}$, and indeed one has

$$T_C = \frac{1}{2}{\rm tr}(f_1 f_2)(k_1-k_2)\cdot\varepsilon_3 = -\frac{1}{2}{\rm tr}(f_1 f_2)\rho_3(k_1^2-k_2^2)\,, \tag{4.77}$$

leading to the identification

$$C(k_1^2,k_2^2;k_3^2) \quad = \quad -C(r)\frac{d_0 g^2}{(4\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\frac{I^D_{2{\rm pt},B}(k_1^2)-I^D_{2{\rm pt},B}(k_2^2)}{k_1^2-k_2^2}\,. \tag{4.78}$$

Note that F and C are indeed symmetric functions in the first two arguments, and that C is actually independent of k_3^2. Also, A is the only one of the functions having an UV divergence (since in B the expression in square brackets is $O(\epsilon)$), and B and C are simply related by [147]

$$2B = (k_1^2-k_2^2)C\,. \tag{4.79}$$

Passing from the scalar to the spinor and gluon loop cases using (4.63) and (4.64) will obviously not change anything essential in this analysis.
For the (massive) spinor loop case we have also verified the above correspondences explicitly, using the formulas for the functions A to H given in [147] (to be precise, we have done this check for A,B,C with arbitrary momentum, for F specializing to $k_4^2=0$ and for H specializing to $k_1^2=k_3^2=0$. This provides also a check on the much more involved calculations of [147]).
From this comparison one can easily see that our analysis for the three-gluon vertex is more general than Ball-Chiu, in other words we have generalized the three-gluon vertex which for some especial choice of the 4-momentum r_i, it coincides perfectly with the Ball-Chiu form factor decomposition. In Appendix F we use the conventional Feynman rules to do some calculation for the scalar case to compare with our results.

4.7 Comparison with the effective action

One-loop effective actions [54, 193] are space-time integrals of Lorentz and gauge-invariant expressions. They are closely related to traces of heat kernels, known as Schwinger-DeWitt [194], Gilkey-Seeley [195] or Hadamard [196] coefficients. In flat space-time gauge theory, these coefficients are polynomials consisting of a matrix potential, the gauge-field strength tensor, and gauge-covariant derivatives. There are many equivalent forms for these coefficients essentially due to the Bianchi identity and the product rule for covariant derivatives. Furthermore, the physically interesting functional trace allows one to cyclically exchange matrix factors and to integrate by parts. Increasing use of computer algebra [197] and new methods to calculate effective actions [37, 84, 113, 198] extend our knowledge of heat kernel coefficients to higher and higher order. To manage the increasing number of terms in them and compare results obtained by different methods, a minimal set of invariants is needed in terms of which all results can be expressed. In addition, an algorithm to expand a given gauge-invariant Lorentz scalar into this minimal set should be provided.

This problem was mainly considered in general relativity where the tensor polynomials consist of the Riemann tensor, the metric tensor, and covariant derivatives. This is more complicated than flat space-time gauge theory due to the symmetry properties of the Riemann tensor. However, Fulling et al. [199] used group representation methods to determine the numbers of independent monomials, so that an appropriate subset of all monomials can be chosen to be a basis. Nevertheless, this subset has to be chosen by hand because there is no known general construction principle.

For gauge theory in flat space-time, van de Ven [200] constructed a basis up to mass dimension ten (fifth order of the inverse mass expansion). But he also chose the basis elements by hand and did not present a general construction principle. For the expansion of effective actions of matter, gauge fields, and gravity in terms of Barvinsky-Vilkovisky form factors [201], a basis set of non-local invariants up to third order in the curvature was defined [202]. Analyses the formal structure of Lorentz and gauge-invariant monomials in nonabelian gauge theories with matter in flat space-time has been done in [203].

It will be instructive to compare our results for the three-point amplitude with the low energy expansion of the one-loop QCD effective action induced by a loop particle of mass m. The general form of this expansion is

$$\Gamma_0[F] = \int_0^\infty \frac{dT}{T}\,\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^{D/2}}\,\mathrm{tr}\int dx_0 \sum_{n=2}^{\infty}\frac{(-T)^n}{n!}\,O_n[F]\,, \tag{4.80}$$

where $O_n(F)$ is a Lorentz and gauge invariant expression of mass dimension $2n$. For the scalar loop, in [113, 198] this expansion was obtained to order $O(T^6)$

$$\begin{aligned} O_1 &= 0\,,\\ O_2 &= -\frac{1}{6}g^2F_{\mu\nu}F_{\mu\nu}\,,\\ O_3 &= -\frac{2}{15}\,ig^3\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\kappa} - \frac{1}{20}g^2D_\lambda F_{\mu\nu}D^\lambda F^{\mu\nu}\,, \end{aligned} \tag{4.81}$$

where the nonabelian covariant derivative is $D_\mu \equiv \partial_\mu + igA^a_\mu T^a$, with $[T^a, T^b] = if^{abc}T^c$.
To see the relation with our form factor decomposition, it will be sufficient to consider the $n = 2$ and $n = 3$ terms:

$$
\begin{aligned}
O_2 &= -\frac{1}{6}g^2 F_{\mu\nu}F_{\mu\nu}\,,\\
O_3 &= -\frac{2}{15}\,ig^3\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\kappa} - \frac{1}{20}g^2 D_\lambda F_{\mu\nu}D^\lambda F^{\mu\nu}\,.
\end{aligned}
\tag{4.82}
$$

Here changing from the scalar to the spinor or gluon loop will change only the coefficients in the expansion (4.80), not its structure. Comparing with, e.g., (4.67) we easily recognize the correspondences

$$
\begin{aligned}
&\gamma^3_{(\cdot)} \leftrightarrow F_\kappa{}^\lambda F_\lambda{}^\mu F_\mu{}^\kappa = f_\kappa{}^\lambda f_\lambda{}^\mu f_\mu{}^\kappa + \text{higher point terms}\,,\\
&\gamma^2_{(\cdot)} \leftrightarrow (\partial + ig\underbrace{A)F(\partial}+igA)F \equiv (igA)\,F\,\partial F\,,\\
&\gamma^{\rm bt}_{(\cdot)} \leftrightarrow (f + ig\underbrace{[A,A])(f}+ig[A,A]) \equiv (ig)[A,A]\,f\,.
\end{aligned}
\tag{4.83}
$$

Thus all the pieces of our form factor decomposition have a simple meaning in terms of the effective action. Note that commutator terms are always generated by boundary terms in the IBP, and that our three-point results allow us to predict certain terms in the higher-point gluon amplitudes using the knowledge that any "abelian" field strength tensor in the nonabelian effective action must appear as part of the full nonabelian field strength tensor including the commutator term. Note also that the tensor structure multiplying the function S in the Ball-Chiu decomposition does not correspond to anything in the expansion (4.80), which is one way of understanding why S turned out to be zero in the calculations of [124,147]. Since the structure of the effective action is loop-independent, this observation allows us also to predict that the vanishing of S is not a one-loop accident, and will be found to persist at higher loop orders.

4.8 Conclusions

In this section we conclude our results and discuss some extension of this formalism. In this Chapter we have calculated the one-loop QCD three-gluon vertex for scalar, spinor and gluon loop particle in a unifying way which gives us a compact result involving only six different parameter integrals. We have established the precise relation of our result to the standard Ball-Chiu decomposition, and also verified this relation for the massive spinor loop case using the explicit result of [147].

As we already mentioned in the introduction, even in a four-dimensional calculation the use of the vertex as a building block for higher loop calculations will in most cases make it necessary to know its D-dimensional continuation. For that reason we have kept the full D-dependence as much as possible. As one can see in fact our result for the scalar case is complete in this sense and it holds for arbitrary D. For the spinor case the only place where we have used $D = 4$ is in the normalization of the path integral, which however corresponds to the usual fixing of $\mathrm{tr}_\gamma \mathbb{1} = 4$ in dimensional regularization. But in the gluon loop case we have used $D = 4$ in a nontrivial way, namely already in the derivation of (4.45). Here a true extension to other spacetime dimensions would require some more work. For the purpose of dimensional regularization it is, sufficient to note that our result for the gluon loop case corresponds to the dimensional reduction variant of the dimensional regularization proposed in [33].

In this calculation, three main advantages of our approach have emerged.

First, the IBP procedure generates the standard transversality-based Ball-Chiu decomposition of the vertex almost automatically, bypassing the usual tedious analysis of the nonabelian Ward identities. Let us recapitulate how this happens: for the bulk terms, in the IBP all polarization vectors get absorbed into "abelian" field strength tensors $f_i^{\mu\nu}$, and thus become transversal. In the abelian case, there would be no boundary contributions, and one would have achieved manifest transversality at the integrand level. In the nonabelian case there are boundary terms, and those combine into commutator terms that carry all non-transversality, and generally contribute to the covariantization of some lower-point bulk term.

Second, this emergence of field strength tensors in the IBP allows one to maintain a close relation between the momentum space amplitudes and the low energy effective action, and thus to profit from the superior organization of the latter with respect to gauge invariance. This has led us to predict that the vanishing of the coefficient function S of the Ball-Chiu decomposition will be found to persist beyond one-loop.

Third, the integrands of the spinor and gluon loop contributions can be obtained from the scalar loop one trivially using the off-shell extended Bern-Kosower "loop replacement rules" (4.63) and (4.64). The gluon loop result corresponds to a field theory calculation in the BFM with quantum Feynman gauge, and thus to the preferred "gauge-invariant vertex" which fulfills the simple Ward identity (4.6) and the SUSY-related identity (4.8). The latter here appears as a simple consequence of the replacement rules and thus relates to worldline SUSY.

We had taken the validity of the replacement rules off-shell at the beginning. Our method of calculating the off-shell three-gluon vertex would have been even more efficient; in fact, more efficient than the combined efforts of [124, 138, 147] that was necessary to arrive at an explicit results for the scalar, spinor, and gluon loop contribution to the three-gluon vertex with standard field theory methods. Before applying it to higher-point vertices, it will thus be important to show the validity of this off-shell extension in general. With hindsight, this can be done as follows: it is sufficient to show the validity of the replacement rules for the effective action. Let us consider the spinor loop case first. Here it was shown in [50] that the replacement rule in the abelian case holds for the off-shell N - photon amplitudes. Thus it holds also for the abelian effective action. The nonabelian effective action in the low energy expansion can be decomposed into terms that are Lorentz scalars built from covariant derivatives and field strength tensors. Each such term in general will, after Fourier transformation, contribute to momentum space functions with various different numbers of legs; e.g., the term $\mathrm{tr}\,(D_\mu F_{\alpha\beta} D^\mu F^{\alpha\beta})$ will contribute to the N - point functions with N between two and six. Generally, each such term in the nonabelian effective Lagrangian has a "core" term, which has a counterpart already in the abelian case (in the example this would be $\partial_\mu f_{\alpha\beta}\partial^\mu f^{\alpha\beta}$) and a number of "covariantizing" terms that all involve commutators, and belong to amplitudes with more legs than the core term. For the core term the IBP leads from bulk term to bulk term and is formally identical to the abelian case, so that the replacement rule holds. But a core term in the effective action appears combined with all its covariantizing terms, all sharing the same coefficient. The replacement rule induces a change of this coefficient defined through a cycle $\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}$ which is multiplied by a $\mathrm{tr}\,(f_{i_1}f_{i_2}\cdots f_{i_n})$ that in the effective action corresponds to a $\mathrm{tr}\,(f^n)$. Consistency is therefore possible only if the same change of coefficient applies also to all the terms where one or several of the factors $f^{\mu\nu}$ are replaced by a $[A, A]$ term, or where a ∂ acting on some F is replaced by a $[A, F]$. This settles the spinor loop case. For the gluon loop case, it is sufficient to remember that the corresponding master formula before taking the limit $C \to \infty$ and the projector sum

$\sum_{p=P,A}$, eq. (4.45), is still isomorphic to the spinor loop one (4.37).
Based on this general validity of the off-shell replacement rules and the general IBP algorithm developed in [55] we anticipate that with the method presented here a first explicit calculation of the four-point vertex should be well in reach. We will discuss this object in the next Chapter. Less straightforward but very interesting would be the extension of the formalism presented here to the gravitational case. The one-loop three-graviton vertices have already been extensively studied for their conformal properties (see [204] and refs. therein). As far as external gravitons are concerned, suitable string-inspired representations exist already for the (off-shell) one-loop N - graviton amplitudes with a scalar or spinor loop [51, 74] as well as for a photon loop [152, 205]. However, a suitable IBP procedure still remains to be developed.

Chapter 5

Form factor decomposition of the off-shell four-gluon vertex

5.1 Introduction

[1] In this Chapter we continue our investigation which has been started in Chapter 4 for one-loop correction of the off-shell three-gluon vertex, here we extend our calculations for the off-shell four-gluon vertex. Off-shell four-gluon is an interesting object because of the following reasons:

- First of all, this vertex is the only primitively divergent one that allows for the formation of bound state (glueball-) poles, a phenomenon usually restricted to higher, superficially convergent vertices.
- Second, the vertex describes quantum corrections to elementary gluon-gluon scattering, which might be important e.g. for the description of gluon-gluon interactions in the high temperature quark-gluon plasma phase of QCD.
- Third, a number of studies indicate that the infrared structure of the correlation functions of Yang-Mills theory is connected to the confining properties of the theory via the so called "Gribov-Zwanziger" scenario [157, 158].
- One of the basic ingredients to the running coupling $\alpha^{4g}(p^2)$ is the dressing function of the nonperturbative four-gluon vertex [206].

So there are several motivations to study this object. Four-gluon vertex is a highly complex object because of its rich tensor structure which based on four color and four Lorentz indices. This correlation function is poorly understood and there has been limited investigation related to this object.

Schwinger-Dyson equations (SDEs) provide a non-perturbative tool for studying quantum field theories. Being an infinite set of coupled equations for the N-point functions of the theory, they contain in principle all information about the observables of the underlying theory. In addition, since they are derived from the renormalized action, they are fully renormalized equations and,

[1] This Chaper is based on [207, 208].

given a certain regularization and renormalization scheme, all divergences are absorbed in appropriately determined renormalization constants [209]. Since SDEs are an infinite set of coupled equations, in practical calculations, however, truncation of the full infinite system to a finite subset of equations have to be performed. In general these truncations will interfere with the renormalization of SDEs and divergences will reappear. In addition an appropriate choice of a regulator might also introduce divergences, as e.g., a hard cut off in numerical calculation, however in practical calculations divergences have to be properly subtracted. Furthermore, perturbative renormalization is not sufficient due to the self-consistent nature of SDEs. In Schwinger-Dyson studies of Landau gauge Yang-Mills propagator in four dimensions especially spurious quadratic divergences caused problems, since they are related to the breaking of the gauge symmetry the regularization/truncation scheme. Successful treatments were, e.g., identification and subtraction in the relevant tensor-components of the gluon self-energy [210,211], and the construction of explicit subtraction terms within the integral kernels [212–214]. On the other hand, logarithmic divergences can be treated straightforwardly in a MOM-scheme.
In Landau gauge studies gluon propagator SDE truncation were chosen such that all terms containing the four-gluon interaction have been neglected [215]. A special role is played by the tadpole diagram: It only contributes a quadratically divergent constant, which is then removed in the renormalization process. The other terms with a four-gluon interaction are of two-loop order, the so-called sunset and squint diagrams which has been studied recently [209].

In this Chapter we present our recent calculation in one-loop correction to the off-shell four-gluon vertex in scalar, spinor and gluon loop particles in the BFM with the Feynman gauge based on the worldline formalism and string inspired method.

5.2 Integration by part procedure

Again our starting point is the "Bern-Kosower master formula" (4.9):

$$\begin{aligned}
&\Gamma^{a_1\dots a_N}_{\text{scalar}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N]\\
&=(-ig)^N\mathrm{tr}(T^{a_1}\dots T^{a_N})(2\pi)^D i\delta(\sum k_i)\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\
&\times\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\dots\int_0^{\tau_{N-2}}d\tau_{N-1}\\
&\times\exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}G_{Bij}k_i\cdot k_j-i\dot G_{Bij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Bigg|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)},
\end{aligned}\tag{5.1}$$

performing the expansion of the exponential and keep those terms which are linear in polarization vectors up to four ε's we get the following amplitude for the 1PI four-gluon vertex which diagrammatically is shown in Fig. 5.1 for the scalar loop:

$$\begin{aligned}
\Gamma_0^{a_1a_2a_3a_4}[k_1,\varepsilon_1;k_2,\varepsilon_2;k_3,\varepsilon_3;k_4,\varepsilon_4] &= (-ig)^4\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\
&\times\int_0^T d\tau_1 d\tau_2 d\tau_3\,(-i)^4P_4\,e^{\sum_{i<j=1}^4 G_{Bij}k_j\cdot k_j},
\end{aligned}\tag{5.2}$$

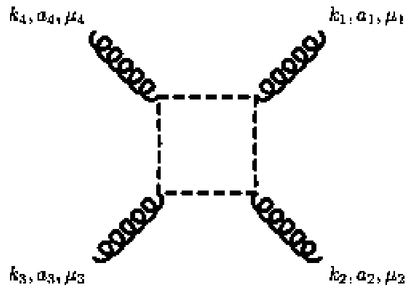


Figure 5.1: Four-gluon vertex for scalar loop particle.

where as before we fixed $\tau_4 = 0$, and the polynomial P_4 is a function of $\dot{G}_{Bij}$'s and $\ddot{G}_{Bij}$'s as:

$$
\begin{aligned}
P_4(\dot{G}_{Bij}, \ddot{G}_{Bij}) = \quad & + \quad \ddot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \, \ddot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 + \ddot{G}_{B13}\varepsilon_1 \cdot \varepsilon_3 \, \ddot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 + \ddot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \, \ddot{G}_{B23}\varepsilon_2 \cdot \varepsilon_3 \\
& - \quad \dot{G}_{B1i}\varepsilon_1 \cdot k_i \, \dot{G}_{B2j}\varepsilon_2 \cdot k_j \, \ddot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 - \dot{G}_{B1i}\varepsilon_1 \cdot k_i \, \dot{G}_{B3j}\varepsilon_3 \cdot k_j \, \ddot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \\
& - \quad \dot{G}_{B1i}\varepsilon_1 \cdot k_i \, \dot{G}_{B4j}\varepsilon_4 \cdot k_j \, \ddot{G}_{B23}\varepsilon_2 \cdot \varepsilon_3 - \dot{G}_{B2i}\varepsilon_2 \cdot k_i \, \dot{G}_{B3j}\varepsilon_3 \cdot k_j \, \ddot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \\
& - \quad \dot{G}_{B2i}\varepsilon_2 \cdot k_i \, \dot{G}_{B4j}\varepsilon_4 \cdot k_j \, \ddot{G}_{B13}\varepsilon_1 \cdot \varepsilon_3 - \dot{G}_{B3i}\varepsilon_3 \cdot k_i \, \dot{G}_{B4j}\varepsilon_4 \cdot k_j \, \ddot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \\
& + \quad \dot{G}_{B1i}\varepsilon_1 \cdot k_i \, \dot{G}_{B2j}\varepsilon_2 \cdot k_j \, \dot{G}_{3k}\varepsilon_3 \cdot k_k \, \dot{G}_{4l}\varepsilon_4 \cdot k_l \, .
\end{aligned}
\tag{5.3}
$$

Here for $N = 4$ as the previous Chapter to apply the "Bern-Kosower replacement rules" one needs to remove all the $\ddot{G}_{Bij}$'s by adding some total derivatives. Here the situation is more complicated than $N = 3$ and $N = 2$ because IBP is not straight forward and it is ambiguous as we have discussed in Chapter 3.

In the following we explain a partial integration algorithm which allows one to remove all the $\ddot{G}_{Bij}$'s contained in the original P_N photon or gluon amplitudes [47].

Such an "impartial" partial integration algorithm was defined in Section 3.4. Since we are working on the case of $N = 4$ let us look at one term in P_4 in (5.3) for instance the first term $\ddot{G}_{B12}\ddot{G}_{B34}$ and try to remove these second derivatives step by step:

$$
\ddot{G}_{B12}\ddot{G}_{B34}e^{(\cdot)} - \frac{1}{4}(\partial_1 - \partial_2)(\partial_3 - \partial_4)\left[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\right], \tag{5.4}
$$

where again $e^{(\cdot)} = e^{\sum_{i<j=1}^{4} G_{Bij} k_i \cdot k_j}$ and $\partial_i = \frac{\partial}{\partial \tau_i}$.

By trivial algebra one can see that $\ddot{G}_{B12}\ddot{G}_{B34}e^{(\cdot)}$ transforms to

$$
\begin{aligned}
\ddot{G}_{B12}\ddot{G}_{B34}e^{(\cdot)} \quad \to \quad & \frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\Big\{ \left[\dot{G}_{B1i}k_1 \cdot k_i - \dot{G}_{B2i}k_2 \cdot k_i\right]\left[\dot{G}_{B3j}k_3 \cdot k_j - \dot{G}_{B4j}k_4 \cdot k_j\right] \\
& - \quad \ddot{G}_{B13}k_1 \cdot k_3 + \ddot{G}_{B23}k_2 \cdot k_3 + \ddot{G}_{B14}k_1 \cdot k_4 - \ddot{G}_{B24}k_2 \cdot k_4 \Big\} e^{(\cdot)} \, .
\end{aligned}
\tag{5.5}
$$

The terms in the second line have to be further processed. Considering just the first one of them, since both variables appearing in $\ddot{G}_{B13}$ were active in the first step, both must also be used in the second one. We need to add the following total derivative to the first term (and similar to the others):

$$\frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\ddot{G}_{B13}e^{(\cdot)}+\frac{1}{8}(\partial_1-\partial_3)\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B13}k_1\cdot k_3e^{(\cdot)}\,. \tag{5.6}$$

This yields:

$$\begin{aligned}\frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\ddot{G}_{B13}e^{(\cdot)} \quad\rightarrow\quad & \frac{1}{8}\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B13}\Big[\dot{G}_{B1i}k_1\cdot k_i-\dot{G}_{B3i}k_3\cdot k_i\Big]e^{(\cdot)}\\ & +\frac{1}{8}\dot{G}_{B13}\Big[\ddot{G}_{B12}\dot{G}_{B34}-\ddot{G}_{B34}\dot{G}_{B12}\Big]e^{(\cdot)}\,.\end{aligned} \tag{5.7}$$

Considering again the first term in the second line, only τ_1 was active in the previous step. Therefore only τ_2 must be used now, and the third step is the final one, so by adding another total derivative:

$$\frac{1}{8}\dot{G}_{B13}\ddot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}+\frac{1}{8}\partial_2\Big[\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B13}e^{(\cdot)}\Big]\,, \tag{5.8}$$

this term transforms to

$$\frac{1}{8}\dot{G}_{B13}\ddot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\rightarrow\frac{1}{8}\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B13}\dot{G}_{B2i}k_2\cdot k_ie^{(\cdot)}\,. \tag{5.9}$$

This prescription treats all variables on the same footing, and therefore must lead to a permutation symmetric result. The nontrivial fact is that the process terminates after a finite number of steps, and does not become cyclic (as would be the case if, for example, one would always treat the indices in a $\ddot{G}_{Bij}$ symmetrically). This is not difficult to derive from the fact that, for any term in P_N , the indices appearing in the $\ddot{G}_{Bij}$'s and the first indices of the $\dot{G}_{Bij}$'s are associated to the polarization vectors, and thus must all take different values.

This algorithm transforms P_4 into Q_4 which has four different polynomials [47] as was also discussed in Chapter 3:

$$\begin{aligned}Q_4 &= Q_4^4+Q_4^3+Q_4^2-Q_4^{22}\,,\\ Q_4^4 &= \dot{G}(1234)+\dot{G}(1243)+\dot{G}(1324)\,,\\ Q_4^3 &= \dot{G}(123)T(4)+\dot{G}(234)T(1)+\dot{G}(341)T(2)+\dot{G}(412)T(3)\,,\\ Q_4^2 &= \dot{G}(12)T(34)+\dot{G}(13)T(24)+\dot{G}(14)T(23)\,,\\ &\quad +\dot{G}(23)T(14)+\dot{G}(24)T(13)+\dot{G}(34)T(12)\,,\\ Q_4^{22} &= \dot{G}(12)\dot{G}(34)+\dot{G}(13)\dot{G}(24)+\dot{G}(14)\dot{G}(23)\,,\end{aligned} \tag{5.10}$$

where we have now employed a more condensed notation:

$$
\begin{aligned}
\dot{G}(i_1 i_2 \cdots i_n) &:= \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}\Big(\frac{1}{2}\Big)^{\delta_{n,2}}\mathrm{tr}(f_{i_1}f_{i_2}\cdots f_{i_n})\,,\\
T(i) &:= \sum_r \dot{G}_{Bir}\varepsilon_i\cdot k_r\,,\\
T(ij) &:= \sum_{r,s}\Big\{\dot{G}_{Bir}\varepsilon_i\cdot k_r\dot{G}_{js}\varepsilon_j\cdot k_s + \frac{1}{2}\dot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\Big[\dot{G}_{Bir}k_i\cdot k_r - \dot{G}_{Bjr}k_j\cdot k_r\Big]\Big\}\,.
\end{aligned}
\tag{5.11}
$$

By replacing P_4 by Q_4 in (5.2) one gets

$$
\begin{aligned}
\Gamma_0^{a_1a_2a_3a_4}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] &= (-ig)^4\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\
&\quad\times\int_0^T d\tau_1 d\tau_2 d\tau_3\,(-i)^4 Q_4\, e^{\sum_{i<j=1}^4 G_{Bij}k_i\cdot k_j}\,.
\end{aligned}
\tag{5.12}
$$

At this level before continuing with our calculations for the amplitude, in the following we discuss the boundary terms which these total derivatives create by going from P_4 to Q_4. For the three-point case we had just one type of boundary term, those which create the two-point functions, but here for this object we have more boundary terms which will be discussed in the next Section. Before starting on the detailed calculation let us clarify that everything will be done for the standard ordering ($\tau_1 > \tau_2 > \tau_3 > \tau_4$). Since we have six inequivalent orderings for the external gluons in the case of $N = 4$, other orderings will be considered only at the moment when we need "the other half of a commutator". And it is understood that the boundary terms will be taken in the circular sense, i.e. the τ_i's do not run from 0 to 1 since we have path ordered integrals, which means their upper and lower limits are different, for example for the standard orderings, see (5.14).

5.2.1 Boundary terms

As we already mentioned, in non-abelian theory in contrast to abelian one, total derivatives creat boundary terms which correspond to lower point functions. Let us go back to the three steps that we have mentioned in the previous Section to remove the $\ddot{G}_{B12}\ddot{G}_{B34}$ and find all boundary terms that each step creates. We have two types of τ-variables, those which are adjacent and those which are not, for example in the case of $\tau_1 > \tau_2 > \tau_3 > \tau_4$ orderings, variables τ_1 and τ_2 are adjacent but τ_1 with τ_3 are non-adjacent, see Fig. 5.1. Eq. (5.4) can be written as

$$
-\frac{1}{4}(\partial_1-\partial_2)(\partial_3-\partial_4)\Big[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big] = -\frac{1}{4}(\partial_1\partial_3-\partial_1\partial_4-\partial_2\partial_3+\partial_2\partial_4)\Big[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big]\,.
\tag{5.13}
$$

Here the first and the last terms contain derivatives respect to non-adjacen variables ($\partial_1\partial_3$ and $\partial_2\partial_4$), second and third terms contain derivatives respect to adjacent variables ($\partial_1\partial_4$ and $\partial_2\partial_3$), so these two different terms must be distinguished since they make different boundary terms.

Let us first look at the adjacent variables. For this ordering, the τ parameters run as

$$
\begin{aligned}
\partial_1(\cdots) &\rightarrow (\cdots)\Big|_{\tau_2}^{\tau_4}, \\
\partial_2(\cdots) &\rightarrow (\cdots)\Big|_{\tau_3}^{\tau_1}, \\
\partial_3(\cdots) &\rightarrow (\cdots)\Big|_{\tau_4}^{\tau_2}, \\
\partial_4(\cdots) &\rightarrow (\cdots)\Big|_{\tau_1}^{\tau_3}.
\end{aligned}
\tag{5.14}
$$

After imposing these limits to the total derivatives, ∂_3 gives

$$
\begin{aligned}
\partial_1\left(\partial_3\left[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\right]\right) &= \partial_1\left(\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big|_{\tau_3=\tau_4}^{\tau_3=\tau_2}\right) \\
&= \partial_1\left(\dot{G}_{B12}\dot{G}_{B24}\, e^{(3\rightarrow 2)}\right),
\end{aligned}
\tag{5.15}
$$

where we used the fact that $\dot{G}_{Bii} = 0$, now if we consider the ∂_1 we get

$$
\begin{aligned}
\partial_1\partial_3\left[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\right] &= \partial_1\left[\dot{G}_{B12}\dot{G}_{B24}\, e^{(3\rightarrow 2)}\right] \\
&= \dot{G}_{B12}\dot{G}_{B24}\, e^{(3\rightarrow 2)}\Big|_{\tau_1=\tau_2}^{\tau_1=\tau_4} \\
&= \dot{G}_{B42}\dot{G}_{B24}e^{(3\rightarrow 2, 1\rightarrow 4)},
\end{aligned}
\tag{5.16}
$$

where we used the compact notation $e^{i\rightarrow j} \equiv e^{\tau_i \rightarrow \tau_j}$ for convenience and they represent the pinched gluons where two external gluon interact with the loop particle at the same point.

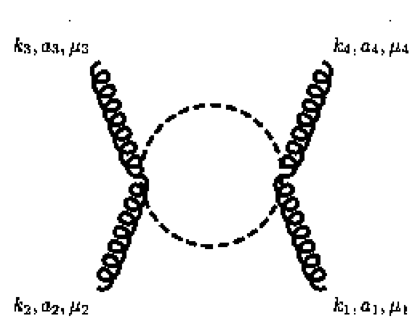


Figure 5.2: Two-gluon vertex for scalar loop particle from the double boundary terms corresponds to $(1 \rightarrow 4, 2 \rightarrow 3)$,

Non-adjacent variables create the following double boundary terms

$$\begin{aligned}
\partial_1\partial_3\Big[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big] &\;\rightarrow\; \dot{G}_{B42}\dot{G}_{B24}e^{(3\to 2,1\to 4)}\,, \\
\partial_2\partial_4\Big[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big] &\;\rightarrow\; \dot{G}_{B13}\dot{G}_{B31}e^{(4\to 1,2\to 3)}\,.
\end{aligned} \tag{5.17}$$

By adding them one gets

$$N_1 = \dot{G}_{B42}\dot{G}_{B24}e^{(3\to 2,1\to 4)} + \dot{G}_{B13}\dot{G}_{B31}e^{(4\to 1,2\to 3)}\,, \tag{5.18}$$

where

$$\begin{aligned}
e^{(3\to 2)} &= e^{G_{B12}k_1\cdot(k_2+k_3)+G_{B24}k_4\cdot(k_2+k_3)+G_{B14}k_1\cdot k_4}\,, \\
e^{(3\to 2,1\to 4))} &= e^{G_{B24}(k_2+k_3)\cdot(k_1+k_4)}\,,
\end{aligned} \tag{5.19}$$

and $e^{(3\to 2,1\to 4))} \equiv e^{(4\to 1,2\to 3)}$.
These kind of boundary terms represent the collapse of four-point diagram in Fig. 5.1 to the two point one (vacuum polarization diagram) which is shown in Fig. 5.2.

For adjacent variables like $\partial_1\partial_4$ and $\partial_2\partial_3$ the situation is different from the non-adjacent ones. Here after imposing the limit for one of the variables the other one is not a total derivative anymore since it shares its index with the boundary of the first one. In this case, one has to work more to make it a total derivative. In the following we show how one can treat this situation, let us look at $\partial_1\partial_4$ term.
First of all, since one is a total derivative but the other not, one has to symmetrised them to preserve the permutation symmetry, i.e.

$$\partial_1\partial_4\Big[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big] \rightarrow \frac{1}{2}\Big(\partial_1\partial_4 + \partial_4\partial_1\Big)\Big[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big]\,, \tag{5.20}$$

where the first derivative from left is a total derivative.

We start with the first one

$$\begin{aligned}
\frac{1}{2}\partial_1\partial_4\Big[\dot{G}_{B12}\dot{G}_{B34}e^{(\cdot)}\Big] &= \frac{1}{2}\partial_1\Big[-\dot{G}_{B12}\ddot{G}_{B34}e^{(\cdot)} + \dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B4i}k_4\cdot k_i e^{(\cdot)}\Big] \\
&= \frac{1}{2}\Big[-\dot{G}_{B12}\ddot{G}_{B34}e^{(\cdot)} + \dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B4i}k_4\cdot k_i e^{(\cdot)}\Big]\Big|_{\tau_1=\tau_2}^{\tau_1=\tau_4} \\
&= \frac{1}{2}\Big[-\dot{G}_{B42}\ddot{G}_{B34}e^{(1\to 4)} + \dot{G}_{B42}\dot{G}_{B34}(\dot{G}_{B42}k_4\cdot k_2 + \dot{G}_{B43}k_4\cdot k_3)e^{(1\to 4)}\Big]\,.
\end{aligned} \tag{5.21}$$

We remove the $\ddot{G}_{B34}$ by adding one more total derivative but now respect to τ_3 since τ_4 was active in the last step,

$$
\begin{aligned}
\frac{1}{2}\Big[-\dot{G}_{B42}\ddot{G}_{B34}e^{(1\to 4)}\Big] &= -\frac{1}{2}\partial_3\Big[\dot{G}_{B42}\dot{G}_{B34}e^{(1\to 4)}\Big] \\
&\quad +\frac{1}{2}\dot{G}_{42}\dot{G}_{B34}\Big[\dot{G}_{B34}k_3\cdot(k_1+k_4)+\dot{G}_{B32}k_3\cdot k_2\Big]e^{(1\to 4)} \\
&= -\frac{1}{2}\dot{G}_{B42}\dot{G}_{B24}e^{(1\to 4,3\to 2)} \\
&\quad +\frac{1}{2}\dot{G}_{42}\dot{G}_{B34}\Big[\dot{G}_{B34}k_3\cdot(k_1+k_4)+\dot{G}_{B32}k_3\cdot k_2\Big]e^{(1\to 4)}\,.
\end{aligned}
\tag{5.22}
$$

Then, by substituting (5.22) in (5.21) we get

$$
\begin{aligned}
&\frac{1}{2}\partial_1\partial_4\Big[\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4 e^{(\cdot)}\Big] = -\frac{1}{2}\dot{G}_{B42}\dot{G}_{B24}e^{(1\to 4,3\to 2)}\,, \\
&+\frac{1}{2}\dot{G}_{B42}\dot{G}_{B34}\Big[\dot{G}_{B34}k_3\cdot k_1+\dot{G}_{B42}k_4\cdot k_2+\dot{G}_{B32}k_3\cdot k_2\Big]e^{(1\to 4)}\,,
\end{aligned}
\tag{5.23}
$$

in a similar way the other half can be written as

$$
\begin{aligned}
&\frac{1}{2}\partial_4\partial_1\Big[\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4 e^{(\cdot)}\Big] = -\frac{1}{2}\dot{G}_{B13}\dot{G}_{B31}e^{(4\to 1,2\to 3)}\,, \\
&-\frac{1}{2}\dot{G}_{B12}\dot{G}_{B31}\Big[\dot{G}_{B21}k_2\cdot k_4+\dot{G}_{B23}k_2\cdot k_3+\dot{G}_{B13}k_1\cdot k_3\Big]e^{(4\to 1)}\,.
\end{aligned}
\tag{5.24}
$$

For adjacent variables after performing the integrals and imposing the limits we get single boundary terms which represent the three-point function, Fig. 5.3, and double boundary terms as before.

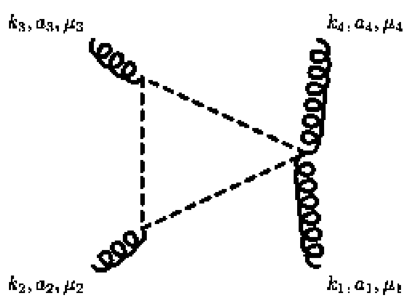


Figure 5.3: Three-gluon vertex for scalar loop particle from single boundary term corresponds to $(4 \to 1)$.

Here we have more boundary terms, at each step whenever we add a total derivative we should find the corresponding boundary terms. The remaining steps to remove the $\ddot{G}_{B12}\ddot{G}_{B34}$ will produce just single and double boundaries. There are two more non-trivial boundaries which

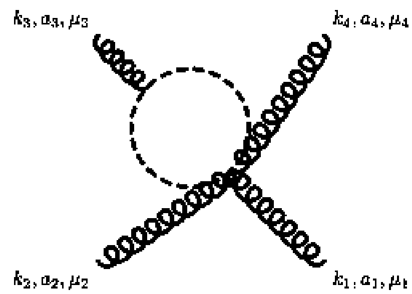


Figure 5.4: Non-trivial boundary terms which corresponds to $(\tau_1 \to \tau_2, \tau_4 \to \tau_1)$.
.

will be discussed in the following. By continuing these steps to remove all the $\ddot{G}_{Bij}$'s in P_4 two new structures arise. For example, if we add the following total derivative to remove $\ddot{G}_{B13}\ddot{G}_{B24}$

$$\partial_1\partial_2\Big[\dot{G}_{B13}\dot{G}_{B24}e^{(\cdot)}\Big], \tag{5.25}$$

it creates

$$\begin{aligned}\partial_1\partial_2\Big[\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4e^{(\cdot)}\Big] &= \Big[\dot{G}_{B13}\dot{G}_{B21}e^{(1\to4,4\to1)} + \dot{G}_{B23}\dot{G}_{B23}e^{(1\to2,4\to3)} \\ &\quad -\dot{G}_{B23}\dot{G}_{B21}e^{(1\to2,4\to1)} - \dot{G}_{B42}\dot{G}_{B24}e^{(1\to4,3\to2)}\Big] \\ &\quad +\dot{G}_{B23}\dot{G}_{B24}\Big[\dot{G}_{B24}k_4\cdot k_1 + \dot{G}_{B32}k_3\cdot k_2 + \dot{G}_{B34}k_3\cdot k_4\Big]e^{(1\to2)}.\end{aligned} \tag{5.26}$$

In (5.26) as before we have single and double pinches, but we have two new structures which we have not seen before, $e^{(1\to4,4\to1)}$ which belongs to three-point function and $e^{(1\to2,4\to1)}$ which can be classified as two-point function. This structure corresponds to Fig. 5.4. These kind of diagrams appear just in the process of removing $\ddot{G}_{B13}\ddot{G}_{B24}$ part of P_4, but they vanish since they contain $\dot{G}_{Bii}$ which is the coincidence limit of the first derivative of the bosonic Green's function and it is zero. For example, in (5.26) this diagram appears as $\dot{G}_{B23}\dot{G}_{B21}e^{(1\to2,4\to1)}$ which contains $\dot{G}_{B22}$ and vanishes. So it seems our IBP procedure does not capture this structure. Let us clarify an interesting point here. In our IBP method all the boundary terms which are produced by this method come from the commutator of two gauge fields, i.e $[A_\mu^{b_1}, A_\nu^{c_1}]$ which are in the definition of the non-abelian field strength tensor $F_{\mu\nu}$. But if we compare our final structures with the low energy effective action as we did for the three-gluon case in Chapter 4, one can see this kind of boundary terms which are produced by IBP. But since gluon is in the adjoint representation we know that the gauge field in the definition of covariant derivative acts on whatever comes after, i.e. $D_\mu f_{\nu\lambda} = \partial_\mu f_{\nu\lambda} + ig[A_\mu, f_{\nu\lambda}]$, and it makes some commutator like $[A_\mu, f_{\nu\lambda}]$ which is not seen in the IBP, it means that the above boundary term Fig. 5.4 actually exists and it can be seen as a commutator like $[A_\mu, f_{\nu\lambda}]$ not $[A_\mu^{b_1}, A_\nu^{c_1}]$. This point and also the comparison of the final structures for four-gluon vertex with the low energy effective action will be explained in details in the upcoming publication [207].

What remains here to be discussed before going to final classification of the boundary terms are the single boundary terms like $e^{(1\to 4,4\to 1)}$ in (5.26). During the whole process of IBP we get eight similar terms for different pinches, which they cancel out by using cyclic permutation of the variables, i.e

$$2\int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 \Big[\dot G_{B13}\dot G_{B14} e^{(1\to 2,2\to 1)} - \dot G_{B13}\dot G_{B21} e^{(1\to 4,4\to 1)} \\ -\dot G_{B12}\dot G_{B24} e^{(2\to 3,3\to 2)} + \dot G_{B13}\dot G_{B23} e^{(3\to 4,4\to 3)}\Big] = 0\,, \tag{5.27}$$

5.2.2 Classifying boundary terms

In the preceding Section we have discussed the way one gets the boundary terms from all those total derivatives that we have used to remove all the $\ddot G_B$'s. If one goes through all these calculations, one may see a large number of boundary terms, but the nice fact about these terms are the way they group together where one can organize them as structures of two and three-point functions.
As we have mentioned before, the single boundary terms correspond to the three-point function which its bulk term is given by Q_3 polynomials. By adding all those terms which have three-cycle, one gets the following modified Q_3^3

$$\begin{aligned} Q_3^3(4[12]3) &= \dot G_{B13}\dot G_{B34}\dot G_{B41} e^{(1\to 2)}\,\mathrm{tr}\Big(4[12]3\Big)\,,\\ Q_3^3(1[23]4) &= \dot G_{B12}\dot G_{B24}\dot G_{B41} e^{(2\to 3)}\,\mathrm{tr}\Big(1[23]4\Big)\,,\\ Q_3^3(1[34]2) &= \dot G_{B13}\dot G_{B32}\dot G_{B21} e^{(3\to 4)}\,\mathrm{tr}\Big(1[34]2\Big)\,,\\ Q_3^3(3[41]2) &= \dot G_{B42}\dot G_{B23}\dot G_{B34} e^{(4\to 1)}\,\mathrm{tr}\Big(3[41]2\Big)\,, \end{aligned} \tag{5.28}$$

where

$$\mathrm{tr}\Big(1[23]4\Big) = f_1^{\mu\nu}\Big[\varepsilon_2^\nu\varepsilon_3^\alpha - \varepsilon_3^\nu\varepsilon_2^\alpha\Big] f_4^{\alpha\mu}\,, \tag{5.29}$$

and also for two-cycle the following modified Q_3^2

$$\begin{aligned} Q_3^2(1\to 2) &= \dot G_{B13}\dot G_{B31}\mathrm{tr}\Big(\varepsilon_2 f_3\varepsilon_1\Big)\dot G_{B4i}\varepsilon_4\cdot k_i e^{(1\to 2)} + \dot G_{B14}\dot G_{B41}\mathrm{tr}\Big(\varepsilon_2 f_4\varepsilon_1\Big)\dot G_{B3i}\varepsilon_3\cdot k_i e^{(1\to 2)}\,,\\ Q_3^2(1\to 4) &= \dot G_{B12}\dot G_{B21}\mathrm{tr}\Big(\varepsilon_1 f_2\varepsilon_4\Big)\dot G_{B3i}\varepsilon_3\cdot k_i e^{(1\to 4)} + \dot G_{B13}\dot G_{B31}\mathrm{tr}\Big(\varepsilon_1 f_3\varepsilon_4\Big)\dot G_{B2i}\varepsilon_2\cdot k_i e^{(1\to 4)}\,,\\ Q_3^2(2\to 3) &= \dot G_{B13}\dot G_{B31}\mathrm{tr}\Big(\varepsilon_3 f_1\varepsilon_2\Big)\dot G_{B4i}\varepsilon_4\cdot k_i e^{(2\to 3)} + \dot G_{B24}\dot G_{B42}\mathrm{tr}\Big(\varepsilon_3 f_4\varepsilon_2\Big)\dot G_{B1i}\varepsilon_1\cdot k_i e^{(2\to 3)}\,,\\ Q_3^2(3\to 4) &= \dot G_{B13}\dot G_{B31}\mathrm{tr}\Big(\varepsilon_4 f_1\varepsilon_3\Big)\dot G_{B2i}\varepsilon_2\cdot k_i e^{(3\to 4)} + \dot G_{B24}\dot G_{B42}\mathrm{tr}\Big(\varepsilon_4 f_2\varepsilon_3\Big)\dot G_{B1i}\varepsilon_1\cdot k_i e^{(3\to 4)}\,. \end{aligned} \tag{5.30}$$

Now, if we look at the double boundary terms which contain two different permutations, $e^{(1\to 2,3\to 4)}$ and $e^{(2\to 3,4\to 1)}$, they combine to give a simple structure which corresponds to Q_2^2 of two point function:

$$Q_2^2(12,34) = \dot G_{B13}\dot G_{B31} e^{(1\to 2,3\to 4)}\Big(\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4 - \varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3\Big)\,, \tag{5.31}$$

and

$$Q_2^2(23,14) = \dot{G}_{B13}\dot{G}_{B31}e^{(2\to3,1\to4)}\Big(\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4 - \varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4\Big)\,. \tag{5.32}$$

Eq. (5.31) and (5.32) can be written as

$$\begin{aligned} Q_2^2(12,34) &= \frac{1}{2}\dot{G}_{B13}\dot{G}_{B31}\mathrm{tr}(f_1f_2)e^{(1\to2,3\to4)}\,,\\ Q_2^2(23,14) &= \frac{1}{2}\dot{G}_{B13}\dot{G}_{B31}\mathrm{tr}(f_1f_2)e^{(2\to3,1\to4)}\,, \end{aligned} \tag{5.33}$$

where for $Q_2^2(12,34)$ the definition of the field strength tensor f_1 and f_2 are

$$\begin{aligned} f_1 &= k_1\otimes\varepsilon_1 - \varepsilon_1\otimes k_1 \to \varepsilon_1\otimes\varepsilon_2 - \varepsilon_2\otimes\varepsilon_1\,,\\ f_2 &= k_2\otimes\varepsilon_2 - \varepsilon_2\otimes k_2 \to \varepsilon_3\otimes\varepsilon_4 - \varepsilon_4\otimes\varepsilon_3\,. \end{aligned} \tag{5.34}$$

and for $Q_2^2(23,14)$

$$\begin{aligned} f_1 &= k_1\otimes\varepsilon_1 - \varepsilon_1\otimes k_1 \to \varepsilon_1\otimes\varepsilon_4 - \varepsilon_4\otimes\varepsilon_1\,,\\ f_2 &= k_2\otimes\varepsilon_2 - \varepsilon_2\otimes k_2 \to \varepsilon_2\otimes\varepsilon_3 - \varepsilon_3\otimes\varepsilon_2\,. \end{aligned} \tag{5.35}$$

We have some left over terms from single boundary terms which can be combined in a nice way to be some bulk terms of the three point function (Q_3^2). They combine to be written as

$$\begin{aligned} Q_{\mathrm{ext}} \;=\; & \Big(\dot{G}_{B13}\dot{G}_{B31}\mathrm{tr}(f_1f_3)\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4 + \dot{G}_{B14}\dot{G}_{B41}\mathrm{tr}(f_1f_4)\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\\ & -\dot{G}_{B23}\dot{G}_{B32}\mathrm{tr}(f_2f_3)\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4 - \dot{G}_{B24}\dot{G}_{B42}\mathrm{tr}(f_2f_4)\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\Big)e^{(1\to2)}\\ & +\Big(\dot{G}_{B12}\dot{G}_{B21}\mathrm{tr}(f_1f_2)\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4 + \dot{G}_{B43}\dot{G}_{B34}\mathrm{tr}(f_3f_4)\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\\ & +\dot{G}_{B13}\dot{G}_{B31}\mathrm{tr}(f_1f_3)\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4 + \dot{G}_{B24}\dot{G}_{B42}\mathrm{tr}(f_2f_4)\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\Big)e^{(1\to4)}\\ & +\Big(\dot{G}_{B12}\dot{G}_{B21}\mathrm{tr}(f_1f_2)\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4 + \dot{G}_{B34}\dot{G}_{B43}\mathrm{tr}(f_3f_4)\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\\ & -\dot{G}_{B13}\dot{G}_{B31}\mathrm{tr}(f_1f_3)\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4 - \dot{G}_{B24}\dot{G}_{B42}\mathrm{tr}(f_2f_4)\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\Big)e^{(2\to3)}\\ & +\Big(\dot{G}_{B14}\dot{G}_{B41}\mathrm{tr}(f_1f_4)\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3 + \dot{G}_{B24}\dot{G}_{B42}\mathrm{tr}(f_2f_4)\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\\ & -\dot{G}_{B13}\dot{G}_{B31}\mathrm{tr}(f_1f_3)\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4 - \dot{G}_{B23}\dot{G}_{B32}\mathrm{tr}(f_2f_3)\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\Big)e^{(3\to4)}\,. \end{aligned} \tag{5.36}$$

In the following Sections we discus the structure of the off-shell four gluon amplitude for scalar, spinor and gluon loop cases.

5.3 The scalar loop case

The scalar loop case for four gluon can be written as

$$\Gamma_0 = \frac{g^4}{(4\pi)^{\frac{D}{2}}}\Big(\Gamma_0^4 + \Gamma_0^3 + \Gamma_0^2 + \Gamma_0^{22} + \Gamma_0^{\mathrm{bt},33} + \Gamma_0^{\mathrm{bt},32} + \Gamma_0^{\mathrm{bt},3\mathrm{ext}} + \Gamma_0^{\mathrm{bt},2}\Big)\,, \tag{5.37}$$

where

$$
\begin{aligned}
\Gamma_0^4 \;&=\; \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3 Q_4^4\Big|_{\tau_4=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_4 Q_4^4\Big|_{\tau_3=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_3\int_0^{\tau_3}d\tau_2 Q_4^4\Big|_{\tau_4=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_3\int_0^{\tau_3}d\tau_4 Q_4^4\Big|_{\tau_2=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_4\int_0^{\tau_4}d\tau_2 Q_4^4\Big|_{\tau_3=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_4\int_0^{\tau_4}d\tau_3 Q_4^4\Big|_{\tau_2=0}e^{(\cdot)}\,,\\
\Gamma_0^3 \;&=\; \Gamma_0^4(Q_4^4\to Q_4^3)\,,\\
\Gamma_0^2 \;&=\; \Gamma_0^4(Q_4^4\to Q_4^2)\,,\\
\Gamma_0^{22} \;&=\; \Gamma_0^4(Q_4^4\to Q_4^{22})\,,\\
\Gamma_0^{\mathrm{bt},33} \;&=\; \mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\, Q_3^3(i[jk]l)\,,\\
\Gamma_0^{\mathrm{bt},32} \;&=\; \mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\, Q_3^2(i\to j)\,,\\
\Gamma_0^{\mathrm{bt},3\mathrm{ext}} \;&=\; \mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\, Q_{\mathrm{ext}}\,,\\
\Gamma_0^{\mathrm{bt},2} \;&=\; \mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\,\Big(Q_2^2(12,34)+Q_2^2(23,14)\Big)\,.
\end{aligned}
\tag{5.38}
$$

For four external gluon we have six inequivalent orderings which are

$$
\begin{aligned}
&T^{a_1}T^{a_2}T^{a_3}T^{a_4}, T^{a_1}T^{a_2}T^{a_4}T^{a_3}, T^{a_1}T^{a_3}T^{a_2}T^{a_4}\\
&T^{a_1}T^{a_3}T^{a_4}T^{a_2}, T^{a_1}T^{a_4}T^{a_2}T^{a_3}, T^{a_1}T^{a_4}T^{a_3}T^{a_2}\,,
\end{aligned}
\tag{5.39}
$$

which should be considered, here for simplicity we present one ordering ($T^{a_1}T^{a_2}T^{a_3}T^{a_4}$), and it is understood now that all the boundary terms should include the commutator of color factors. One pinched leg represent three-point function then it contains one commutator for instance $T^{a_1}T^{a_2}[T^{a_3},T^{a_4}]$ which is obtained after including all orderings. And also double pinchs must contain two commutators like $[T^{a_1},T^{a_2}][T^{a_3},T^{a_4}]$ which indicate two point functions and it is obtained after including all other orderings.

5.4 The spinor loop case

For the spinor loop case as we have already discussed in Chapter 4, it is convenient to use the super worldline formalism, we recall the "Bern-Kosower master formula" for the spinor case which is written in super formalism:

$$\begin{aligned}\Gamma^{a_1\dots a_N}_{\frac{1}{2}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -2(-ig)^N \mathrm{tr}(T^{a_1}\dots T^{a_N})\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\\ &\quad\times\prod_{k=1}^{N}\int_0^T d\tau_k\int d\theta_k\,\delta\Big(\frac{\tau_N}{T}\Big)\vartheta(\hat\tau_{1N})\prod_{l=1}^{N-1}\vartheta(\hat\tau_{l(l+1)})\\ &\quad\times\exp\Bigg\{\sum_{i,j=1}^{N}\bigg[\frac{1}{2}\hat G_{ij}k_i\cdot k_j+iD_i\hat G_{ij}\varepsilon_i\cdot k_j+\frac{1}{2}D_iD_j\hat G_{ij}\varepsilon_i\cdot\varepsilon_j\bigg]\Bigg\}\Bigg|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)}.\end{aligned}\tag{5.40}$$

where

$$\begin{aligned}D &= \frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial\tau}\,,\\ \hat\tau_{ij} &:= \tau_i-\tau_j+\theta_i\theta_j\,.\end{aligned}\tag{5.41}$$

are super derivative and super proper-time accordingly and

$$\begin{aligned}\hat G(\tau_i,\theta_i;\tau_j,\theta_j) &\equiv G_B(\tau_i,\tau_j)+\theta_i\theta_j G_F(\tau_i,\tau_j)\,,\\ \vartheta(\hat\tau_{ij}) &= \vartheta(\tau_i-\tau_j)+\theta_i\theta_j\delta(\tau_i-\tau_j)\,,\end{aligned}\tag{5.42}$$

are super Green function and super Heaviside step function. Note that if one starts expanding the (5.40) to perform the spinor loop case directly, one needs to consider that all ε_j^ν, θ_k and $d\theta_l$ anticummute with each other. In this case we use the "Bern-Kosower master formula" to obtain the corresponding result for the spinor loop case. According to this replacement rule:

$$\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}\to\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}-G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\,.\tag{5.43}$$

As we have discussed in Chapter 3 we should use the Q' representation of four-point case eq. (3.38), that is suitable for the application of the Bern-Kosower rules, see Section 3.4. The Q'_4 polynomial will change to a new one $\hat Q'_4$ which is

$$\begin{aligned}\hat Q'_4 &= \hat Q'^4_4+\hat Q'^3_4+\hat Q'^2_4+\hat Q'^{22}_4\,,\\ \hat Q'^4_4 &= \dot{\hat G}(1234)+\dot{\hat G}(1243)+\dot{\hat G}(1324)\,,\\ \hat Q'^3_4 &= \dot{\hat G}(123)T(4)+\dot{\hat G}(234)T(1)+\dot{\hat G}(341)T(2)+\dot{\hat G}(412)T(3)\,,\\ \hat Q'^2_4 &= \dot{\hat G}(12)T(34)+\dot{\hat G}(13)T(24)+\dot{\hat G}(14)T(23)\,,\\ &\quad+\dot{\hat G}(23)T(14)+\dot{\hat G}(24)T(13)+\dot{\hat G}(34)T(12)\,,\\ \hat Q'^{22}_4 &= \dot{\hat G}(12)\dot{\hat G}(34)+\dot{\hat G}(13)\dot{\hat G}(24)+\dot{\hat G}(14)\dot{\hat G}(23)\,,\end{aligned}\tag{5.44}$$

(Note that as was discussed in Chapter 3, for Q' representation we have used $Q_4^2=Q_4^{'2}+2Q_4^{22}$, where $Q_4^{'2}$ is obtained from Q_4^2 by eliminating the term with $r=j$ and $s=i$ from the sum over dummy indices, for the remaining component $Q_4^{(\cdot)}=Q_4^{(\cdot)}$), where we need to change the definition of $\dot G(i_1i_2\cdots i_n)$ in (5.11) to a new one as

$$\hat{\dot{G}}(i_1 i_2 \cdots i_n) := \Big(\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\Big)\,(\frac{1}{2})^{\delta_{n,2}}\mathrm{tr}(f_{i_1}f_{i_2}\cdots f_{i_n})\,. \tag{5.45}$$

So for spinor loop contribution to the four-gluon amplitude we find

$$\Gamma_{\frac{1}{2}} = -2\frac{g^4}{(2\pi)^{\frac{D}{2}}}\Big(\Gamma^4_{\frac{1}{2}} + \Gamma^3_{\frac{1}{2}} + \Gamma^2_{\frac{1}{2}} + \Gamma^{22}_{\frac{1}{2}} + \Gamma^{\mathrm{bt},33}_{\frac{1}{2}} + \Gamma^{\mathrm{bt},32}_{\frac{1}{2}} + \Gamma^{\mathrm{bt},2}_{\frac{1}{2}}\Big)\,, \tag{5.46}$$

where

$$\begin{aligned}
\Gamma^4_{\frac{1}{2}} &= \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3 \hat{Q}'^4_4\Big|_{\tau_4=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_4 \hat{Q}^4_4\Big|_{\tau_3=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_3\int_0^{\tau_3}d\tau_2 \hat{Q}^4_4\Big|_{\tau_4=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_3\int_0^{\tau_3}d\tau_4 \hat{Q}^4_4\Big|_{\tau_2=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_4\int_0^{\tau_4}d\tau_2 \hat{Q}^4_4\Big|_{\tau_3=0}e^{(\cdot)}\\
&\quad+\mathrm{tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_4\int_0^{\tau_4}d\tau_3 \hat{Q}^4_4\Big|_{\tau_2=0}e^{(\cdot)}\,,\\
\Gamma^3_{\frac{1}{2}} &= \Gamma^4_0(\hat{Q}'^4_4 \to \hat{Q}'^3_4)\,,\\
\Gamma^2_{\frac{1}{2}} &= \Gamma^4_0(\hat{Q}'^4_4 \to \hat{Q}'^2_4)\,,\\
\Gamma^{22}_{\frac{1}{2}} &= \Gamma^4_0(\hat{Q}''^4_4 \to \hat{Q}'^{22}_4)\,,\\
\Gamma^{\mathrm{bt},33}_{\frac{1}{2}} &= \mathrm{tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\Big(\dot{G}_{Bij}\dot{G}_{Bjk}\dot{G}_{Bki} \to \dot{G}_{Bij}\dot{G}_{Bjk}\dot{G}_{Bki} - G_{Fij}G_{Fjk}G_{Fki}\Big)\,,\\
\Gamma^{\mathrm{bt},32}_{\frac{1}{2}} &= \mathrm{tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\Big(\dot{G}_{Bij}\dot{G}_{Bji} \to \dot{G}_{Bij}\dot{G}_{Bji} - G_{Fij}G_{Fji}\Big)\,,\\
\Gamma^{\mathrm{bt},3\mathrm{ext}}_{\frac{1}{2}} &= \mathrm{tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\Big(\dot{G}_{Bij}\dot{G}_{Bji} \to \dot{G}_{Bij}\dot{G}_{Bji} - G_{Fij}G_{Fji}\Big)\,,\\
\Gamma^{\mathrm{bt},2}_{\frac{1}{2}} &= \mathrm{tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1\Big(\dot{G}_{Bij}\dot{G}_{Bji} \to \dot{G}_{Bij}\dot{G}_{Bji} - G_{Fij}G_{Fji}\Big)\,.
\end{aligned} \tag{5.47}$$

Fig.5.5 shows all the possible bulk and boundary terms for the spinor loop-case diagrammatically.

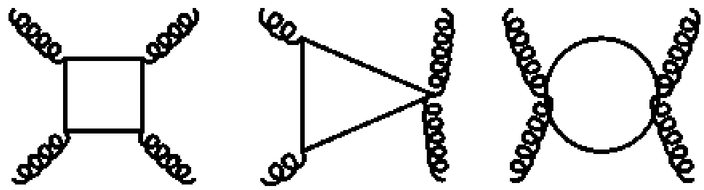


Figure 5.5: Four, three and two gluon diagrams for spinor loop case.

5.5 The gluon loop case

We continue our calculations in this Section for the gluon loop case, as we was shown for the three-point case in Chapter (4), the gluon loop case is more subtle and one is faced with the issue of gauge (in)dependent not only for the background field but also for the loop particle. The path integral representation for gluon loop is discussed in the Chapter 1, see also [50]. As we have mentioned the only known generalization of the worldline path integral representation of the effective action for the gluon loop is based on the BFM with quantum Feynman gauge. We recall the master formula for the gluon-loop case as

$$\begin{aligned}
\Gamma^{a_1\dots a_N}_{\text{gluon}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -\frac{(-ig)^N}{4}\mathrm{tr}(T^{a_1}\dots T^{a_N})\lim_{C\to\infty}\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-CT} \\
&\times \prod_{k=1}^{N}\int_0^T d\tau_k \int d\theta_k\,\delta\Big(\frac{\tau_N}{T}\Big)\vartheta(\hat{\tau}_{1N})\prod_{l=1}^{N-1}\vartheta(\hat{\tau}_{l(l+1)})\sum_{p=P,A}\sigma_p Z_p \\
&\times \exp\Bigg\{\sum_{i,j=1}^{N}\Bigg[\frac{1}{2}\hat{G}^C_{p,ij}k_i\cdot k_j + iD_i\hat{G}^C_{p,ij}\varepsilon_i\cdot k_j + \frac{1}{2}D_iD_j\hat{G}^C_{p,ij}\varepsilon_i\cdot\varepsilon_j\Bigg]\Bigg\}\Bigg|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)},
\end{aligned} \tag{5.48}$$

where T^a are the group generators in the adjoint representation and

$$\begin{aligned}
Z_A &= (2\cosh[CT/2])^4 \quad , \quad \sigma_A = -1 \\
Z_P &= (2\sinh[\mathrm{CT}/2])^4 \quad , \quad \sigma_P = 1
\end{aligned} \tag{5.49}$$

$$\hat{G}^C_{P,A}(\tau_1,\theta_1;\tau_2,\theta_2) = G_B(\tau_1,\tau_2) + \theta_1\theta_2 G^C_{P,A}(\tau_1,\tau_2)\,, \tag{5.50}$$

where

$$\begin{aligned}
G^C_P(\tau_1,\tau_2) &= 2\mathrm{sign}(\tau_1-\tau_2)\frac{\sinh[C(\frac{T}{2}-|\tau_1-\tau_2|)]}{\sinh[CT/2]}\,, \\
G^C_A(\tau_1,\tau_2) &= 2\mathrm{sign}(\tau_1-\tau_2)\frac{\cosh[C(\frac{T}{2}-|\tau_1-\tau_2|)]}{\cosh[CT/2]}\,.
\end{aligned} \tag{5.51}$$

A and P are stand for anti-periodic and periodic boundary condition respectively in the original path integral. For this case again we have the bulk diagrams and the three and two point diagrams which come from the boundary terms, see, Fig. 5.6.

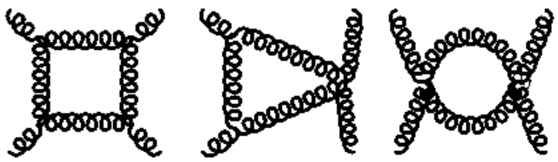


Figure 5.6: Four, three and two gluon diagrams for the gluon loop case.

As was discussed in Chapters 1 and 4, the gluon loop case is isomorphic to the spinor loop one before taking the limit $C \to \infty$ and the projector sum $\sum_{p=P,A}$ in eq. (5.48). So similarly in the four-point case because of this isomorphism we can generalize the decomposition as

$$\begin{aligned}\Gamma_{\text{gluon}} \quad = \quad & -\frac{1}{4}\frac{g^4}{(4\pi)^{\frac{D}{2}}}\lim_{C\to\infty}\sum_{p=P,A}\sigma_p\Big(\Gamma^4_{\text{gluon}}(C,p)+\Gamma^3_{\text{gluon}}(C,p)+\Gamma^2_{\text{gluon}}(C,p)+\Gamma^{22}_{\text{gluon}}(C,p) \\ & +\Gamma^{\text{bt},33}_{\text{gluon}}(C,p)+\Gamma^{\text{bt},32}_{\text{gluon}}(C,p)+\Gamma^{\text{bt},2}_{\text{gluon}}(C,p)\Big)\,,\end{aligned} \tag{5.52}$$

where $\Gamma^{(\cdot)}_{\text{gluon}}(C,p)$ is different from the corresponding $\Gamma^{(\cdot)}_{\frac{1}{2}}$ in eq. (5.46) only by replacement of m^2 by C, G_{Fij} by $G^C_{p,ij}$ and the insertion of Z_p under the T integral.

What remains to be analyzed is the $C\to\infty$ limit and sum over boundary condition, for four-gluon vertex, one finds the following limits

$$\begin{aligned}\lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_p Z_p &= -8\,,\\ \lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_p Z_p\tilde{G}^C_{p,ij} &= 16\,,\\ \lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_p Z_p\tilde{G}^C_{p,ijk} &= 16\,,\\ \lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_p Z_p\tilde{G}^C_{p,1234} &= 32\,,\\ \lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_p Z_p\tilde{G}^C_{p,1243} &= 0\,,\\ \lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_p Z_p\tilde{G}^C_{p,1324} &= 0\,,\\ \lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_p Z_p\tilde{G}^C_{p,ij}\tilde{G}^C_{p,kl} &= 0\,,\end{aligned} \tag{5.53}$$

where

$$\tilde{G}^C_{p,i_1 i_2 \cdots i_n} \;=\; G^C_{p,i_1 i_2} G^C_{p,i_2 i_3} \cdots G^C_{p,i_n i_1}\,. \tag{5.54}$$

Note that the only nonzero integral for four-cycle is the one that follows the ordering of the external gluons. Now, by having these limits one can write down the gluon loop decomposition from the spinor loop one,

$$\Gamma_{\rm gluon} \;=\; 2\frac{g^4}{(4\pi)^{D/2}}\Big(\Gamma^4_{\rm gluon} + \Gamma^3_{\rm gluon} + \Gamma^2_{\rm gluon} + \Gamma^{22}_{\rm gluon} + \Gamma^{\rm bt,33}_{\rm gluon} + \Gamma^{\rm bt,32}_{\rm gluon} + \Gamma^{\rm bt,2}_{\rm gluon}\Big)\,, \tag{5.55}$$

where

$$\begin{aligned}
\Gamma^4_{\rm gluon} &= \Gamma^4_{\frac12}\left(G_{F12}G_{F23}G_{F34}G_{F41} \to -4\right)\,,\\
\Gamma^3_{\rm gluon} &= \Gamma^3_{\frac12}\left(G_{Fij}G_{Fjk}G_{Fki} \to -2\right)\,,\\
\Gamma^2_{\rm gluon} &= \Gamma^2_{\frac12}\left(G_{Fij}G_{Fji} \to -2\right)\,,\\
\Gamma^{22}_{\rm gluon} &= \Gamma^{22}_{\frac12}\left(G_{Fij}G_{Fji}G_{Fkl}G_{Flk} \to 0, G_{Fij}G_{Fji} \to -2\right)\,,\\
\Gamma^{\rm bt,33}_{\rm gluon} &= \Gamma^{\rm bt,33}_{\frac12}\left(G_{Fij}G_{Fjk}G_{Fki} \to -2\right)\,,\\
\Gamma^{\rm bt,32}_{\rm gluon} &= \Gamma^{\rm bt,32}_{\frac12}\left(G_{Fij}G_{Fjk}G_{Fki} \to -2\right)\,,\\
\Gamma^{\rm bt,ext}_{\rm gluon} &= \Gamma^{\rm bt,ext}_{\frac12}\left(G_{Fij}G_{Fjk}G_{Fki} \to -2\right)\,,\\
\Gamma^{\rm bt,2}_{\rm gluon} &= \Gamma^{\rm bt,2}_{\frac12}\left(G_{Fij}G_{Fji} \to -2\right)\,.
\end{aligned} \tag{5.56}$$

For the ghost loop contribution of this amplitude which is needed to subtract the unphysical degrees of freedom of the gluon loop, it is not contained in (5.48), as has been discussed before it is equal to the scalar loop case with opposite sign.

5.6 Ward identity

5.6.1 Off-shell four-gluon Ward-identity

Off-shell, the Ward identities for the gluon amplitudes are inhomogeneous and map the N-vertex to $N-1$ point amplitudes, e.g. for the four point case one finds [216]

$$k_1^\mu \Gamma^{abcd}_{\mu\nu\alpha\beta} = -igf_{abe}\Gamma^{cde}_{\nu\alpha\beta}(k_1+k_2,k_3,k_4) + \text{perm}\,, \tag{5.57}$$

where f_{abc} is the structure constant of the gauge group. These identities as they stand hold for the scalar and spinor loop unambiguously, but for the gluon loop only if one uses the pinch techniques, or equivalently the BFM with quantum Feynman gauge. Other gauge fixing of the gluon will generally lead to more complicated right-hand side involving ghosts. We recall four gluon amplitude in the level of vertex operators

$$\begin{aligned}
\Gamma^4_{\rm scal}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] \;&=\; (-ig)^4\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\int \mathcal{D}x\, e^{-\int_0^T d\tau \frac14 \dot{x}^2}\\
&\quad\times\Big\langle V^g_{\rm scal}[k_1,\varepsilon_1]V^g_{\rm scal}[k_2,\varepsilon_2]V^g_{\rm scal}[k_3,\varepsilon_3]V^g_{\rm scal}[k_4,\varepsilon_4]\Big\rangle\,,
\end{aligned} \tag{5.58}$$

where

$$V^{g}_{\text{scal}}[k,\varepsilon] = T^{a}\int_{0}^{T} d\tau\varepsilon\cdot\dot{x}(\tau)\,e^{ik\cdot x(\tau)}\,, \tag{5.59}$$

is the gluon vertex operator for scalar loop case. Now we analyze the equation (5.57) for four-gluon vertex, let us replace a polarization vector with its momentum, i.e. $\varepsilon_i \to k_i$ and see what happens to the vertex operator

$$V^{g}_{\text{scal}}[k_i,\varepsilon_i] \overset{\varepsilon_i \to k_i}{=} T^{a}\int_{\tau_{i-1}}^{\tau_{i+1}} d\tau_i\, k_i\cdot\dot{x}(\tau_i)\,e^{ik_i\cdot x(\tau_i)} = -iT^{a}\int_{\tau_{i-1}}^{\tau_{i+1}} d\tau_i\,\frac{\partial}{\partial\tau_i}\,e^{ik_i\cdot x(\tau_i)}\,, \tag{5.60}$$

which under this change, the vertex operator turns to a total derivative respect to τ_i. Now let us specify one of the vertex operators in (5.58) to make this change, for example $V^{g}_{\text{scal}}[k_4,\varepsilon_4]$, and we consider the normal ordering which is $\tau_1 > \tau_2 > \tau_3 > \tau_4$,

$$\begin{aligned} V^{g}_{\text{scal}}[k_4,\varepsilon_4] &\overset{\varepsilon_4 \to k_4}{=} -iT^{a_4}\int_{0}^{\tau_3} d\tau_4\,\frac{\partial}{\partial\tau_4}\,e^{ik_4\cdot x(\tau_4)} = -iT^{a_4}\left(e^{ik_4\cdot x(\tau_4)}\right)\Big|^{\tau_4=\tau_3}_{\tau_4=\tau_1} \\ &= -iT^{a_4}\Big(e^{ik_4\cdot x(\tau_3)} - e^{ik_4\cdot x(\tau_1)}\Big)\,. \end{aligned} \tag{5.61}$$

If we plug this vertex operator into (5.58), we get:

$$\begin{aligned} &\Gamma^{4}_{\text{scal}}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] \overset{\varepsilon_4 \to k_4}{=} -i(-ig)^4\,\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\int \mathcal{D}x\,e^{-\int_0^T d\tau\frac{1}{4}\dot{x}^2} \\ &\times\Big\{\int_0^T d\tau_1\varepsilon_1\cdot\dot{x}(\tau_1)\,e^{ik_1\cdot x(\tau_1)}\int_0^{\tau_1} d\tau_2\varepsilon_2\cdot\dot{x}(\tau_2)\,e^{ik_2\cdot x(\tau_2)}\int_0^{\tau_2} d\tau_3\varepsilon_3\cdot\dot{x}(\tau_3)\,e^{i(k_3+k_4)\cdot x(\tau_3)} \\ &-\int_0^T d\tau_1\varepsilon_1\cdot\dot{x}(\tau_1)\,e^{i(k_1+k_4)\cdot x(\tau_1)}\int_0^{\tau_1} d\tau_2\varepsilon_2\cdot\dot{x}(\tau_2)\,e^{ik_2\cdot x(\tau_2)}\int_0^{\tau_2} d\tau_3\varepsilon_3\cdot\dot{x}(\tau_3)\,e^{ik_3\cdot x(\tau_3)}\Big\}\,. \end{aligned} \tag{5.62}$$

If we look at the other orderings (note that we have six inequivalent orderings) for example $\tau_1 > \tau_2 > \tau_4 > \tau_3$ for the same change we get:

$$\begin{aligned} &\Gamma^{4}_{\text{scal}}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] \overset{\varepsilon_4 \to k_4}{=} -i(-ig)^4\,\mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\int \mathcal{D}x\,e^{-\int_0^T d\tau\frac{1}{4}\dot{x}^2} \\ &\times\Big\{\int_0^T d\tau_1\varepsilon_1\cdot\dot{x}(\tau_1)\,e^{ik_1\cdot x(\tau_1)}\int_0^{\tau_1} d\tau_2\varepsilon_2\cdot\dot{x}(\tau_2)\,e^{i(k_2+k_4)\cdot x(\tau_2)}\int_0^{\tau_2} d\tau_3\varepsilon_3\cdot\dot{x}(\tau_3)\,e^{ik_3\cdot x(\tau_3)} \\ &-\int_0^T d\tau_1\varepsilon_1\cdot\dot{x}(\tau_1)\,e^{ik_1\cdot x(\tau_1)}\int_0^{\tau_1} d\tau_2\varepsilon_2\cdot\dot{x}(\tau_2)\,e^{ik_2\cdot x(\tau_2)}\int_0^{\tau_2} d\tau_3\varepsilon_3\cdot\dot{x}(\tau_3)\,e^{i(k_3+k_4)\cdot x(\tau_3)}\Big\}\,. \end{aligned} \tag{5.63}$$

For these two different orderings if we look at the terms that contain $\varepsilon_3 \cdot \dot{x}(\tau_3) e^{i(k_3+k_4)\cdot x(\tau_3)}$ in (5.62) and (5.63) and add them up, we get

$$\Gamma^4_{\rm scal}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] \stackrel{\varepsilon_4 \to k_4}{=} -i(-ig)^4 {\rm tr}(T^{a_1}T^{a_2}[T^{a_3},T^{a_4}]) \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int \mathcal{D}x\, e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} \times \int_0^T d\tau_1 \varepsilon_1 \cdot \dot{x}(\tau_1)\, e^{ik_1\cdot x(\tau_1)} \int_0^{\tau_1} d\tau_2 \varepsilon_2 \cdot \dot{x}(\tau_2)\, e^{ik_2\cdot x(\tau_2)} \int_0^{\tau_2} d\tau_3 \varepsilon_3 \cdot \dot{x}(\tau_3)\, e^{i(k_3+k_4)\cdot x(\tau_3)} \,. \tag{5.64}$$

Now, if we use $[T^a, T^b] = if^{abc}T^c$, then (5.64) can be written as

$$\Gamma^4_{\rm scal}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] \stackrel{\varepsilon_4 \to k_4}{=} (-ig)^4 {\rm tr}(T^{a_1}T^{a_2}T^{c}) f^{a_3a_4c} \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int \mathcal{D}x\, e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} \times \int_0^T d\tau_1 \varepsilon_1 \cdot \dot{x}(\tau_1)\, e^{ik_1\cdot x(\tau_1)} \int_0^{\tau_1} d\tau_2 \varepsilon_2 \cdot \dot{x}(\tau_2)\, e^{ik_2\cdot x(\tau_2)} \int_0^{\tau_2} d\tau_3 \varepsilon_3 \cdot \dot{x}(\tau_3)\, e^{i(k_3+k_4)\cdot x(\tau_3)} \,. \tag{5.65}$$

The right hand side of (5.65) is proportional to the three-gluon vertex in the following way

$$\Gamma^4_{\rm scal}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] \stackrel{\varepsilon_4 \to k_4}{=} -igf^{abc}\Gamma^{3abc}_{\rm scal}[k_1,k_2,k_3+k_4] \,. \tag{5.66}$$

Above calculations was for two inequivalent orderings, but if we continue with other orderings we will get the other permutations in eq. (5.57), which indicates that the four-gluon amplitude satisfies the Ward identity

$$k_1^\mu \Gamma^{abcd}_{\mu\nu\alpha\beta} = -igf_{abe}\Gamma^{cde}_{\nu\alpha\beta}(k_1+k_2,k_3,k_4) + {\rm perm}\,. \tag{5.67}$$

Eq. (5.57) was determined for the off-shell four-gluon vertex after considerable calculations from the pinched techniques in [216] and since then there is no generalization for the Ward identity for N-gluon off-shell. In the next Section we will present the generalization of the Ward identity for N-gluon vertex in our formalism.

5.6.2 Off-shell N-gluon Ward-identity

In this Section we generalize the above Ward identity which was obtained for four-gluon vertex to N-gluon vertex, which according to our knowledge is not in the literature. To do so we first recall N-gluon amplitude in the vertex operator level

$$\Gamma^{a_1\cdots a_N}_{\rm scal}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (ig)^N \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int \mathcal{D}x\, e^{-\int_0^T \frac{1}{4}\dot{x}^2} \Big\langle V^g_{\rm scal}[k_1,\varepsilon_1]\cdots V^g_{\rm scal}[k_N,\varepsilon_N] \Big\rangle \,. \tag{5.68}$$

Now if we replace a polarization of one of the external gluons with its momenta, namely $\varepsilon_i \to k_i$ which makes the vertex operator a total derivative as we have discussed above,

$$V^g_{\rm scal}[k_i,\varepsilon_i] \stackrel{\varepsilon_i \to k_i}{=} -iT^{a_i}\Big[e^{ik_i\cdot x(\tau_{i+1})} - e^{ik_i\cdot x(\tau_{i-1})}\Big] \,. \tag{5.69}$$

By plug it into eq. (5.68) one gets (for $\tau_1 > \tau_2 > \cdots > \tau_N$)

$$\begin{aligned}
&\Gamma^{a_1\cdots a_N}_{\rm scal}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] \overset{\varepsilon_i\to k_i}{=} -i(ig)^N\,\mathrm{tr}(T^{a_1}\cdots T^{a_{i-1}}T^{a_i}T^{a_{i+1}}\cdots T^{a_N})\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\\
&\times\int\mathcal{D}x\,e^{-\int_0^T\frac14\dot x^2}\bigg\{\int_0^T d\tau_1\varepsilon_1\cdot\dot x(\tau_1)e^{ik_1\cdot x(\tau_1)}\cdots\int_0^{\tau_{i-2}}d\tau_{i-1}\varepsilon_{i-1}\cdot\dot x(\tau_{i-1})e^{i(k_{i-1}+k_i)\cdot x(\tau_{i-1})}\\
&\times\int_0^{\tau_{i-1}}d\tau_{i+1}\varepsilon_{i+1}\cdot\dot x(\tau_{i+1})\,e^{ik_{i+1}\cdot x(\tau_{i+1})}\cdots\int_0^{\tau_{N-1}}d\tau_N\varepsilon_N\cdot\dot x(\tau_N)e^{ik_N\cdot x(\tau_N)}\\
&\qquad\qquad-\int_0^T d\tau_1\varepsilon_1\cdot\dot x(\tau_1)e^{ik_1\cdot x(\tau_1)}\cdots\int_0^{\tau_{i-2}}d\tau_{i-1}\varepsilon_{i-1}\cdot\dot x(\tau_{i-1})e^{ik_{i-1}\cdot x(\tau_{i-1})}\\
&\times\int_0^{\tau_{i-1}}d\tau_{i+1}\varepsilon_{i+1}\cdot\dot x(\tau_{i+1})\,e^{i(k_i+k_{i+1})\cdot x(\tau_{i+1})}\cdots\int_0^{\tau_{N-1}}d\tau_N\varepsilon_N\cdot\dot x(\tau_N)e^{ik_N\cdot x(\tau_N)}\bigg\}\,.
\end{aligned}\tag{5.70}$$

Now if we do the same for other inequivalent orderings which in the case of general N is $(N-1)!$, for example for let us do the same change to $\tau_1 > \tau_2 > \cdots\tau_{i+1} > \tau_i > \cdots\tau_N$ ordering, which leads to:

$$\begin{aligned}
&\Gamma^{a_1\cdots a_N}_{\rm scal}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] \overset{\varepsilon_i\to k_i}{=} -i(ig)^N\,\mathrm{tr}(T^{a_1}\cdots T^{a_{i+1}}T^{a_i}T^{a_{i-1}}\cdots T^{a_N})\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\\
&\times\int\mathcal{D}x\,e^{-\int_0^T\frac14\dot x^2}\bigg\{\int_0^T d\tau_1\varepsilon_1\cdot\dot x(\tau_1)e^{ik_1\cdot x(\tau_1)}\cdots\int_0^{\tau_{i-2}}d\tau_{i-1}\varepsilon_{i-1}\cdot\dot x(\tau_{i-1})e^{ik_{i-1}\cdot x(\tau_{i-1})}\\
&\times\int_0^{\tau_{i-1}}d\tau_{i+1}\varepsilon_{i+1}\cdot\dot x(\tau_{i+1})\,e^{i(k_i+k_{i+1})\cdot x(\tau_{i+1})}\cdots\int_0^{\tau_{N-1}}d\tau_N\varepsilon_N\cdot\dot x(\tau_N)e^{ik_N\cdot x(\tau_N)}\\
&\qquad\qquad-\int_0^T d\tau_1\varepsilon_1\cdot\dot x(\tau_1)e^{ik_1\cdot x(\tau_1)}\cdots\int_0^{\tau_{i-2}}d\tau_{i-1}\varepsilon_{i-1}\cdot\dot x(\tau_{i-1})e^{ik_{i-1}\cdot x(\tau_{i-1})}\\
&\times\int_0^{\tau_{i-1}}d\tau_{i+1}\varepsilon_{i+1}\cdot\dot x(\tau_{i+1})\,e^{ik_{i+1}\cdot x(\tau_{i+1})}\int_0^{\tau_{i+1}}d\tau_{i+2}\varepsilon_{i+2}\cdot\dot x(\tau_{i+2})\,e^{i(k_i+k_{i+2})\cdot x(\tau_{i+1})}\\
&\times\cdots\int_0^{\tau_{N-1}}d\tau_N\varepsilon_N\cdot\dot x(\tau_N)e^{ik_N\cdot x(\tau_N)}\bigg\}\,.
\end{aligned}\tag{5.71}$$

Again by adding the second term of (5.70) and the first term of (5.71) inside the curly brackets in their lhs, one gets

$$\begin{aligned}
&i(ig)^N\mathrm{tr}(T^{a_1}T^{a_2}\cdots[T^{a_i},T^{a_{i+1}}]\cdots T^{a_N})\int_0^\infty\frac{dT}{T}\,e^{-m^2T}\int\mathcal{D}x\,e^{-\int_0^T\frac14\dot x^2}\\
&\int_0^T d\tau_1\varepsilon_1\cdot\dot x(\tau_1)e^{ik_1\cdot x(\tau_1)}\cdots\int_0^{\tau_{i-2}}d\tau_{i-1}\varepsilon_{i-1}\cdot\dot x(\tau_{i-1})e^{ik_{i-1}\cdot x(\tau_{i-1})}\\
&\times\int_0^{\tau_{i-1}}d\tau_{i+1}\varepsilon_{i+1}\cdot\dot x(\tau_{i+1})\,e^{i(k_i+k_{i+1})\cdot x(\tau_{i+1})}\cdots\int_0^{\tau_{N-1}}d\tau_N\varepsilon_N\cdot\dot x(\tau_N)e^{ik_N\cdot x(\tau_N)}\,.
\end{aligned}\tag{5.72}$$

Eq. (5.72) presents $N-1$-vertex operator with a change of the momentum of $(i+1)^{\rm th}$ vertex operator which modifies to $V^g_{\rm scal}[k_i+k_{i+1},\varepsilon_{i+1}]$. Again by using $[T^a,T^b]=iT^cf^{abc}$, eq. (5.72) can be written as

$$-ig\,(ig)^{N-1} f^{a_i a_{i+1} c}\,\mathrm{tr}(T^{a_1}T^{a_2}\cdots T^{c}\cdots T^{a_N}) \int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int \mathcal{D}x\, e^{-\int_0^T \frac{1}{4}\dot{x}^2} \times \Big\langle V^{g}_{\rm scal}[k_1,\varepsilon_1]\cdots V^{g}_{\rm scal}[k_i+k_{i+1},\varepsilon_{i+1}]\cdots V^{g}_{\rm scal}[k_N,\varepsilon_N]\Big\rangle\,. \tag{5.73}$$

If we consider all inequivalent orderings of the external gluons, and considering the same change for all of them ($\varepsilon_i \to k_i$), we obtain the change of the other vertex operators. In general one can write down the Ward identity for N off-shell gluon as

$$k_1^{\mu_1}\Gamma^{a_1a_2\cdots a_N}_{\mu_1\mu_2\cdots\mu_N} = -igf_{a_1a_2c}\Gamma^{ca_3a_4\cdots a_N}_{\mu_2\mu_3\cdots\mu_N}[k_1+k_2,k_3,\cdots,k_N] + \mathrm{perm}\,, \tag{5.74}$$

As one can see in (5.73) beside of the structure constant f^{abc} we have the trace over $N-1$ color-matrices which can be written in term of totally symmetric and antisymmetric trace structures, see [217], for example

$$\mathrm{tr}(T^aT^bT^c) = d^{abc} + \frac{1}{2}f^{abc}\,, \tag{5.75}$$

where d^{abc} is the case of $n=3$ of the more general totally symmetric trace structure, summed over all $n!$ permutations $\mathcal{P}(a_1\cdots a_n)$,

$$d^{a_1\cdots a_n} = \frac{1}{n!}\sum_{\sigma\in\mathcal{P}(a_1\cdots a_n)} \mathrm{tr}(T^{a_{(\sigma_1)}}\cdots T^{a_{(\sigma_n)}})\,. \tag{5.76}$$

As another example one can decompose the trace of four color-matrix as

$$\mathrm{tr}(T^aT^bT^cT^d) = d^{abcd} + \frac{1}{2}(f^{bce}d^{ead} - f^{ade}d^{ebc}) - \frac{1}{6}(f^{ade}f^{ebc} - f^{abe}f^{ecd})\,. \tag{5.77}$$

5.7 Form factors of the off-shell four-gluon vertex

In this Section we present the final structure of the off-shell four-gluon vertex for the scalar case in the Q-representation. As we know the boundary terms reprint the two- and three- gluon vertex and from Chapter 4 they have just five form factors which are known from our calculations in Chapter 4. So the non-trivial part of the form factors of the off-shell four- gluon at one-loop come from the bulk term, what we called Q_4 in the Q-representation. A nice way to see the real different structures from Q_4 is drawing some diagrams and show them in a diagrammatic way. Since we are in the non-abelian theory and since we have ordered integrals and also color ordering for the external gluons the number of different integrals in (5.37) for scalar loop are more than abelian case. In the abelian case all the integral-variables u_i run from 0 to 1 and they are not ordered since we deal with photons. But here we have different story and we should take care of these orderings. We recall the Q_4 polynomials as

$$
\begin{aligned}
Q_4 &= Q_4^4 + Q_4^3 + Q_4^2 - Q_4^{22}\,, \\
Q_4^4 &= \dot{G}(1234) + \dot{G}(1243) + \dot{G}(1324)\,, \\
Q_4^3 &= \dot{G}(123)T(4) + \dot{G}(234)T(1) + \dot{G}(341)T(2) + \dot{G}(412)T(3)\,, \\
Q_4^2 &= \dot{G}(12)T(34) + \dot{G}(13)T(24) + \dot{G}(14)T(23)\,, \\
& \quad + \dot{G}(23)T(14) + \dot{G}(24)T(13) + \dot{G}(34)T(12)\,, \\
Q_4^{22} &= \dot{G}(12)\dot{G}(34) + \dot{G}(13)\dot{G}(24) + \dot{G}(14)\dot{G}(23)\,.
\end{aligned}
\tag{5.78}
$$

To start and to find the number of different integrals we begin with Q_4^4, we have three diagrams for this polynomial which are shown in Fig. 5.7, Fig. 5.8 and Fig. 5.9. They represent the following parameter integrals (after rescaling $\tau_i = Tu_i$)

$$
\begin{aligned}
&\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \int_0^{u_3} du_4 \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}\,, \\
&\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \int_0^{u_3} du_4 \dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31}\,, \\
&\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \int_0^{u_3} du_4 \dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41}\,.
\end{aligned}
\tag{5.79}
$$

Now if we do the following changes of u_i variables $u_1 \to u_2 \to u_3 \to u_4 \to u_1$ (cyclic symmetry) the last two integrals are the same and they just represent one structure as it is obvious from Fig. 5.8 and Fig. 5.9, so for Q_4^4 we have only two inequivalent structures, we call them **planar** and **non-planar**.

These two diagrams in the scalar loop case can be represented as

$$
\begin{aligned}
Q^4_{4\text{plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\text{tr}(f_1f_2f_3f_4)\int_0^\infty dT T^{3-\frac{D}{2}}e^{-m^2T} \\
&\times \int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \int_0^{u_3} du_4 \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41} e^{T\sum_{i<j} G_{Bij}k_i\cdot k_j}\,,
\end{aligned}
\tag{5.80}
$$

$$
\begin{aligned}
Q^4_{4\text{non-plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\text{tr}(f_1f_2f_4f_3)\int_0^\infty dT T^{3-\frac{D}{2}}e^{-m^2T} \\
&\times \int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \int_0^{u_3} du_4 \dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31} e^{T\sum_{i<j} G_{Bij}k_i\cdot k_j}\,.
\end{aligned}
\tag{5.81}
$$

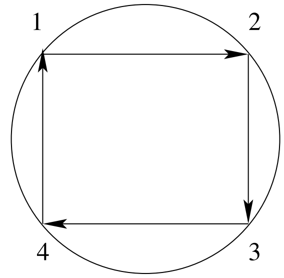


Figure 5.7: This diagram correspond to four-cycle $\dot{G}(1234)$, we call it **Planar**.

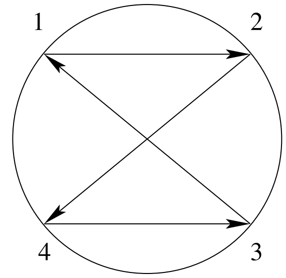


Figure 5.8: This diagram correspond to four-cycle $\dot{G}(1243)$, we call it **Non-planar**.

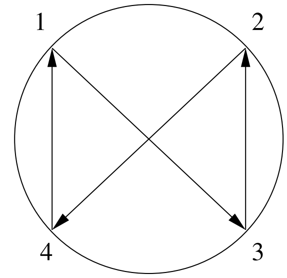


Figure 5.9: This diagram correspond to four-cycle $\dot{G}(1324)$, we call it **Non-planar**.

After the performance of the T-integral, they can be written as

$$
\begin{aligned}
Q^4_{4\text{plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\mathrm{tr}(f_1f_2f_3f_4)\Gamma\Big(4-\frac{D}{2}\Big) \\
&\quad\times\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\int_0^{u_3}du_4\frac{\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}}{[m^2-\sum_{i<j}G_{Bij}k_i\cdot k_j]^{4-\frac{D}{2}}}\,,
\end{aligned}
\tag{5.82}
$$

$$
\begin{aligned}
Q^4_{4\text{non}-\text{plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\mathrm{tr}(f_1f_2f_4f_3)\Gamma\Big(4-\frac{D}{2}\Big) \\
&\quad\times\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\int_0^{u_3}du_4\frac{\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31}}{[m^2-\sum_{i<j}G_{Bij}k_i\cdot k_j]^{4-\frac{D}{2}}}\,.
\end{aligned}
\tag{5.83}
$$

In the following we transform our integrals to the Schwinger/Feynman parameter integrals α_i, so we fixe $u_4 = 0$ and define

$$
\begin{aligned}
u_1 &= \alpha_2+\alpha_2+\alpha_3 = 1-\alpha_1\,, \\
u_2 &= \alpha_3+\alpha_4 = 1-\alpha_1-\alpha_2\,, \\
u_3 &= \alpha_4 = 1-\alpha_1-\alpha_2-\alpha_3\,,
\end{aligned}
\tag{5.84}
$$

where $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 1$.

According to these changes the Green functions and its derivatives can be written as

$$\begin{aligned}
G_{B12} &= \alpha_2(1-\alpha_2) \quad &, \quad \dot{G}_{B12} = 1-2\alpha_2\,,\\
G_{B13} &= (\alpha_1+\alpha_4)(1-\alpha_1-\alpha_4) \quad &, \quad \dot{G}_{B31} = 1-2(\alpha_1+\alpha_4)\,,\\
G_{B14} &= \alpha_1(1-\alpha_1) \quad &, \quad \dot{G}_{B41} = 1-2\alpha_1\,,\\
G_{B23} &= \alpha_3(1-\alpha_3) \quad &, \quad \dot{G}_{B23} = 1-2\alpha_3\,,\\
G_{B24} &= (\alpha_1+\alpha_2)(1-\alpha_1-\alpha_2) \quad &, \quad \dot{G}_{B24} = 1-2(\alpha_3+\alpha_4)\,,\\
G_{B34} &= \alpha_4(1-\alpha_4) \quad &, \quad \dot{G}_{B34} = 1-2\alpha_4\,,
\end{aligned} \tag{5.85}$$

and also the $e^{(\cdot)}$ can be transformed to

$$\begin{aligned}
\sum_{i<j=1}^{4} G_{Bij}k_i\cdot k_j &= -\alpha_1\alpha_2k_1^2-\alpha_2\alpha_3k_2^2-\alpha_3\alpha_4k_3^2-\alpha_1\alpha_4k_4^2-\alpha_2\alpha_4(k_1+k_4)^2-\alpha_1\alpha_3(k_1+k_2)^2\\
&= -\alpha_1\alpha_2k_1^2-\alpha_2\alpha_3k_2^2-\alpha_3\alpha_4k_3^2-\alpha_1\alpha_4k_4^2-\alpha_2\alpha_4u-\alpha_1\alpha_3s\,,
\end{aligned} \tag{5.86}$$

where s and u are the Mandelstam variables.

Eq. (5.80) and (5.81) in term of α_i parameters can be written as

$$\begin{aligned}
Q^4_{4\text{plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_1f_2f_3f_4)\\
&\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}\,,
\end{aligned} \tag{5.87}$$

$$\begin{aligned}
Q^4_{4\text{non-plan}} &= -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_1f_2f_4f_3)\\
&\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_1)(1-2\alpha_3)(1-2\alpha_4)(1-2\alpha_3-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}\,,
\end{aligned} \tag{5.88}$$

which are two structures from the Q_4^4 part of the bulk term.

For Q_4^3 we have two different integers which represent two different form factors as are shown in Fig. 5.10 and Fig. 5.11 which can be written as

$$\begin{aligned}
Q^4_{3\text{plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_1f_2f_3)(\varepsilon_4\cdot k_1)\\
&\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}\,,
\end{aligned} \tag{5.89}$$

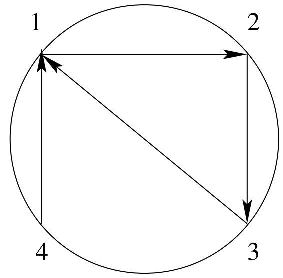


Figure 5.10: This diagram correspond to $\dot{G}(123)\dot{G}_{B41}\varepsilon_4 \cdot k_1$, under cyclic permutation of indices it appears eight times, we call it **Planar**.

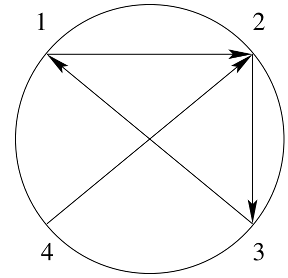


Figure 5.11: This diagram correspond to $\dot{G}(123)\dot{G}_{B42}\varepsilon_4 \cdot k_2$, under cyclic permutation of indices it appears four times, we call it **Non-planar**.

$$\begin{aligned} Q^4_{3\text{non}-\text{plan}} &= -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_1f_2f_3)(\varepsilon_4\cdot k_2) \\ &\times \int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_3-2\alpha_4)(1-2\alpha_1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \end{aligned} \tag{5.90}$$

For Q_4^3 we have two different integers which represent two different form factors as are shown in Fig. 5.10 and Fig. 5.11.
For Q_4^2 we have eight different structures which are shown in Fig. 5.12 - Fig. 5.19 those can be written as

$$\begin{aligned} Q^2_{4\text{qopp}} &= -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_2f_4)(\varepsilon_3\cdot k_4)(\varepsilon_4\cdot k_1)T^{a_1}T^{a_2}T^{a_3}T^{a_4} \\ &\times \int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_3-2\alpha_4)^2(1-2\alpha_1)(1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \end{aligned} \tag{5.91}$$

$$\begin{aligned} Q^2_{4\text{NPopp}} &= -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_2f_4)(\varepsilon_3\cdot k_4)(\varepsilon_1\cdot k_2)T^{a_1}T^{a_2}T^{a_3}T^{a_4} \\ &\times \int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_3-2\alpha_4)^2(1-2\alpha_2)(1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \end{aligned} \tag{5.92}$$

$$\begin{aligned} Q^2_{4\text{Zopp}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_2f_4)(\varepsilon_1\cdot \varepsilon_3)(k_3\cdot k_4)T^{a_1}T^{a_2}T^{a_3}T^{a_4} \\ &\times \int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_3-2\alpha_4)^2(1-2\alpha_1-2\alpha_4)(1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \end{aligned} \tag{5.93}$$

$$Q^2_{4\mathrm{qadj}} = -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\mathrm{tr}(f_1f_2)(\varepsilon_3\cdot k_1)(\varepsilon_4\cdot k_1)T^{a_1}T^{a_2}T^{a_3}T^{a_4}$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_2)^2(1-2\alpha_1-2\alpha_4)(1-2\alpha_1)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \tag{5.94}$$

$$Q^2_{4\mathrm{NPadj}} = \frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\mathrm{tr}(f_1f_2)(\varepsilon_3\cdot k_1)(\varepsilon_4\cdot k_2)T^{a_1}T^{a_2}T^{a_3}T^{a_4}$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_2)^2(1-2\alpha_1-2\alpha_4)(1-2\alpha_3-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \tag{5.95}$$

$$Q^2_{4\mathrm{Zadj}} = \frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\mathrm{tr}(f_1f_2)(\varepsilon_4\cdot k_3)(\varepsilon_3\cdot k_1)T^{a_1}T^{a_2}T^{a_3}T^{a_4}$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_2)^2(1-2\alpha_1-2\alpha_4)(1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \tag{5.96}$$

$$Q^2_{4\mathrm{Eadj}} = -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\mathrm{tr}(f_1f_2)(\varepsilon_3\cdot\varepsilon_4)(k_4\cdot k_1)T^{a_1}T^{a_2}T^{a_3}T^{a_4}$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_2)^2(1-2\alpha_1)(1-2\alpha_4)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}, \tag{5.97}$$

$$Q^2_{4\mathrm{Aadj}} = \frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\mathrm{tr}(f_1f_2)(\varepsilon_3\cdot\varepsilon_2)(k_4\cdot k_1)T^{a_1}T^{a_2}T^{a_3}T^{a_4}$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_2)^2(1-2\alpha_1)(1-2\alpha_3)}{[m^2+\alpha_1\alpha_2k_1^2+\alpha_2\alpha_3k_2^2+\alpha_3\alpha_4k_3^2+\alpha_1\alpha_4k_4^2+\alpha_2\alpha_4u+\alpha_1\alpha_3s]^{4-\frac{D}{2}}}. \tag{5.98}$$

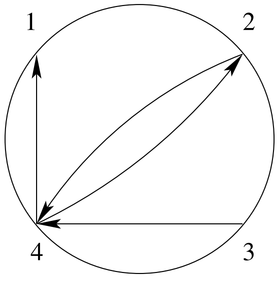


Figure 5.12: This diagrams corresponds to $\dot{G}(24)\dot{G}_{B34}\dot{G}_{B41}\varepsilon_3 \cdot k_4\varepsilon_4 \cdot k_1$, under cyclic permutation of indices it appears four times, we call it **qopp**.

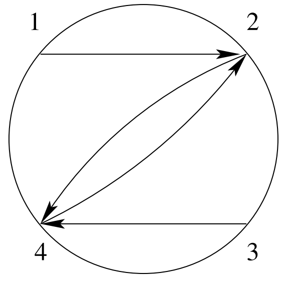


Figure 5.13: This diagram corresponds to $\dot{G}(24)\dot{G}_{B34}\dot{G}_{B12}\varepsilon_3 \cdot k_4\varepsilon_1 \cdot k_2$, under cyclic permutation of indices it appears four times, we call it **NPopp**.

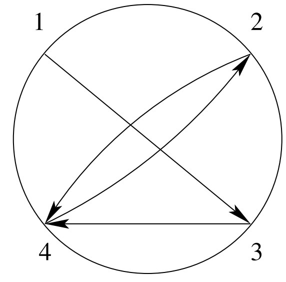


Figure 5.14: This diagram corresponds to $\dot{G}(24)\dot{G}_{B13}\dot{G}_{B34}\varepsilon_1 \cdot \varepsilon_3 k_3 \cdot k_4$, under cyclic permutation of indices it appears sixteen times, we call it **Zopp**.

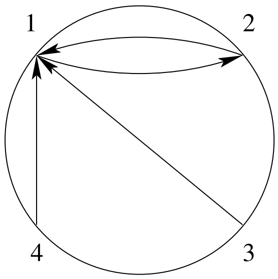


Figure 5.15: This diagram corresponds to $\dot{G}(12)\dot{G}_{B31}\dot{G}_{B41}\varepsilon_3 \cdot k_1\varepsilon_4 \cdot k_1$, under cyclic permutation of indices it appears eight times, we call it **qadj**.

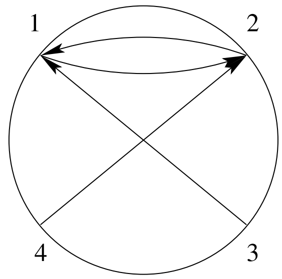


Figure 5.16: This diagram corresponds to $\dot{G}(12)\dot{G}_{B31}\dot{G}_{B42}\varepsilon_3 \cdot k_1\varepsilon_4 \cdot k_2$, under cyclic permutation of indices it appears four times, we call it **NPadj**.

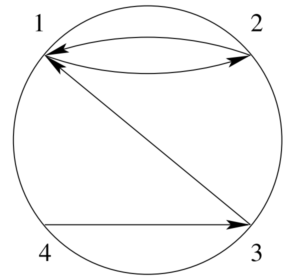


Figure 5.17: This diagram corresponds to $\dot{G}(12)\dot{G}_{B43}\dot{G}_{B31}\varepsilon_4 \cdot k_4\varepsilon_3 \cdot k_1$, under cyclic permutation of indices it appears sixteen times, we call it **Zadj**.

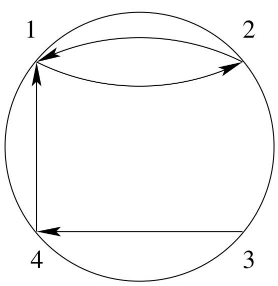


Figure 5.18: This diagram corresponds to $\dot{G}(12)\dot{G}_{B34}\dot{G}_{B41}\varepsilon_3 \cdot \varepsilon_4 k_4 \cdot k_1$, under cyclic permutation of indices it appears sixteen times, we call it **Eadj**.

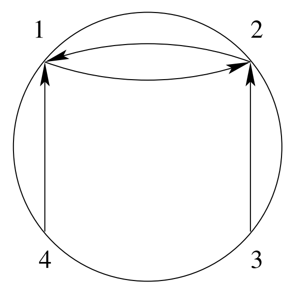


Figure 5.19: This diagram corresponds to $\dot{G}(12)\dot{G}_{B32}\dot{G}_{B41}\varepsilon_3 \cdot k_3\varepsilon_4 \cdot k_1$, under cyclic permutation of indices it appears four times, we call it **Aadj**.

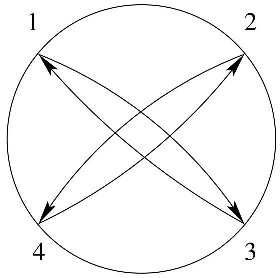


Figure 5.20: This corresponds to $\dot{G}(13)\dot{G}(24)$, it appears once, we call it **Non-planar**.

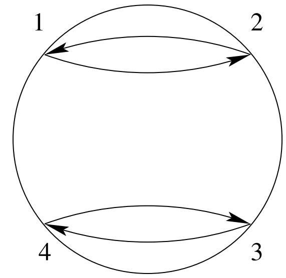


Figure 5.21: This corresponds to $\dot{G}(12)\dot{G}(34)$, under cyclic permutation of indices it appears twice, we call it **Planar**.

And finally the last part of the bulk term Q_4^{22} has two inequivalent integrals that are shown in Fig. 5.20 and Fig. 5.21 which can be written as

$$\begin{aligned} Q^{22}_{4\text{plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_1 f_2)\text{tr}(f_3 f_4)T^{a_1}T^{a_2}T^{a_3}T^{a_4} \\ &\times \int_0^1 \prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4 \alpha_i)(1-2\alpha_2)^2(1-2\alpha_4)^2}{[m^2+\alpha_1\alpha_2 k_1^2+\alpha_2\alpha_3 k_2^2+\alpha_3\alpha_4 k_3^2+\alpha_1\alpha_4 k_4^2+\alpha_2\alpha_4 u+\alpha_1\alpha_3 s]^{4-\frac{D}{2}}}\,, \end{aligned} \tag{5.99}$$

$$\begin{aligned} Q^{22}_{4\text{Non-plan}} &= \frac{g^4}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\text{tr}(f_1 f_3)\text{tr}(f_2 f_4)T^{a_1}T^{a_2}T^{a_3}T^{a_4} \\ &\times \int_0^1 \prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4 \alpha_i)(1-2\alpha_1-2\alpha_4)^2(1-2\alpha_3-2\alpha_4)^2}{[m^2+\alpha_1\alpha_2 k_1^2+\alpha_2\alpha_3 k_2^2+\alpha_3\alpha_4 k_3^2+\alpha_1\alpha_4 k_4^2+\alpha_2\alpha_4 u+\alpha_1\alpha_3 s]^{4-\frac{D}{2}}}\,. \end{aligned} \tag{5.100}$$

As a summary, we have shown that the bulk term for the off-shell four-gluon vertex has only fourteen form-factors which have been shown diagrammatically and also their parameter integral representations. Note that the bulk term of four gluon vertex is totally finite in $D = 4$ dimension.

As we have discussed before the boundary terms contribute to the covariantization of the two-and three-point amplitudes should give five inequivalent parameter integral representations or form factors since the bulk terms of two- and three-point amplitudes have just five form factors.

For $Q_3^3(i[j,k]l)$ we have just one independent diagram which is shown in Fig. 5.22. The parameter integral representation for this form factor is written as

$$Q_3^3 = -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(3-\frac{D}{2}\Big)\big(f_4\varepsilon_2 f_3\varepsilon_1 - f_4\varepsilon_1 f_3\varepsilon_2\big)$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_1-2\alpha_4)(1-2\alpha_1)(1-2\alpha_4)}{[m^2+\alpha_4(\alpha_1+\alpha_2)k_4^2+\alpha_3(\alpha_1+\alpha_2)s+\alpha_3\alpha_4k_3^2]^{3-\frac{D}{2}}}\,, \tag{5.101}$$

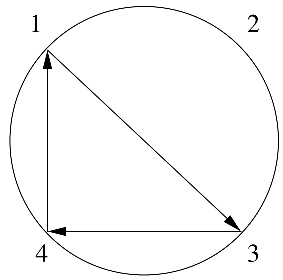


Figure 5.22: This diagram corresponds to $\dot{G}_{B13}\dot{G}_{B34}\dot{G}_{B41}$.

In total for $Q_3^2(i\to j)$ and Q_{ext} we have three form factors or diagrams, Fig. 5.23, 5.24 and 5.25, where their parameter integral representations can be written correspondingly as

$$Q_3^2(1\to 2) = -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(3-\frac{D}{2}\Big)(\varepsilon_2 f_4\varepsilon_1)(\varepsilon_3\cdot k_1)$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_1-2\alpha_4)(1-2\alpha_1)^2}{[m^2+\alpha_4(\alpha_1+\alpha_2)k_4^2+\alpha_3(\alpha_1+\alpha_2)s+\alpha_3\alpha_4k_3^2]^{3-\frac{D}{2}}}\,, \tag{5.102}$$

$$Q_3^2(2\to 3) = \frac{g^4}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(3-\frac{D}{2}\Big)(\varepsilon_3 f_4\varepsilon_2)(\varepsilon_1\cdot k_3)$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_3-2\alpha_4)^2(1-2\alpha_1-2\alpha_4)}{[m^2+\alpha_4(\alpha_2+\alpha_3)u+\alpha_1(\alpha_2+\alpha_3)k_1^2+\alpha_1\alpha_4k_4^2]^{3-\frac{D}{2}}}\,, \tag{5.103}$$

$$Q_{\text{ext}}(1\to 2) = \frac{g^4}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(3-\frac{D}{2}\Big)\mathrm{tr}(f_2f_3)(\varepsilon_1\cdot\varepsilon_4)$$
$$\times\int_0^1\prod_{i=1}^4 d\alpha_i \frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_3)^2(1-2\alpha_1)}{[m^2+\alpha_4(\alpha_1+\alpha_2)k_4^2+\alpha_3(\alpha_1+\alpha_2)s+\alpha_3\alpha_4k_3^2]^{3-\frac{D}{2}}}\,, \tag{5.104}$$

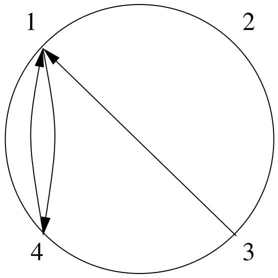


Figure 5.23: This diagram corresponds to $\dot{G}_{B14}\dot{G}_{B41}\dot{G}_{B31}$.

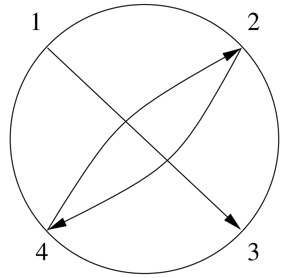


Figure 5.24: This diagram corresponds to $\dot{G}_{B24}\dot{G}_{B42}\dot{G}_{B13}$.

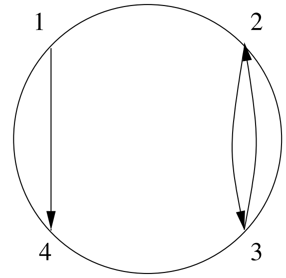


Figure 5.25: This diagram corresponds to $\dot{G}_{B23}\dot{G}_{B32}\dot{G}_{B14}$.

where in the above we used the following exponentials in term of α_i parameters for one-pinch boundary terms

$$
\begin{aligned}
e^{(1\to 2)} &= e^{G_{B23}k_3\cdot(k_1+k_2)+G_{B24}k_4\cdot(k_1+k_2)+G_{B34}k_3\cdot k_4} \to e^{-\alpha_3(\alpha_1+\alpha_2)s-\alpha_4(\alpha_1+\alpha_2)k_4^2-\alpha_3\alpha_4 k_3^2}\,,\\
e^{(2\to 3)} &= e^{G_{B13}k_1\cdot(k_2+k_3)+G_{B34}k_4\cdot(k_2+k_3)+G_{B14}k_1\cdot k_4} \to e^{-\alpha_4(\alpha_2+\alpha_3)u-\alpha_1(\alpha_2+\alpha_3)k_1^2-\alpha_4\alpha_1 k_4^2}\,,\\
e^{(3\to 4)} &= e^{G_{B14}k_1\cdot(k_3+k_4)+G_{B24}k_2\cdot(k_3+k_4)+G_{B12}k_1\cdot k_2} \to e^{-\alpha_1(\alpha_3+\alpha_4)s-\alpha_2(\alpha_3+\alpha_4)k_2^2-\alpha_1\alpha_2 k_1^2}\,,\\
e^{(4\to 1)} &= e^{G_{B21}k_2\cdot(k_1+k_4)+G_{B31}k_3\cdot(k_1+k_4)+G_{B23}k_2\cdot k_3} \to e^{-\alpha_2(\alpha_1+\alpha_4)u-\alpha_3(\alpha_1+\alpha_4)k_3^2-\alpha_2\alpha_3 k_2^2}\,.
\end{aligned}
\tag{5.105}
$$

And finally for the double pinched Q_2^2 we have just one form factor, which it is shown in Fig. 5.26 and its parameter integral representation can be written as

$$
\begin{aligned}
Q_2^2(1\to 2, 3\to 4) &= -\frac{g^4}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(2-\frac{D}{2}\Big)\big(\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4-\varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3\big)\\
&\quad\times\int_0^1\prod_{i=1}^4 d\alpha_i\,\frac{\delta(1-\sum_{i=1}^4\alpha_i)(1-2\alpha_1-2\alpha_4)^2}{[m^2+\alpha_3(\alpha_1+\alpha_2+\alpha_4)s]^{2-\frac{D}{2}}}\,,
\end{aligned}
\tag{5.106}
$$

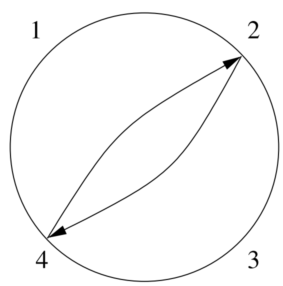


Figure 5.26: This diagram corresponds to $\dot{G}_{B42}\dot{G}_{B24}$.

$$
\begin{aligned}
e^{(1\to 2,3\to 4)} &= e^{G_{B23}(k_1+k_2)\cdot(k_3+k_4)} \to e^{\alpha_3(1-\alpha_3)s} = e^{-(\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_3\alpha_4)s}\,, \\
e^{(1\to 4,2\to 3)} &= e^{G_{B34}(k_1+k_4)\cdot(k_2+k_4)} \to e^{\alpha_4(1-\alpha_4)u} = e^{-(\alpha_1\alpha_4+\alpha_2\alpha_4+\alpha_3\alpha_4)u}\,.
\end{aligned}
\tag{5.107}
$$

In this Section we presented the final form factors of the off-shell four-gluon vertex and also the lower point cases (two and three) diagrammatically as well as their parameter integral representations in D-dimension. In the next Section we will discuss off-shell four-gluon vertex in the $\mathcal{N}=4$ super Yang-Mills theory.

5.8 Four gluon in $\mathcal{N}=4$ Super Yang-Mills

In this Section we continue our discussionin the off-shell four-gluon vertex but in the $\mathcal{N}=4$ super Yang-Mills (here after simply $\mathcal{N}=4$ SYM). In the following first we make a brief review on $\mathcal{N}=4$ SYM theory and later we will discuss the four gluon vertex in this theory.
In recent years, tremendous progress has been made towards a more complete understanding of the scattering amplitude in $\mathcal{N}=4$ SYM [218]. $\mathcal{N}=4$ SYM is special in that it has more symmetry than any other gauge theory [219, 220]. More than three decades the S-matrix of this theory has been under study [221]. Numerous discoveries have been made like the application of the AdS/CFT correspondence [218] to gluon scattering at strong coupling [222], a hidden dual superconformal symmetry of the planar theory [223], and a dual description of the leading singularities of the S-matrix as integrals over periods in a Grassmann manifold [224].

The $\mathcal{N}=4$ SYM model consists of a gauge field A_μ, four Majorana fermions ψ_i, three real scalars X_p and three real pseudo-scalars Y_p. These fields are in the adjoint representation of a compact gauge group, G. The Lagrange density of $\mathcal{N}=4$ SYM is given by [225]

$$
\begin{aligned}
\mathcal{L} &= \mathrm{tr}\Big\{ -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_i \not{D}\psi_i + D^\mu X_p D_\mu X_p + D^\mu Y_q D_\mu Y_q \\
&\quad -ig\bar{\psi}_i\alpha^p_{ij}[X_p,\psi_j] + g\bar{\psi}_i\gamma_5\beta^q_{ij}[Y_q,\psi_j] \\
&\quad +\frac{g^2}{2}\big([X_l,X_k][X_l,X_k] + [Y_l,Y_k][Y_l,Y_k] + 2[X_l,Y_k][X_l,Y_k]\big)\Big\}
\end{aligned}
\tag{5.108}
$$

where α^p and β^q are are 4×4 matrices given as [2]

$$
\begin{aligned}
\alpha^1 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \alpha^2 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \alpha^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \\
\beta^1 &= \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \beta^2 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \beta^3 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}.
\end{aligned}
\tag{5.109}
$$

The remarkable fact about the classical Lagrangian of $\mathcal{N} = 4$ SYM (5.108) is that it is scale invariance which remains a symmetry of the quantized theory [226], which implies the vanishing of the β functions to all order in perturbation theory [3].
It is understood that the theory is UV finite in perturbation theory [227, 228]. As we already mentioned $\mathcal{N} = 4$ SYM scattering amplitudes are free of UV divergences, so one may think that, the one-loop four-gluon amplitudes in $\mathcal{N} = 4$ SYM is built out of triangle and box diagrams but not the bubbles, since they are the source of UV divergences at one-loop level in dimensional regularization, i.e. $D = 4 - 2\epsilon$. But it turns out that in one-loop $\mathcal{N} = 4$ SYM theory the triangle and bubbles vanish and the above statement is not true. Interestingly if one drops the contributions of $\mathcal{O}(\epsilon)$ and higher orders, this conclusion even holds for N-gluon scattering amplitude [229] [4].

To continue our discussion for the case in hand, the four-gluon off-shell amplitude according to the above statement where in $\mathcal{N} = 4$ SYM the one-loop two- and three- gluon amplitudes vanish makes our calculation extremely simple. To apply this statement in our case our boundary terms vanish (since they contribute to the covariantization of the two-and three-point amplitudes), so what remains is the bulk term. We recall the integral form of Q'-representation (5.2) which is (just for $T^{a_1}T^{a_2}T^{a_3}T^{a_4}$ ordering)

$$
\begin{aligned}
\Gamma_0^{a_1a_2a_3a_4}[k_1, \varepsilon_1; \cdots; k_4, \varepsilon_4] &= (-ig)^4 \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) \int_0^\infty dT (4\pi T)^{-D/2} e^{-m^2 T} \\
&\times \int_0^T d\tau_1 d\tau_2 d\tau_3 \, (-i)^4 Q_4' \, e^{\sum_{i<j=1}^4 G_{Bij} k_i \cdot k_j} ,
\end{aligned}
\tag{5.110}
$$

where as before the τ-integrals are ordered, and we use Q'-representation in (3.38) where in its form all cycle factors are explicit and we removed the cycles from the tails, and moreover all the coefficients in the decomposition turn out to be unity to apply the "Bern-Kosower replacement

[2] σ_0 is the 2×2 identity matrix.

[3] In theoretical physics, specifically quantum field theory, a beta function, $\beta(g)$, encodes the dependence of a coupling parameter, g, on the energy scale, μ, of a given physical process described by quantum field theory. It is defined as $\beta(g) = \frac{\partial g}{\partial \log(\mu)}$.

[4] It is now known that this conclusion holds for one-loop $\mathcal{N} = 4$ scattering amplitude with arbitrary external states.

rule". After rescaling and performance of T-integral, it can be written as

$$\begin{aligned}\Gamma_0^{a_1a_2a_3a_4}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] &= g^4\frac{1}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(4-\frac{D}{2}\Big)\\ &\times\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\frac{Q_4'}{(m^2-\sum_{i<j}G_{Bij}k_i\cdot k_j)^{4-\frac{D}{2}}}\,.\end{aligned} \tag{5.111}$$

The bulk term Q'^4 contains four different structures and we need to consider all of them separately. We start with one- and two-tail polynomials, i.e. $Q_4^{'3}$ and $Q_4^{'2}$. For $\mathcal{N}=4$ in 4-dimension we have the relation between scalar (Γ_0), spinor ($\Gamma_{\frac{1}{2}}$) and gluon plus ghost (Γ_1) satisfy the (4.8) which we recall again

$$3\Gamma_0+2\Gamma_{\frac{1}{2}}+\Gamma_1=0\,. \tag{5.112}$$

We firs show that the contribution of $Q_4^{'3}$ and $Q_4^{'2}$ for $\mathcal{N}=4$ SYM are zero. To do so we need to look at these polynomials for all scalar, spinor and gluon (after subtraction of the ghost loop). For spinor loop

$$\begin{aligned}\Gamma_{\frac{1}{2}}^{a_1a_2a_3a_4}[k_1,\varepsilon_1;\cdots;k_4,\varepsilon_4] &= -2g^4\frac{1}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(4-\frac{D}{2}\Big)\\ &\times\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\frac{\hat{Q}'_4}{(m^2-\sum_{i<j}G_{Bij}k_i\cdot k_j)^{4-\frac{D}{2}}}\,,\end{aligned} \tag{5.113}$$

and for gluon loop we have

$$\begin{aligned}\Gamma_1[\varepsilon_1,k_1;\cdots;\varepsilon_4,k_4] &= -\frac{1}{4}g^4\frac{1}{(4\pi)^{\frac{D}{2}}}\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Gamma\Big(4-\frac{D}{2}\Big)\\ &\times\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\frac{\tilde{Q}'_4}{(m^2-\sum_{i<j}G_{Bij}k_i\cdot k_j)^{4-\frac{D}{2}}}\,.\end{aligned} \tag{5.114}$$

Now we look at $Q_4^{'3}$ and $Q_4^{'2}$ for all these three cases, for scalar one

$$Q_4^{'3}\equiv Q_4^3=\dot{G}(123)T(4)+\dot{G}(234)T(1)+\dot{G}(341)T(2)+\dot{G}(412)T(3)\,. \tag{5.115}$$

Now, its counterpart for the spinor case $\hat{Q}_4^3$ would be:

$$\hat{Q}_4^3=\hat{\dot{G}}(123)T(4)+\hat{\dot{G}}(234)T(1)+\hat{\dot{G}}(341)T(2)+\hat{\dot{G}}(412)T(3)\,, \tag{5.116}$$

where

$$\hat{\dot{G}}(ijk)=\big(\dot{G}_{Bij}\dot{G}_{Bjk}\dot{G}_{Bki}-G_{Fij}G_{Fjk}G_{Fki}\big)Z_3(ijk)\,. \tag{5.117}$$

For $\tau_1>\tau_2>\tau_3>\tau_4$ orderings we have

$$\begin{aligned}\hat{Q}_4^3 &= \big(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}+1\big)Z_3(123)T(4)+\big(\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}+1\big)Z_3(234)T(1)\\ &+\big(\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}+1\big)Z_3(341)T(2)+\big(\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}+1\big)Z_3(412)T(3)\,.\end{aligned} \tag{5.118}$$

For gluon loop, $\tilde{Q}'_4$ is obtained after taking the limit $C \to \infty$, namely

$$\begin{aligned} \lim_{C\to\infty} e^{-CT} \sum_{p=P,A} \sigma_p Z_p \tilde{G}^C_{p,ijk} &= 16\,, \\ \lim_{C\to\infty} e^{-CT} \sum_{p=P,A} \sigma_p Z_p &= -8\,. \end{aligned} \tag{5.119}$$

Then for $\tilde{Q}^3_4$ we have,

$$\begin{aligned} \tilde{Q}^3_4 &= -8\Big[\big(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}+2\big)Z_3(123)T(4)+\big(\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}+2\big)Z_3(234)T(1) \\ &\quad +\big(\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}+2\big)Z_3(341)T(2)+\big(\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}+2\big)Z_3(412)T(3)\Big]\,, \end{aligned} \tag{5.120}$$

Now if we consider the degrees of freedom in $D=4$ for $\mathcal{N}=4$ SYM, one observes that

$$3Q^3_4+2(-2\hat{Q}^3_4)+\big[-\frac{1}{4}(\tilde{G}^3_4)\big]=Q^3_4\,, \tag{5.121}$$

where we substituted (5.115), (5.118) and (5.120), but we should not forget about the ghost contribution, which as we already mentioned is equal to scalar contribution but with different sign $(-Q^3_4)$. It means that Q^3_4-contribution of all loop particles vanishes and they satisfy

$$3\Gamma^3_0+2\Gamma^3_{\frac{1}{2}}+\Gamma^3_1=0\,. \tag{5.122}$$

The same argument holds for Q'^2_4, which it means scalar, spinor, gluon and ghost contributions cancel each other because they satisfy

$$3\Gamma^2_0+2\Gamma^2_{\frac{1}{2}}+\Gamma^2_1=0\,. \tag{5.123}$$

Now we look at the other two parts of the bulk term which do not vanish for $\mathcal{N}=4$ SYM. Q'^4_4 gives

$$Q'^4_4\equiv Q^4_4=\dot{G}(1234)+\dot{G}(1243)+\dot{G}(1324)\,. \tag{5.124}$$

The spinor counterpart gives

$$\begin{aligned} \hat{Q}^4_4 &= \big(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}+1\big)Z_4(1234)+\big(\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31}-1\big)Z_4(1243) \\ &\quad +\big(\dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41}-1\big)Z_4(1324)\,. \end{aligned} \tag{5.125}$$

And for gluon case, since

$$\lim_{C\to\infty} e^{-CT} \sum_{p=P,A} \sigma_p Z_p \tilde{G}^C_{p,1234} = 32\,, \tag{5.126}$$

that leads to

$$\begin{aligned} \tilde{Q}^4_4 &= -8\Big[\big(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}+4\big)Z_4(1234)+\big(\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31}\big)Z_4(1243) \\ &\quad +\big(\dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41}\big)Z_4(1324)\Big]\,. \end{aligned} \tag{5.127}$$

Note that those indices in four-cycle which do not follow the external orderings do not contribute to the $C \to \infty$ limit. By adding these terms one gets

$$3Q_4^4 + 2(-2\hat{Q}_4^4) + \big[-\frac{1}{4}(\tilde{G}_4^4)\big] = Q_4^4 + 4Z_4(1234) + 4Z_4(1243) + 4Z_4(1324)\,, \tag{5.128}$$

which again by considering the ghost part we get

$$3\Gamma_0^4 + 2\Gamma_{\frac{1}{2}}^4 + \Gamma_1^4 = 4\Big[\mathrm{tr}(f_1f_2f_3f_4) + \mathrm{tr}(f_1f_2f_4f_3) + \mathrm{tr}(f_1f_3f_2f_4)\Big]\,, \tag{5.129}$$

where we used the fact that $Z_4(ijkl) = \mathrm{tr}(f_if_jf_kf_l)$.

For $Q_4^{'22}$ scalar case,

$$Q_4^{'22} \equiv Q_4^{22} = \dot{G}(12)\dot{G}(34) + \dot{G}(13)\dot{G}(24) + \dot{G}(14)\dot{G}(23)\,. \tag{5.130}$$

Spinor counterpart can be written as

$$\begin{aligned}\hat{Q}_4^{22} \;&=\; \dot{G}(12)\dot{G}(34) + \dot{G}(12)Z_2(34) + \dot{G}(34)Z_2(12) + Z_2(12)Z_2(34)\\ &+\; \dot{G}(13)\dot{G}(24) + \dot{G}(13)Z_2(24) + \dot{G}(24)Z_2(13) + Z_2(13)Z_2(24)\\ &+\; \dot{G}(14)\dot{G}(23) + \dot{G}(14)Z_2(23) + \dot{G}(23)Z_2(14) + Z_2(14)Z_2(23)\,.\end{aligned} \tag{5.131}$$

Since

$$\begin{aligned}\lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_pZ_p\tilde{G}^C_{p,ij} &= 16\,,\\ \lim_{C\to\infty} e^{-CT}\sum_{p=P,A}\sigma_pZ_p\tilde{G}^C_{p,ij}\,\tilde{G}^C_{p,kl} &= 0\,,\end{aligned} \tag{5.132}$$

which means that only the mixed terms contribute, and it leads to

$$\begin{aligned}\tilde{Q}_4^{22} \;=\; -8\Big\{&\dot{G}(12)\dot{G}(34) + 2\dot{G}(12)Z_2(34) + 2\dot{G}(34)Z_2(12)\\ &+\dot{G}(13)\dot{G}(24) + 2\dot{G}(13)Z_2(24) + 2\dot{G}(24)Z_2(13)\\ &+\dot{G}(14)\dot{G}(23) + 2\dot{G}(14)Z_2(23) + 2\dot{G}(23)Z_2(14)\Big\}\,.\end{aligned} \tag{5.133}$$

Again by adding them, one gets

$$\begin{aligned}3Q_4^4 + 2(-2\hat{Q}_4^4) + (-\frac{1}{4}\tilde{Q}_4^4) \;=\; &\dot{G}(12)\dot{G}(34) + \dot{G}(13)\dot{G}(24) + \dot{G}(14)\dot{G}(23)\\ &-4Z_2(12)Z_2(34) - 4Z_2(13)Z_2(24) - 4Z_2(14)Z_2(23)\,.\end{aligned} \tag{5.134}$$

Note that $Z_2(ij) = \frac{1}{2}\mathrm{tr}(f_if_j)$. By taking into account the ghost contribution, (5.134) leads to

$$3\Gamma_0^{22} + 2\Gamma_{\frac{1}{2}}^{22} + \Gamma_1^{22} = -\Big[\mathrm{tr}(f_1f_2)\mathrm{tr}(f_3f_4) + \mathrm{tr}(f_1f_3)\mathrm{tr}(f_2f_4) + \mathrm{tr}(f_1f_4)\mathrm{tr}(f_2f_3)\Big]\,. \tag{5.135}$$

By putting together (5.129) and (5.135) one gets

$$
\begin{aligned}
3(\Gamma_0^4+\Gamma_0^{22})+2(\Gamma_{\frac{1}{2}}^4+\Gamma_{\frac{1}{2}}^{22})+(\Gamma_1^4+\Gamma_1^{22}) \quad = \quad & 4\Big[\mathrm{tr}(f_1f_2f_3f_4)+\mathrm{tr}(f_1f_2f_4f_3)+\mathrm{tr}(f_1f_3f_2f_4) \\
& -\frac{1}{4}\mathrm{tr}(f_1f_2)\mathrm{tr}(f_3f_4)-\frac{1}{4}\mathrm{tr}(f_1f_3)\mathrm{tr}(f_2f_4)-\frac{1}{4}\mathrm{tr}(f_1f_4)\mathrm{tr}(f_2f_3)\Big]\,.
\end{aligned}
\tag{5.136}
$$

As a summary of this Section, we showed that in $\mathcal{N}=4$ SYM the one-loop two- and three-gluon amplitudes vanish (this relates to the conformal invariance and finiteness of the theory). The one-loop four-gluon vertex is the first non-vanishing one, and it is extremely simple:
all boundary terms cancel out (since they would covariantize the nonexisting lower point amplitudes) and the bulk terms factors [207, 208] as

$$
\Gamma^{a_1a_2a_3a_4}_{\mathcal{N}=4}=4g^4\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,F_{ss}^4B(1234)+\mathrm{non}-\mathrm{cyclic\ permutations}\,, \tag{5.137}
$$

where the whole Lorentz structure is contained in the invariant

$$
\begin{aligned}
F_{ss} \quad = \quad & \mathrm{tr}(f_1f_2f_3f_4)+\mathrm{tr}(f_1f_2f_4f_3)+\mathrm{tr}(f_1f_3f_2f_4) \\
& -\frac{1}{4}\mathrm{tr}(f_1f_2)\mathrm{tr}(f_3f_4)-\frac{1}{4}\mathrm{tr}(f_1f_3)\mathrm{tr}(f_2f_4)-\frac{1}{4}\mathrm{tr}(f_1f_4)\mathrm{tr}(f_2f_3)\,,
\end{aligned}
\tag{5.138}
$$

and $B(1234)$ is the off-shell box integral with momenta $k_1,\cdots,k_4$

$$
\begin{aligned}
B(1234) \quad &= \quad \frac{1}{(4\pi)^{\frac{D}{2}}}\int_0^\infty dT\,e^{-m^2T}T^{3-\frac{D}{2}}\int_0^! du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\,e^{T\sum_{i<j}G_{Bij}k_i\cdot k_j} \\
&= \quad \frac{1}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(4-\frac{D}{2}\Big)\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\frac{1}{\big(m^2-\sum_{i<j}G_{Bij}k_i\cdot k_j\big)^{4-\frac{D}{2}}}\,.
\end{aligned}
\tag{5.139}
$$

Although this explicit form of the massive off-shell one-loop four-gluon amplitude in $\mathcal{N}=4$ appears to be totally new, the Lorentz tensor F_{ss} is well-known to string theorists, since it appears in the low energy expansion of the effective action of the open superstring; see [217] and refs therein.

5.9 Conclusion

Among all of the elementary vertices that appear in the QCD Lagrangian, the four-gluon vertex is the most poorly understood. From the point of view of continuum studies, this fact may be regarded as a consequence of the rich tensorial structure of this object generated by the presence of four color and four Lorentz indices. The analytic studies dedicated of this vertex are very scarce especially at loop orders. From the point of view of lattice simulations, the situation is simpler, in the sense that, to the best of our knowledge, no simulations of four-gluon vertex have been performed. Because of these complications and its rich tensorial structure, the number of form factors is enormous, it is claimed to be around 138 [230], which is out of interests especially from lattice point of view. In this Chapter as we have shown the bulk term for four-gluon vertex produces only (in Feynman gauge) fourteen form factors which have been presented in its final parameter-integrals. This number is quite surprising and we believe that the other form factors which have been studies from other authors are not all independent and as we have shown it is not more than fourteen form factors. In this study we also showed that our four-gluon satisfies the Ward identity, and we also extract the Ward-identity for any N external gluons which is very simple and easy to be seen from our method. At the last Section of this Chapter, we have shown the structure of the off-shell four-gluon vertex in the $\mathcal{N} = 4$ SYM, which turns out to be incredibly simple. The S representation and also complete discussion of the off-shell four-gluon vertex will be published soon [207].

Chapter 6

Photon-Graviton amplitude in worldine formalism

6.1 Introduction

[1] Newton's law of gravity is long ranged, and this suggests the existence of a gravity force mediated by a massless particle called a graviton. In view of the fact that even-spin particle exchanges are fundamentally attractive, observed universal gravitational attraction selects the graviton spin to be an even integer. Until now, all observations have supported a spin-2 graviton picture [231–233]. At present, there does not exist a complete theory of quantum gravity. The main problem is that Einstein gravity is nonrenormalizable [234, 235] because the gravitational constant G has dimension of inverse mass square. In this respect the theory of gravitation is more like other nonrenormalizable effective theories such as the Fermi theory of weak interaction. However, Weinberg [236] showed that it is quite impossible to construct a Lorentz- invariant quantum theory of particles of mass zero and helicity ± 2 without introducing some sort of gauge invariance into the theory. It is well known that the classical theory of gravitational radiation in the linearized version of general relativity has gauge invariance, which is related to the general covariance of the full theory. To gain further insight concerning the massless spin-2 graviton, we consider here physical processes at tree-level and one-loop in the context of the linearized gravity coupled to QED using worldline formulation of QFT.

Among the four fundamental interactions in nature, the gravitational interaction has not yet been successfully quantized. But the challenge of combining the quantum principle with the elegant theory of general relativity, based on general covariance, has been made ceaselessly. While the very small gravitational coupling constant might reduce the importance of theoretical and experimental investigation of quantum gravity, gravity becomes as strong as the other forces near the Planck scale, and it is believed to be crucial in a consistent description of the birth of the Universe according to the Big Bang scenario. Furthermore, the successful unification of electromagnetic and weak interactions in the standard model makes unavoidable the thought that further unifications might be realized for all other fundamental interactions. During past decades, developments of supergravity [237] and superstring theories [153] were inspired by the hope of constructing a consistent unified quantum theory including gravity. In all cases, any common aspect of gravity and other interactions is very much worth exploring.

[1] This Chapter is based on [238–240].

In recent years, much effort has been devoted to the study of the structure of graviton amplitudes. This was largely due to developments in string theory, which led to the prediction that such amplitudes should be much more closely related to gauge theory amplitudes than one would suspect by comparing the Lagrangians or Feynman rules of gravitational and gauge theories. Specifically, the KLT relations in string theory imply that graviton amplitudes should "squares" of gauge theory amplitudes [241, 242]. At tree-level there are some classical results on amplitudes involving gravity [243, 244]. The tree-level Compton-type amplitudes involving gravitons and spin zero, half and one particles were computed [245,246] to verify another remarkable factorization property [247] of the graviton-graviton scattering amplitudes in terms of the photonic Compton amplitudes which will be discussed in the following Sections from worldline viewpoint. At the level of one (matter) loop, mixed graviton-gauge boson amplitudes that have been computed are the graviton-photon- photon vertex [248–252], its nonabelian generalization [253], and the related amplitude for photon-graviton conversion in an external field [254–258]. We believe that new insight into the structural relations between photon and graviton amplitudes might be obtained by studying the N graviton amplitudes involving a massive loop, and more generally the mixed one-loop graviton-photon amplitudes. This series of studies have been started by Schubert et.al and the first study on one-loop photon-photon-graviton was obtained in [75].

In this Chapter we discus the worldline formalism in the presence of external gravitational field. In the worldline formalism a path integral is used to quantize the worldline coordinates of the particle. Contrary to the simpler cases of scalar and vector background, external gravity requires a precise definition of the UV regularization of the path integral [51] as was discussed in the introduction. In the following we discus this issue and we will present some calculations in tree-level Compton scattering for photon-graviton in scalar case, we also represent results for the vacuum polarization diagram (one-loop two graviton). Actually here we report and on-going study of the one-loop photon-graviton amplitudes, using both effective action and worldline techniques. The emphasis is on Kawai-Lewellen-Tye-like relations [241]. In 1986 Kawai, Lewellen and Tye [241] found a relation between tree-level amplitudes in open and closed string theory which in the field theory limit implies relations between amplitudes in gauge theory and in gravity. This way at tree-level e.g. graviton scattering amplitudes are expressed by gluon amplitudes. More precisely, a graviton amplitude can be expressed as a sum of squares of partial color ordered gluon amplitudes. Schematically, such relations are of the form

$$\text{(gravity amplitude)} \sim \text{(gauge amplitude)}^2$$

Their existence is related to the factorization of vertex operators for open and closed strings

$$V^{\text{closed}} = V^{\text{open}}_{\text{left}} \bar{V}^{\text{open}}_{\text{right}} \,. \tag{6.1}$$

In the low-energy effective action this duality leads to drastic simplifications of gravitational interactions. In more technical terms the absence of interactions of left- and right-moving worldsheet fields of the closed string allows to factorize any closed string tree-level amplitude into products of disk amplitudes of left-moving fields and right-moving fields.

For example, at the four and five point level one has [242]

$$\begin{aligned} M_4(1,2,3,4) &= -is_{12}A_4(1,2,3,4)A_4(1,2,4,3)\,, \\ M_5(1,2,3,4,5) &= is_{12}s_{34}A_5(1,2,3,4,5)A_5(2,1,4,3,5) + is_{13}s_{24}A_5(1,3,2,4,5)A_5(3,1,4,2,5)\,, \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} M_n &= \text{tree} - \text{level graviton amplitudes} \\ A_n &= \text{tree} - \text{level gauge theory amplitudes} \\ s_{ij} &= (k_i + k_j)^2 \,. \end{aligned} \tag{6.3}$$

Usually such relations are studied contrasting purely gravitational amplitudes with pure gauge theory amplitudes (see, however, [259]). In the line of work presented here we more generally study mixed amplitudes involving both gravitons and photons.

This Chapter is organized as follow: in Section 6.2 for completeness our discussion we present tadpole and self-energy calculation of graviton which was already studied in [51]. In Section 6.3, we present the results of Graviton-photon-photon amplitude to begin our discussion in the mixed background which is base on [75]. In the rest of this Chapter we discuss our main work in photon-graviton amplitudes which concerned the one-graviton- four-photon amplitude in one loop and also tree-level Compton scattering involving graviton in worldline formalism, and later we compare our results with the previous studies.

6.2 One- and two-point functions at one-loop

We begin with the rather simple one-point function which gives the scalar particle contribution to the cosmological constant in Fig 6.1.

Figure 6.1: Graviton tadpole

This diagram is just graviton vertex operator (coupling to scalar loop) self contraction, which produces

$$\begin{aligned} \Gamma[k_0,\varepsilon_0]_{\text{tadpole}} &= \int_0^\infty \frac{dT}{T} e^{-m^2T} \int \mathcal{D}x \, e^{-\frac{1}{4}\int_0^T d\tau_0 \dot{x}^2} \langle V^h_{\text{scal}}[k_0,\varepsilon_0] \rangle \\ &= -\frac{\kappa}{4}\varepsilon_{0\mu\nu} \int_0^\infty \frac{dT}{T} e^{-m^2T} \int \mathcal{D}x \, e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} \int_0^T d\tau_0 \langle (\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) \, e^{ik_0\cdot x} \rangle \,, \end{aligned} \tag{6.4}$$

and after separating out the constant zero mode $y^\mu(\tau) = x^\mu(\tau) - x_0^\mu$, we get

$$\Gamma[k_0,\varepsilon_0]_{\text{tadpole}} = -\frac{\kappa}{4}\varepsilon_{0\mu\nu} \int_0^\infty \frac{dT}{T} e^{-m^2T} \frac{1}{(4\pi T)^{\frac{D}{2}}} \int d^D x_0 \int_0^T d\tau_0 \langle (\dot{y}^\mu \dot{y}^\nu + a^\mu a^\nu + b^\mu c^\nu) \, e^{ik_0\cdot(y+x_0)} \rangle \,. \tag{6.5}$$

The integration over d^Dx_0 gives the momentum delta function $(2\pi)^D\delta^D(k_0)$, which for simplicity we factorize together with the polarization tensor $\varepsilon_{0\mu\nu}$. We use the following Wick contractions

$$\begin{aligned}\langle y^\mu(\tau_1)\, y^\nu(\tau_2)\rangle &= -G_{B12}(\tau_1,\tau_2)\delta^{\mu\nu}\,,\\ \langle \dot{y}^\mu(\tau_1)\, \dot{y}^\nu(\tau_2)\rangle &= \ddot{G}_{B12}(\tau_1,\tau_2)\delta^{\mu\nu}\,,\end{aligned}\tag{6.6}$$

where we recall

$$\begin{aligned}G_B(\tau_i,\tau_j) &= |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T}\,,\\ \dot{G}_B(\tau_i,\tau_j) &= \mathrm{sign}(\tau_i-\tau_j) - 2\frac{(\tau_i-\tau_j)}{T}\,,\\ \ddot{G}_B(\tau_i,\tau_j) &= 2\delta(\tau_i-\tau_j) - \frac{2}{T}\,,\end{aligned}\tag{6.7}$$

which dots are derivatives respect to the first variable. After Wick contraction

$$\Gamma^{\mu\nu}[k_0,\varepsilon_0]_{\text{tadpole}} = -\frac{\kappa}{4}\int_0^\infty \frac{dT}{T}e^{-m^2T}\frac{1}{(4\pi T)^{\frac{D}{2}}}\int_0^T d\tau_0\big[\ddot{G}_B(\tau_0,\tau_0) - G_{\text{gh}}(\tau_0,\tau_0)\big]\delta^{\mu\nu}\,.\tag{6.8}$$

But note that

$$\ddot{G}_B(\tau_0,\tau_0) - G_{\text{gh}}(\tau_0,\tau_0) = -\frac{2}{T}\,.\tag{6.9}$$

By substituting (6.9) in (6.10) and rescale the τ_0 variable as $\tau_0 = Tu_0$ we get

$$\begin{aligned}\Gamma^{\mu\nu}[k_0,\varepsilon_0]_{\text{tadpole}} &= \frac{\kappa}{2}\delta^{\mu\nu}\int_0^\infty \frac{dT}{T}e^{-m^2T}\frac{1}{(4\pi T)^{\frac{D}{2}}}\int_0^1 du_0\\ &= \frac{\kappa}{2}\delta^{\mu\nu}\frac{(m^2)^{\frac{D}{2}}}{(4\pi)^{\frac{D}{2}}}\Gamma\big(-\frac{D}{2}\big)\,.\end{aligned}\tag{6.10}$$

As one can see, this tadpole diagram diverges at even dimensions D and must be renormalized. We now discuss the more interesting two-point function. This is special (for $\bar{\xi} = 0$) since only vertices with one graviton are present, see Fig. 6.2. This diagrams contains a Wick contraction

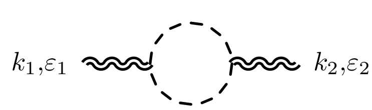


Figure 6.2: Graviton self energy.

of two graviton vertex operators, which produces

$$
\begin{aligned}
\Gamma_{\rm self-energy}[k_1,\varepsilon_1;k_2,\varepsilon_2] &= \int_0^\infty \frac{dT}{T}e^{-m^2T}\frac{1}{(4\pi T)^{\frac{D}{2}}}\int d^Dx_0\langle V^h_{\rm scal}[k_1,\varepsilon_1]V^h_{\rm scal}[k_2,\varepsilon_2]\rangle \\
&= \frac{\kappa^2}{16}\varepsilon_{1\mu\nu}\varepsilon_{2\alpha\beta}\int_0^\infty \frac{dT}{T}e^{-m^2T}\frac{1}{(4\pi T)^{\frac{D}{2}}}\int d^Dx_0\int_0^T d\tau_1\int_0^T d\tau_2 \\
&\quad \Big\langle\big(\dot y^\mu(\tau_1)\dot y^\nu(\tau_1)+a^\mu(\tau_1)a^\nu(\tau_1)+b^\mu(\tau_1)c^\nu(\tau_1)\big)e^{ik_1\cdot(x_0+y(\tau_1))} \\
&\times \quad \big(\dot y^\alpha(\tau_2)\dot y^\beta(\tau_2)+a^\alpha(\tau_2)a^\beta(\tau_2)+b^\alpha(\tau_2)c^\beta(\tau_2)\big)e^{ik_2\cdot(x_0+y(\tau_2))}\Big\rangle\,.
\end{aligned}
\tag{6.11}
$$

As before the zero mode integration gives a delta function for momentum conservation, which we factorize again for notational simplicity. Then the straightforward application of the Wick contraction theorem, we get the following terms

$$
\begin{aligned}
\Big\langle\dot y^\mu(\tau_1)\dot y^\nu(\tau_1)\dot y^\alpha(\tau_2)\dot y^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= (A+B+C+D+E+F+G+H)\,e^{(\cdot)}\,, \\
\Big\langle\dot y^\mu(\tau_1)\dot y^\nu(\tau_1)a^\alpha(\tau_2)a^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= G_{\rm gh22}\delta^{\alpha\beta}\Big[\ddot G_{B11}\delta^{\mu\nu}+k^\mu k^\nu(2\dot G_{B11}\dot G_{B12}-\dot G^2_{B11}-\dot G^2_{B12})\Big]\,e^{(\cdot)}\,, \\
\Big\langle\dot y^\mu(\tau_1)\dot y^\nu(\tau_1)b^\alpha(\tau_2)c^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= -2G_{\rm gh22}\delta^{\alpha\beta}\Big[\ddot G_{B11}\delta^{\mu\nu}+k^\mu k^\nu(2\dot G_{B11}\dot G_{B12}-\dot G^2_{B11}-\dot G^2_{B12})\Big]\,e^{(\cdot)}\,, \\
\Big\langle a^\mu(\tau_1)a^\nu(\tau_1)\dot y^\alpha(\tau_2)\dot y^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= G_{\rm gh11}\delta^{\mu\nu}\Big[\ddot G_{B22}\delta^{\alpha\beta}+k^\alpha k^\beta(2\dot G_{B22}\dot G_{B12}-\dot G^2_{B22}-\dot G^2_{B21})\Big]\,e^{(\cdot)}\,, \\
\Big\langle a^\mu(\tau_1)a^\nu(\tau_1)a^\alpha(\tau_2)a^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= \Big[G_{\rm gh11}\delta^{\mu\nu}G_{\rm gh22}\delta^{\alpha\beta}+G^2_{\rm gh12}(\delta^{\mu\alpha}\delta^{\nu\beta}+\delta^{\mu\beta}\delta^{\nu\alpha})\Big]e^{(\cdot)}\,, \\
\Big\langle a^\mu(\tau_1)a^\nu(\tau_1)b^\alpha(\tau_2)c^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= \Big[2G_{\rm gh11}\delta^{\mu\nu}G_{\rm gh22}\delta^{\alpha\beta}\Big]e^{(\cdot)}\,, \\
\Big\langle b^\mu(\tau_1)c^\nu(\tau_1)\dot y^\alpha(\tau_2)\dot y^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= -2G_{\rm gh11}\delta^{\mu\nu}\Big[\ddot G_{B22}\delta^{\alpha\beta}+k^\alpha k^\beta(2\dot G_{B22}\dot G_{B12}-\dot G^2_{B22}-\dot G^2_{B21})\Big]\,e^{(\cdot)}\,, \\
\Big\langle b^\mu(\tau_1)c^\nu(\tau_1)a^\alpha(\tau_2)a^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= \Big[-2G_{\rm gh11}\delta^{\mu\nu}G_{\rm gh22}\delta^{\alpha\beta}\Big]e^{(\cdot)}\,, \\
\Big\langle b^\mu(\tau_1)c^\nu(\tau_2)b^\alpha(\tau_2)c^\beta(\tau_2)\,e^{ik_1\cdot y(\tau_1)}\,e^{ik_2\cdot y(\tau_2)}\Big\rangle &= 4\left[G_{\rm gh11}G_{\rm gh22}\delta^{\mu\nu}\delta^{\alpha\beta}-G^2_{\rm gh12}\delta^{\mu\beta}\delta^{\nu\alpha}\right]e^{(\cdot)}\,,
\end{aligned}
\tag{6.12}
$$

where

$$
\begin{aligned}
A &= \ddot G_{B11}\delta^{\mu\nu}\Big[\ddot G_{B22}\delta^{\alpha\beta}+k^\alpha k^\beta(2\dot G_{B12}\dot G_{B22}-\dot G^2_{B21}-\dot G^2_{B22})\Big]\,, \\
B &= \ddot G_{B12}\delta^{\mu\alpha}\Big[\ddot G_{B12}\delta^{\nu\beta}+k^\nu k^\beta(\dot G_{B11}\dot G_{B22}+\dot G_{B21}\dot G_{B12}-\dot G_{B22}\dot G_{B12}-\dot G_{B11}\dot G_{B21})\Big]\,, \\
C &= \ddot G_{B12}\delta^{\mu\beta}\Big[\ddot G_{B12}\delta^{\nu\alpha}+k^\nu k^\alpha(\dot G_{B11}\dot G_{B22}+\dot G_{B21}\dot G_{B12}-\dot G_{B22}\dot G_{B12}-\dot G_{B11}\dot G_{B21})\Big]\,, \\
D &= \ddot G_{B12}\delta^{\nu\alpha}\Big[k^\mu k^\beta(\dot G_{B11}\dot G_{B22}+\dot G_{B21}\dot G_{B12}-\dot G_{B22}\dot G_{B12}-\dot G_{B11}\dot G_{B21})\Big]\,, \\
E &= \ddot G_{B12}\delta^{\nu\beta}\Big[k^\mu k^\alpha(\dot G_{B11}\dot G_{B22}+\dot G_{B21}\dot G_{B12}-\dot G_{B22}\dot G_{B12}-\dot G_{B11}\dot G_{B21})\Big]\,, \\
F &= \ddot G_{B22}\delta^{\alpha\beta}\Big[k^\mu k^\nu(2\dot G_{B11}\dot G_{B12}-\dot G^2_{B11}-\dot G^2_{B12})\Big]\,, \\
G &= k^\mu k^\nu k^\alpha k^\beta\Big[\dot G^2_{B11}\dot G^2_{B22}+\dot G_{B11}\dot G_{B21}\dot G_{B12}\dot G_{B22}+\dot G^2_{B2baszir11}\dot G^2_{B12}\Big]\,.
\end{aligned}
\tag{6.13}
$$

And for notational simplicity we have abbreviated $e^{(\cdot)} \equiv e^{-G_{B12}k^2}$ and $G_{Bij} \equiv G_B(\tau_i, \tau_j)$ and so on, and we have used $k = k_1 = -k_2$.

By following [51], we define $r_i = \varepsilon_{1\mu\nu} R_i^{\mu\nu\alpha\beta} \varepsilon_{2\alpha\beta}$ where

$$
\begin{aligned}
R_1^{\mu\nu\alpha\beta} &= \delta^{\mu\nu}\delta^{\alpha\beta}\,, \\
R_2^{\mu\nu\alpha\beta} &= \delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha}\,, \\
R_3^{\mu\nu\alpha\beta} &= \frac{1}{k^2}(\delta^{\mu\alpha}k^\nu k^\beta + \delta^{\nu\alpha}k^\mu k^\beta + \delta^{\mu\beta}k^\nu k^\alpha + \delta^{\nu\beta}k^\mu k^\alpha)\,, \\
R_4^{\mu\nu\alpha\beta} &= \frac{1}{k^2}(\delta^{\mu\nu}k^\alpha k^\beta + \delta^{\alpha\beta}k^\mu k^\nu)\,, \\
R_5^{\mu\nu\alpha\beta} &= \frac{1}{k^4}k^\mu k^\nu k^\alpha k^\beta\,,
\end{aligned}
\tag{6.14}
$$

while the integrals coming from the quantum mechanical correlation functions are given by (after rescaling)

$$
\begin{aligned}
I_1 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\ddot{G}_B(u_1,u_1) - G_{\rm gh}(u_1,u_1)\Big)\Big(\ddot{G}_B(u_2,u_2) - G_{\rm gh}(u_2,u_2)\Big) e^{-TG_B(u_1,u_2)k^2}\,, \\
I_2 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\ddot{G}_B^2(u_1,u_2) - G_{\rm gh}^2(u_1,u_2)\Big) e^{-TG_B(u_1,u_2)k^2}\,, \\
I_3 &= \int_0^1 du_1 \int_0^1 du_2 \dot{G}_B(u_2,u_1)\ddot{G}_B(u_1,u_2)\dot{G}_B(u_1,u_2) e^{-TG_B(u_1,u_2)k^2}\,, \\
I_4 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\ddot{G}_B(u_1,u_1) - G_{\rm gh}(u_1,u_1)\Big)\Big(\dot{G}_B(u_1,u_2)\Big)^2 e^{-TG_B(u_1,u_2)k^2}\,, \\
I_5 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\dot{G}_B(u_1,u_2)\Big)^4 e^{-TG_B(u_1,u_2)k^2}\,.
\end{aligned}
\tag{6.15}
$$

Note that we have used the fact that $\dot{G}_B(u_i, u_i) = 0$.

By the above definitions and integral representations, this diagram can be written as

$$
\Gamma_{\rm self-energy}[k_1,\varepsilon_1;k_2,\varepsilon_2] = \frac{\kappa^2}{16}\frac{1}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-m^2T}\Big[r_1I_1 + r_2I_2 - Tk^2(r_3I_3 + r_4I_4) + T^2k^4r_5I_5\Big]\,.
\tag{6.16}
$$

Translational invariant can be used (as we have used before) to fixe $u_2 = 0$ and also $u_1 = u$. Then one obtains the following results by using dimensional regularization when necessary (the details of these integrals are presented in Appendix G),

$$
\begin{aligned}
I_1 &= 4\int_0^1 du e^{-Tk^2(u-u^2)}\,, \\
I_2 &= -2k^2T - I_1 + 4\,, \\
I_3 &= -(1+\frac{I_1-4}{k^2T})\,, \\
I_4 &= \frac{I_1-4}{k^2T}\,, \\
I_5 &= \frac{2}{k^2T} + \frac{3}{k^4T^2}(I_1-4)\,,
\end{aligned}
\tag{6.17}
$$

At this stage the proper time (T) integral can be carried out at D-dimensional which leads to

$$
\begin{aligned}
\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} r_1 I_1 &= 4r_1\Gamma(-\frac{D}{2})(P^2)^{\frac{D}{2}}\,, \\
\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} r_2 I_2 &= r_2\Big[-2k^2\Gamma(1-\frac{D}{2})(m^2)^{\frac{D}{2}-1} - 4(P^2)^{\frac{D}{2}}\Gamma(-\frac{D}{2}) + 4\Gamma(-\frac{D}{2})(m^2)^{\frac{D}{2}}\Big]\,, \\
\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} r_3 I_3(-Tk^2) &= r_3\Big[k^2\Gamma(1-\frac{D}{2})(m^2)^{\frac{D}{2}-1} + 4(P^2)^{\frac{D}{2}}\Gamma(-\frac{D}{2}) + 4\Gamma(-\frac{D}{2})(m^2)^{\frac{D}{2}}\Big]\,, \\
\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} r_4 I_4(-Tk^2) &= r_4\Big[4\Gamma(-\frac{D}{2})(m^2)^{\frac{D}{2}} - 4\Gamma(-\frac{D}{2})(P^2)^{\frac{D}{2}}\Big]\,, \\
\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} r_5 I_5(T^2k^4) &= r_5\Big[2k^2\Gamma(1-\frac{D}{2})(m^2)^{\frac{D}{2}-1} + 12\Gamma(-\frac{D}{2})(P^2)^{\frac{D}{2}} - 12\Gamma(-\frac{D}{2})(m^2)^{\frac{D}{2}}\Big]\,.
\end{aligned}
\tag{6.18}
$$

From these results, the contribution of Fig. 6.2 for graviton self-energy can be written as

$$
\begin{aligned}
\Gamma_{\mathrm{self-energy}}[k_1,\varepsilon_1;k_2,\varepsilon_2] &= \frac{\kappa^2}{16}\frac{1}{(4\pi)^{\frac{D}{2}}}\Big\{4\Gamma(-\frac{D}{2})\Big[(P^2)^{\frac{D}{2}}(r_1-r_2+r_3-r_4+3r_5) \\
&\quad +(m^2)^{\frac{D}{2}}(r_2+r_3+r_4-3r_5)\Big] + k^2\Gamma(1-\frac{D}{2})(m^2)^{\frac{D}{2}-1}(r_3-2r_2+2r_5)\Big\}\,,
\end{aligned}
\tag{6.19}
$$

where we used the expression

$$
(P^2)^x = \int_0^1 du(m^2+k^2(u-u^2))^x\,. \tag{6.20}
$$

There is one additional diagram to this amplitude which comes from the case $\bar{\xi} \neq 0$, correspond to Fig. (6.3) but since this diagram represent a coupling of two graviton with scalar field in the loop, we do not have a vertex operator for this coupling so what we should do is going back to the main action which was written in term of the curvature tensor $R(x)$

$$
\Gamma[g] = -\int_0^\infty \frac{dT}{T}\int_P \mathcal{D}x\, e^{-S[x^\mu]}\,, \tag{6.21}
$$

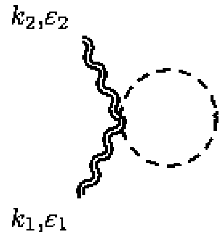


Figure 6.3: Additional diagram for graviton self-energy.

where the action is defined as

$$S[x^\mu] = \int_0^T d\tau\Big[\frac{1}{4}\dot{x}^\mu\dot{x}^\nu + m^2 + \xi R(x)\Big]\,, \tag{6.22}$$

where we use the following conventions for the curvature tensors [51]

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]V^\lambda &= R_{\mu\nu}{}^\lambda{}_\rho V^\rho\,, \\ R_{\mu\nu} &= R_{\lambda\mu}{}^\lambda{}_\nu\,, \\ R &= R^\mu{}_\mu > 0 \ \text{ on sphere}\,. \end{aligned} \tag{6.23}$$

In a short hand notation the scalar curvature is given by the sum of the following seven terms

$$\begin{aligned} R &= \partial^\mu\partial^\nu g_{\mu\nu} - g^{\mu\nu}\partial^2 g_{\mu\nu} + \frac{3}{4}(\partial_\alpha g_{\mu\nu})^2 - \frac{1}{2}(\partial^\mu g^{\nu\alpha})(\partial_\nu g_{\mu\alpha}) - \frac{1}{4}(g^{\mu\nu}\partial_\alpha g_{\mu\nu})^2 \\ &\quad + (g^{\mu\nu}\partial^\alpha g_{\mu\nu})(\partial^\beta g_{\beta\alpha}) - (\partial^\mu g_{\mu\nu})^2\,, \end{aligned} \tag{6.24}$$

and for this diagram we need the linear and quadratic terms in the expansion around flat space that read as

$$\begin{aligned} R &= R^{(1)} + R^{(2)}\,, \\ R^{(1)} &= \kappa\Big[\partial^\mu\partial^\nu h_{\mu\nu} - \Box h\Big]\,, \\ R^{(2)} &= \kappa^2\Big[h^{\mu\nu}\Box h_{\mu\nu} - 2h^{\mu\nu}\partial_\mu\Big(\partial^\alpha h_{\alpha\nu} - \frac{1}{2}\partial_\nu h\Big) - \Big(\partial^\alpha h_{\alpha\mu} - \frac{1}{2}\partial_\mu h\Big)^2\,, \\ &\quad + \frac{3}{4}(\partial_\alpha h_{\mu\nu})^2 - \frac{1}{2}(\partial_\mu h_{\nu\alpha})(\partial^\nu h^{\mu\alpha})\Big]\,, \end{aligned} \tag{6.25}$$

where we recall that $\kappa h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$ and $h = \delta^{\mu\nu}h_{\mu\nu}$. Since in this Chapter we consider amplitudes in tree-level and one-loop which contains an emission or absorption of only one graviton $\bar{\xi} = 0$, we do not go further in the detail of this calculation. The strategy would be the following: without talking about vertex operators, by inserting for the metric a sums of N planes waves directly into the worldline action (6.22), expanding the worldline action that contains R as well, and then expanding the exponential of the worldline action, keeping at the end only the term proportional to the N plane waves of interest. This produces all the terms for the calculation of the N-graviton correlation function which here would be just $N = 2$. For more detail see [51].

6.3 Graviton-photon-photon amplitude

As was discussed in the Introduction 6.1, the photon-graviton amplitudes have been investigated in recent years recently, the main objective are KLT-type relations between amplitudes involving one-graviton on one side, and N photons on the other.
We believe that new insight into the structural relations between photon and graviton amplitudes might be obtained by studying the N graviton amplitudes involving a massive loop, and more generally the mixed one-loop graviton-photon amplitudes. This series of studies have been started by Schubert et.al and the first study on one-loop photon-photon-graviton was obtained in [75]. In this Section we present their final result for completeness. This amplitude is shown in Fig. 6.4 where the single wavy lines represent the external photons and double wavy line the external graviton.

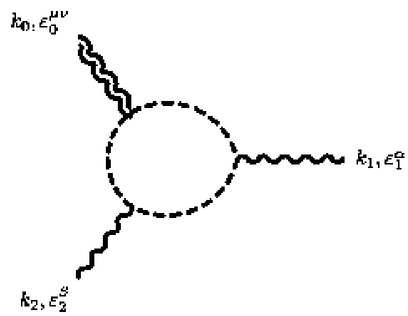


Figure 6.4: Photon-photon-graviton.

Here we just represent the result which was obtained in [75] for both scalar and spinor loop cases for this amplitude. In this case it is actually simplest to use the corresponding low-energy effective Lagrangian [248, 260].

$$\begin{aligned}
\mathcal{L}_{\rm scal}^{h\gamma\gamma} &= \frac{e^2}{180(4\pi)^2m^2}\Big[15(\xi-\frac{1}{6})RF_{\mu\nu}^2 - 2R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} - R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + 3(\nabla^\alpha F_{\alpha\mu})^2\Big]\,,\\
\mathcal{L}_{\rm spin}^{h\gamma\gamma} &= \frac{e^2}{180(4\pi)^2m^2}\Big[5RF_{\mu\nu}^2 - 26R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} + 2R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + 24(\nabla^\alpha F_{\alpha\mu})^2\Big]\,,
\end{aligned} \tag{6.26}$$

Using, as is usual, gravitons that are helicity eigenstates and factorize into vector polarizations,

$$\varepsilon_{0\mu\nu}^{++}(k_0) = \varepsilon_{0\mu}^{+}(k_0)\varepsilon_{0\nu}^{+}(k_0), \quad \varepsilon_{0\mu\nu}^{--}(k_0) = \varepsilon_{0\mu}^{-}(k_0)\varepsilon_{0\nu}^{-}(k_0)\,, \tag{6.27}$$

one finds that the only non-vanishing components of the on-shell amplitude are the ones where all helicities are equal,

$$\begin{aligned}
A_{\rm spin}^{(++;++)} &= \frac{\kappa e^2}{90(4\pi)^2m^2}[01]^2[02]^2\,, \qquad A_{\rm spin}^{(--;--)} = \frac{\kappa e^2}{90(4\pi)^2m^2}\langle 01\rangle^2\langle 02\rangle^2\,,\\
A_{\rm spin}^{(++;++)} &= -2\,A_{\rm scal}^{(++;++)}\,, \qquad A_{\rm spin}^{(--;--)} = -2\,A_{\rm scal}^{(--;--)}\,.
\end{aligned} \tag{6.28}$$

Here we have used the standard spinor helicity notation [139, 173]. Comparison with the low-energy limits of the four-photon amplitudes [140] one indeed finds a KLT-like relation, that is, effectively the graviton has factored into two photons.

6.4 One-graviton four-photon amplitude

We are now working on the one-graviton four-photon amplitude (the one-graviton three-photon one vanishes for parity reasons). Although the corresponding effective Lagrangians are also available [261], the expressions are already quite cumbersome, and we prefer here a direct calculation starting from the worldline expression,

$$\begin{aligned}\Gamma^{h,4\gamma}_{\rm scal} &= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x e^{-\frac{1}{4}\int_0^T \dot{x}^2} \Big\langle V^h_{\rm scal}(k_0) V^\gamma_{\rm scal}(k_1) V^\gamma_{\rm scal}(k_2) V^\gamma_{\rm scal}(k_3) V^\gamma_{\rm scal}(k_4) \Big\rangle\,, \\ &= -\frac{\kappa (ie)^3}{4} \varepsilon_{0\mu\nu}\varepsilon_{1\alpha}\varepsilon_{2\beta}\varepsilon_{3\gamma} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \frac{1}{(4\pi T)^{\frac{D}{2}}} \int_0^T d\tau_0 d\tau_1 d\tau_2 d\tau_3 \\ &\quad \Big\langle \big[\dot{x}_0^\mu(\tau_0)\dot{x}_0^\nu(\tau_0) + a^\mu(\tau_0)a^\nu(\tau_0) + b^\mu(\tau_0)c^\nu(\tau_0)\big] e^{ik_0\cdot x_0(\tau_0)} \big[(\dot{x}_1^\alpha\, e^{ik_1\cdot x_1(\tau_1)})\,(\dot{x}_2^\beta\, e^{ik_2\cdot x_2(\tau_2)})\,(\dot{x}_3^\gamma\, e^{ik_3\cdot x_3(\tau_3)})\big] \Big\rangle\end{aligned} \tag{6.29}$$

where we have used the following photon and graviton vertex operators for scalar case

$$\begin{aligned} V^\gamma_{\rm scal}[k,\varepsilon] &= ie\varepsilon_\mu \int_0^T d\tau \dot{x}^\mu\, e^{ik\cdot x}\,, \\ V^h_{\rm scal}[k_0,\varepsilon_0] &= -\frac{\kappa}{4}\varepsilon_{0\mu\nu} \int_0^T d\tau \Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau)\Big] e^{ik_0\cdot x_0}\,. \end{aligned} \tag{6.30}$$

However, (6.29) comes from the effective action and thus represents only the the one-particle irreducible part of the amplitude. Contrary to the above three-point case, the five point amplitude has already reducible contribution Fig 6.5

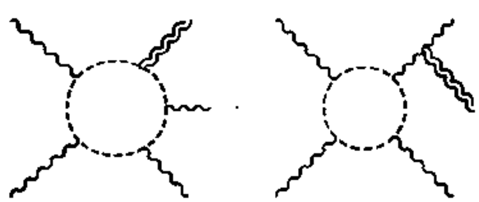


Figure 6.5: One-graviton four-photon amplitude.

Thus, it will be important to find out how to combine the irreducible and reducible diagrams in an algebraically nice way; for a short report see [238]. This problem has been studied at tree-level by various authors [243–245], and remarkable results were found for the sum of diagrams in the case of gravitational Compton scattering [246, 247]. However, we are not aware of similar studies at the one-loop level. To address this issue in the worldline formalism we first should learn more about how to compare Feynman diagram and worldline calculations in quantum gravity. We are in the process of doing this. In the following Section we use worldline formalism to find a nice factorization for tree-level Compton scattering which contains irreducible and reducible diagrams but at tree-level as Fig. 6.5 and to combine them in an efficient way which has been done before in a different way in [246, 247]. This work is an ingoing project [239].

6.5 Worldline representation of the Compton scattering for scalar case

In Section 1.3, we studied worldline formalism in curved space-time at one-loop, and we constructed the graviton vertex operator and its Green function for one-loop calculations.
In this Section we will calculate the gravitational Compton scattering for scalar particle, so for this calculation we will discuses worldline representation of open-line and its propagator in the gravitational background.

We want to describe the propagation of a scalar particle in gravity, which is given by (1.95). The propagator is represented by

$$\begin{aligned}\langle 0|\phi_q(x)\phi_q(y)|0\rangle &= \frac{1}{\Box + m^2 + \xi R} = \int_0^T dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}x\, e^{-\int_0^T d\tau\left[\frac{1}{4}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + m^2 + \bar{\xi}R\right]} \\ &= \int_0^T dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}x\mathcal{D}a\mathcal{D}b\mathcal{D}c\, e^{-\int_0^T d\tau\left[\frac{1}{4}g_{\mu\nu}(x)\left(\dot{x}^\mu\dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu\right) + m^2 + \bar{\xi}R\right]}\,,\end{aligned} \tag{6.31}$$

which describes a propagation of a scalar particle from y to x in the presence of gravity, Fig. 6.6. We compute the path integral by splitting $x^\mu(\tau)$ into a background part $x^\mu_{\rm bg}(\tau)$, which encodes

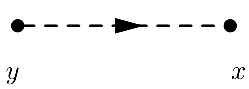


Figure 6.6: Propagating a scalar particle from y to x

the boundary condition, and a quantum part which $q^\mu(\tau)$, which has zero Dirichelet boundary conditions at $\tau = 0, T$

$$\begin{aligned} x(\tau) &= x_{\rm bg}(\tau) + q(\tau)\,, \\ x_{\rm bg}(\tau) &= \frac{(y-x)\tau}{T} + x\,, \\ \dot{x}(\tau) &= \frac{y-x}{T} + \dot{q}(\tau)\,, \\ q(0) &= q(T) = 0\,. \end{aligned} \tag{6.32}$$

For future use we record the propagator for $q^\mu(\tau)$ [52]:

$$\begin{aligned}
\langle q^\mu(\tau_1) q^\nu(\tau_2)\rangle &= -2\delta^{\mu\nu}\Delta(\tau_1,\tau_2)\,,\\
\langle e^{ik_1\cdot q(\tau_1)} e^{ik_2\cdot q(\tau_2)}\rangle &= e^{\sum_i k_i^2(\tau_i^2-\tau_i)+\sum_{i<j} 2k_i\cdot k_j \Delta(\tau_i,\tau_j)}\,,\\
\Delta(\tau_1,\tau_2) &= \frac{\tau_1\tau_2}{T} - \tau_1\theta(\tau_2-\tau_1) - \tau_2\theta(\tau_1-\tau_2) = \frac{\tau_1\tau_2}{T} + \frac{|\tau_1-\tau_2|}{2} - \frac{\tau_1+\tau_2}{2}\,,\\
\Delta(\tau,\tau) &= \frac{\tau^2}{T} - \tau\,.
\end{aligned} \tag{6.33}$$

In order of κ (one-graviton where $\bar{\xi}=0$), we set $h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik\cdot x(\tau)}$ in (6.31) and pick the linear terms in $\varepsilon_{\mu\nu}$.
In the rest of this Chapter we need the following derivatives of the propagator Δ,

$$\begin{aligned}
{}^\bullet\Delta(\tau_1,\tau_2) &= \frac{\tau_2}{T} + \frac{1}{2}\mathrm{sign}(\tau_1-\tau_2) - \frac{1}{2}\,,\\
\Delta^\bullet(\tau_1,\tau_2) &= \frac{\tau_1}{T} - \frac{1}{2}\mathrm{sign}(\tau_1-\tau_2) - \frac{1}{2}\,,\\
{}^\bullet\Delta^\bullet(\tau_1,\tau_2) &= \frac{1}{T} - \delta(\tau_1-\tau_2)\,,
\end{aligned} \tag{6.34}$$

and also after rescaling $\tau_i = Tu_i$ we have

$$\begin{aligned}
\Delta(\tau_1,\tau_2) &= T\Delta(u_1,u_2) = T(u_1u_2 + \frac{|u_1-u_2|}{2} - \frac{u_1+u_2}{2})\,,\\
{}^\bullet\Delta(\tau_1,\tau_2) &= {}^\bullet\Delta(u_1,u_2)\,,\\
\Delta^\bullet(\tau_1,\tau_2) &= \Delta^\bullet(u_1,u_2)\,,\\
{}^\bullet\Delta^\bullet(\tau_1,\tau_2) &= \frac{{}^\bullet\Delta^\bullet(u_1,u_2)}{T}\,.
\end{aligned} \tag{6.35}$$

Now, we have the following vertex operator for graviton coupled to a scalar particle (its form is equivalent to its vertex operator at one-loop in (1.113)),

$$V^h_{\mathrm{scal}}[k,\varepsilon] = -\frac{\kappa}{4}\varepsilon_{\mu\nu}\int_0^T d\tau\Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau)\Big]\, e^{ik\cdot x}\,. \tag{6.36}$$

but with different Green's function (6.33).

Now, let us show here only the simplest example, namely how to reproduce in the worldline formalism the basic graviton-scalar-scalar vertex, which is shown in Fig. 6.7.
From (6.31)by setting $\xi = 0$,

$$\langle\phi(x)\phi(y)\rangle = \int_0^\infty dT e^{-m^2T}\int_{x(0)=x}^{x(T)=y}\mathcal{D}x(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2}\int_0^T d\tau_0 \langle V^h_{\mathrm{scal}}[k_0,\varepsilon_0]\rangle\,, \tag{6.37}$$

Figure 6.7: Gravity vertex from standard QFT Feynman rules.

by inserting the vertex operator and decompose the x-filed to a classical and quantum parts, one gets

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle &= -\frac{\kappa}{4}\varepsilon_{0\mu\nu}\int_0^\infty dT e^{-m^2T}\, e^{-\frac{1}{4T}(x-y)^2}\int_{q(0)=q(T)=0}\mathcal{D}q(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau\dot{q}^2}\\ &\times\int_0^T d\tau_0\, e^{ik_0\cdot x_{\rm bg}(\tau_0)}\Big\langle\Big\{\big[\frac{y-x}{T}+\dot{q}(\tau_0)\big]^\mu\big[\frac{y-x}{T}+\dot{q}(\tau_0)\big]^\nu\\ &\qquad +a^\mu(\tau_0)a^\nu(\tau_0)+b^\mu(\tau_0)c^\nu(\tau_0)\Big\}e^{ik_0\cdot q(\tau_0)}\Big\rangle\ ,\end{aligned} \tag{6.38}$$

where

$$\begin{aligned}\Big\langle\cdots\Big\rangle &= \Big\langle\Big\{\frac{(y-x)^\mu(y-x)^\nu}{T^2}+\frac{(y-x)^\mu}{T}\dot{q}^\nu(\tau_0)+\frac{(y-x)^\nu}{T}\dot{q}^\mu(\tau_0)+\dot{q}^\mu(\tau_0)\dot{q}^\nu(\tau_0)\\ &\qquad +a^\mu(\tau_0)a^\nu(\tau_0)+b^\mu(\tau_0)c^\nu(\tau_0)\Big\}e^{ik_0\cdot q(\tau_0)}\Big\rangle\\ &= \Big\{\frac{(y-x)^\mu(y-x)^\nu}{T^2}-\frac{2i}{T}\big(k_0^\mu(y-x)^\nu+k_0^\nu(y-x)^\mu\big){}^\bullet\Delta(\tau_0,\tau_0)-4k_0^\mu k_0^\nu{}^\bullet\Delta^2(u_0,u_0)\\ &\qquad +\delta^{\mu\nu}\big[-2{}^\bullet\Delta^\bullet(\tau_0,\tau_0)+2\delta(\tau_0-\tau_0)-4\delta(\tau_0-\tau_0)\big]\Big\}e^{\Delta(\tau_0,\tau_0)k_0^2}\ .\end{aligned} \tag{6.39}$$

Since ${}^\bullet\Delta^\bullet(\tau_0,\tau_0)=\frac{1}{T}-\delta(\tau_0-\tau_0)$, then

$$-2{}^\bullet\Delta^\bullet(\tau_0,\tau_0)+2\delta(\tau_0-\tau_0)-4\delta(\tau_0-\tau_0)=-\frac{2}{T}\ , \tag{6.40}$$

which is nice since an ill-defined $\delta(0)$ cancels out by ghost fields. Eq. (6.38) leads to

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle &= -\frac{\kappa}{4}\varepsilon_{0\mu\nu}\int_0^\infty dT e^{-m^2T}\, e^{-\frac{1}{4T}(x-y)^2}\int_{q(0)=q(T)=0}\mathcal{D}q(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau\dot{q}^2}\\ &\times\int_0^T d\tau_0\, e^{ik_0\cdot x_{\rm bg}(\tau_0)}\Big\{\frac{(y-x)^\mu(y-x)^\nu}{T^2}-\frac{2i}{T}\big(k_0^\mu(y-x)^\nu+k_0^\nu(y-x)^\mu\big){}^\bullet\Delta(\tau_0,\tau_0)\\ &\qquad -4k_0^\mu k_0^\nu{}^\bullet\Delta^2(u_0,u_0)-\frac{2}{T}\delta^{\mu\nu}\Big\}e^{\Delta(\tau_0,\tau_0)k_0^2}\ .\end{aligned} \tag{6.41}$$

Fourier transforming to momentum space and doing the following changes

$$\begin{aligned} (y-x) &= x' , \\ \frac{(y+x)}{2} &= y' , \end{aligned} \tag{6.42}$$

and also

$$\begin{aligned} x'^{\mu} e^{ix'\cdot a} &= -i\frac{\partial}{\partial a^{\mu}} e^{ix'\cdot a} , \\ a &= p_2 + k_0 u_0 , \end{aligned} \tag{6.43}$$

then, one gets

$$\begin{aligned} \langle\phi(p_1)\phi(p_2)\rangle &= -\frac{\kappa}{4}\varepsilon_{0\mu\nu}(2\pi)^D\delta^D(p_1+p_2+k_0)\int d^Dx'\int_0^\infty dT\, e^{-m^2T}e^{-\frac{x'^2}{4T}}(4\pi T)^{-\frac{D}{2}} \\ &\times\int_0^1 du_0\Big\{-\frac{1}{T}\frac{\partial^2}{\partial a_\mu \partial a_\nu} - 2\big(k_0^\mu\frac{\partial}{\partial a_\nu} + k_0^\nu\frac{\partial}{\partial a_\mu}\big){}^\bullet\Delta(u_0,u_0) \\ &\qquad -4Tk_0^\mu k_0^\nu {}^\bullet\Delta^2(u_0,u_0) - 2\delta^{\mu\nu}\Big\} e^{T\Delta(u_0,u_0)k_0^2}\, e^{ix'\cdot a} . \end{aligned} \tag{6.44}$$

Note that, the x'-integral is Gaussian and gives

$$\int d^Dx' e^{ix'\cdot a - \frac{x'^2}{4T}} = (4\pi T)^{\frac{D}{2}} e^{-Ta^2} . \tag{6.45}$$

Eq. (6.44) can be written as

$$\begin{aligned} \langle\phi(p_1)\phi(p_2)\rangle &= -\frac{\kappa}{4}\varepsilon_{0\mu\nu}(2\pi)^D\delta^D(p_1+p_2+k_0)\int_0^\infty dT\, e^{-m^2T}\int_0^1 du_0 \\ &\times\Big\{-\frac{1}{T}\frac{\partial^2}{\partial a_\mu \partial a_\nu} - 2\big(k_0^\mu\frac{\partial}{\partial a_\nu} + k_0^\nu\frac{\partial}{\partial a_\mu}\big){}^\bullet\Delta(u_0,u_0) \\ &\qquad -4Tk_0^\mu k_0^\nu {}^\bullet\Delta^2(u_0,u_0) - 2\delta^{\mu\nu}\Big\} e^{T\Delta(u_0,u_0)k_0^2}\, e^{-Ta^2} \\ &= -\frac{\kappa}{4}\varepsilon_{0\mu\nu}(2\pi)^D\delta^D(p_1+p_2+k_0)\int_0^\infty dT\, e^{-m^2T}\int_0^1 du_0 \\ &\times\Big\{2\delta^{\mu\nu} - 4Ta^\mu a^\nu + 8Tk_0^\mu a^\nu {}^\bullet\Delta(u_0,u_0) - 4Tk_0^\mu k_0^\nu {}^\bullet\Delta^2(\tau_0,\tau_0) - 2\delta^{\mu\nu}\Big\} e^{T\Delta(u_0,u_0)k_0^2 - Ta^2} . \end{aligned} \tag{6.46}$$

where

$$\begin{aligned} \Delta(u_0,u_0) &= u_0^2 - u_0 , \\ {}^\bullet\Delta(u_0,u_0) &= u_0 - \frac{1}{2} . \end{aligned} \tag{6.47}$$

Then one can easily get

$$
\begin{aligned}
&4Ta^\mu a^\nu + 8Tk_0^\mu a^{\nu\bullet}\Delta(u_0,u_0) - 4Tk_0^\mu k_0^{\nu\bullet}\Delta^2(\tau_0,\tau_0) = \\
&-4Tp_2^\mu p_2^\nu - Tk_0^\mu k_0^\nu + 8Tu_0(k_0^\mu p_2^\nu - k_0^\nu p_2^\mu) - 4Tk_0^\mu p_2^\nu \,, \qquad (6.48)
\end{aligned}
$$

for transverse and traceless $\varepsilon_{0\mu\nu}$, only the first term contributes which finally gives

$$
\begin{aligned}
\langle\phi(p_1)\phi(p_2)\rangle &= \frac{\kappa}{4}\varepsilon_{0\mu\nu}(2\pi)^D\delta^D(p_1+p_2+k_0)\int_0^1 du_0\int_0^\infty dT(4Tp_2^\mu p_2^\nu)e^{-T\left[m^2+p_2^2+(k_0^2+2k_0\cdot p_2)u_0\right]} \\
&= \frac{\kappa}{4}\varepsilon_{0\mu\nu}(2\pi)^D\delta^D(p_1+p_2+k_0)\int_0^1 du_0\frac{(4p_2^\mu p_2^\nu)}{\left[m^2+p_2^2+(k_0^2+2k_0\cdot p_2)u_0\right]^2} \\
&= (2\pi)^D\delta^D(p_1+p_2+k_0)\frac{1}{p_2^2+m^2}(\kappa\varepsilon_{0\mu\nu}p_2^\mu p_2^\nu)\frac{1}{p_1^2+m^2}\,. \qquad (6.49)
\end{aligned}
$$

This is indeed the correct expression for the vertex [238].

6.5.1 Some cross sections involve graviton

The evaluation of the Compton scattering cross section is a standard exercise in relativistic quantum mechanics, because gauge invariance together with the zero mass of the photon allows the results to be presented in terms of simple analytic forms, see for example [262]. Superficially the same analysis should be applicable to the interaction of graviton. Like photons, such particles are massless and subject to a gauge invariance, so that similar analytic results for graviton cross sections can be expected. Also, just as virtual photon exchange leads to a detailed understanding of electromagnetic interactions between charged systems, a careful treatment of virtual graviton exchange allows an understanding not just of Newtonian gravity, but also of spin-dependent phenomena associated with general relativity. However despite this obvious parallel, quantum mechanic texts do not discuss gravitational interactions in any details. There are at least three reasons for this situation [246]:

- The graviton is a spin two particle in contrast to the spin one photon, so that the interaction forms are more complex, involving symmetric second rank tensors rather than simple Lorentz four-vectors.
- There exist few experimental results with which to compare the theoretical calculations.
- To guarantee gauge invariance in some processes we must include, in addition to the usual Born and seagull diagrams, the contribution from a graviton pole term, which involves a triple-graviton coupling. This vertex is a sixth rank tensor and contains a multitude of kinematic forms.

Recently, using powerful string-based techniques, which simplify conventional quantum field theory calculations, it has been demonstrated that the elastic scattering of gravitons from an elementary target of arbitrary spin must factorize [242], a feature that had been noted earlier based on gauge theory arguments [247]. This factorization permits a relatively painless evaluation of the various graviton amplitudes. In the following we show how this factorization comes about and evaluate some relevant cross sections.

a) Electromagnetic Compton scattering

Before treating the case of gravitons it is useful to review photon interactions, because this familiar formalism can be used as a bridge to our understanding of the gravitational case. For spin zero, the Compton scattering contains three diagrams, see Fig. 6.8, from left to right, two Born and a seagull diagrams,

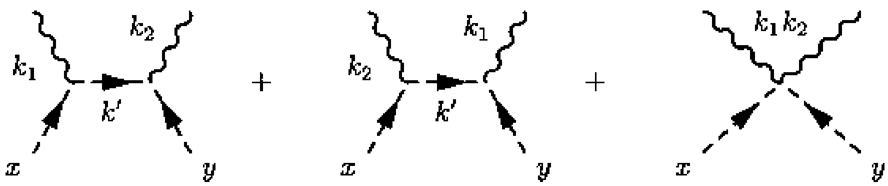


Figure 6.8: Diagrams relevant to Compton scattering, $s+\gamma\to s+\gamma$.

These three diagrams are represented by following worldline amplitude,

$$\Gamma[x,y;\varepsilon,\varepsilon_0] = -e^2\int_0^\infty dT e^{-m^2T}\int_{x(0)=y}^{x(T)=x}\mathcal{D}x e^{-\frac{1}{4}\int_0^T d\tau\dot{x}^2}\int_0^T d\tau_1 d\tau_2\langle V^\gamma_{\rm scal}[k_1,\varepsilon_1]V^\gamma_{\rm scal}[k_2,\varepsilon_2]\rangle\,, \tag{6.50}$$

where

$$V^\gamma_{\rm scal}[k,\varepsilon] = \varepsilon_\mu\int_0^T d\tau\dot{x}^\mu e^{ik\cdot x}\,, \tag{6.51}$$

is the photon vertex operator coupled to scalar particle.

I the following, we will use the on-shell conditions for the photons from the beginning, but the scalar legs will be kept off-shell for technical reasons. By inserting the vertex operators and doing all possible Wick contractions, one gets

$$\begin{aligned}
&\Gamma[k_1,\varepsilon_1;k_2,\varepsilon_2] = -e^2\int_0^\infty dT e^{-m^2T}e^{-\frac{1}{4T}(y-x)^2}\int_{q(0)=q(T)=0}\mathcal{D}q(\tau)e^{-\frac{1}{4}\int_0^T d\tau\dot{q}^2}\\
&\int_0^T d\tau_1 d\tau_2 e^{ik_1\cdot x_{\rm bg}(\tau_1)}e^{ik_2\cdot x_{\rm bg}(\tau_2)}\varepsilon_{1\mu}\varepsilon_{2\nu}\Big\langle\Big[\frac{y-x}{T}+\dot{q}(\tau_1)\Big]^\mu e^{ik_1\cdot q(\tau_1)}\Big[\frac{y-x}{T}+\dot{q}(\tau_2)\Big]^\nu e^{ik_2\cdot q(\tau_2)}\Big\rangle\\
&= -e^2\int_0^\infty dT e^{-m^2T}e^{-\frac{1}{4T}(x-y)^2}(4\pi T)^{-D/2}\int_0^T d\tau_1 d\tau_2 e^{ik_1\cdot x_{\rm bg}(\tau_1)}e^{ik_2\cdot x_{\rm bg}(\tau_2)}\\
&\varepsilon_{1\mu}\varepsilon_{2\nu}\Big[\frac{(y-x)^\mu(y-x)^\nu}{T^2}-\frac{2i(y-x)^\mu}{T}(k_1^\nu\Delta^\bullet(\tau_1,\tau_2)+k_2^\nu\Delta^\bullet(\tau_2,\tau_2))-\frac{2i(y-x)^\nu}{T}(k_1^\mu{}^\bullet\Delta(\tau_1,\tau_1)\\
&\qquad +k_2^\mu{}^\bullet\Delta(\tau_1,\tau_2))-2\delta^{\mu\nu}{}^\bullet\Delta^\bullet(\tau_1,\tau_2)-4{}^\bullet\Delta(\tau_1,\tau_2)\Delta^\bullet(\tau_1,\tau_2)k_1^\nu k_2^\mu\Big]e^{2\Delta(\tau_1,\tau_2)k_1\cdot k_2}\,,
\end{aligned} \tag{6.52}$$

Here we do on-shell calculation for this conventional Compton scattering, which means $k_i^2 = \varepsilon_i\cdot k_i = 0$, which makes third and fourth terms vanish. What remains is the following from

(6.52):

$$\Gamma[k_1,\varepsilon_1;k_2,\varepsilon_2] = -e^2\int_0^\infty dT e^{-m^2T}e^{-\frac{1}{4T}(x-y)^2}(4\pi T)^{-D/2}\int_0^T d\tau_1 d\tau_2 e^{ik_1\cdot x_{\rm bg}(\tau_1)}e^{ik_2\cdot x_{\rm bg}(\tau_2)}$$
$$\varepsilon_{1\mu}\varepsilon_{2\nu}\Big[\frac{(y-x)^\mu(y-x)^\nu}{T^2} - \frac{2i(y-x)^\mu}{T}(k_1^\nu\Delta^\bullet(\tau_1,\tau_2)+k_2^\nu\Delta^\bullet(\tau_2,\tau_2)) - \frac{2i(y-x)^\nu}{T}(k_1^{\mu\bullet}\Delta(\tau_1,\tau_1)$$
$$+k_2^{\mu\bullet}\Delta(\tau_1,\tau_2)) - 2\delta^{\mu\nu\bullet}\Delta^\bullet(\tau_1,\tau_2) - 4^\bullet\Delta(\tau_1,\tau_2)\Delta^\bullet(\tau_1,\tau_2)k_1^\nu k_2^\mu\Big]e^{2\Delta(\tau_1,\tau_2)k_1\cdot k_2}\,. \tag{6.53}$$

Now, we first rescale $\tau_i = Tu_i$, then Fourier transform to momentum space

$$\begin{aligned}\Gamma[p_1,k_1;p_2,k_2] &= \int d^Dx\int d^Dy e^{ip_1\cdot x}\,e^{ip_2\cdot y}\,\Gamma[x,k_1;y,k_2]\\
&= -e^2\int d^Dx\int d^Dy e^{ip_1\cdot x}\,e^{ip_2\cdot y}\,e^{i(k_1+k_2)\cdot x}\int_0^\infty dT\frac{e^{-m^2T}e^{-\frac{(y-x)^2}{4T}}}{(4\pi T)^{D/2}}\\
&\quad\int_0^1 du_1du_2T^2e^{ik_1\cdot(y-x)u_1}e^{ik_2\cdot(y-x)u_2}\Big[\frac{\varepsilon_1\cdot(y-x)\,\varepsilon_2\cdot(y-x)}{T^2}\\
&\quad-\frac{2i}{T}\varepsilon_1\cdot(y-x)\,\varepsilon_2\cdot k_1\Delta^\bullet(u_1,u_2) - \frac{2i}{T}\varepsilon_2\cdot(y-x)\,\varepsilon_1\cdot k_2{}^\bullet\Delta(u_1,u_2)\\
&\quad-2\frac{(\varepsilon_1\cdot\varepsilon_2)^\bullet\Delta^\bullet(u_1,u_2)}{T} - 4(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_1)^\bullet\Delta(u_1,u_2)\Delta^\bullet(u_1,u_2)\Big]e^{2T\Delta(u_1,u_2)k_1\cdot k_2}\,.\end{aligned} \tag{6.54}$$

To do these integrals we do the following change of variables

$$y-x = x'\quad,\quad \frac{y+x}{2} = y' \Rightarrow y = \frac{x'+2y'}{2}\quad,\quad x = \frac{2y'-x'}{2}\,, \tag{6.55}$$

which gives unit *Jacobian*. Our amplitude (6.54) after performing the y'-integral modifies to

$$\begin{aligned}\Gamma[p_1,k_1;p_2,k_2] &= -e^2(2\pi)\varepsilon^4\delta^4(p_1+p_2+k_1+k_2)\int d^Dx'\int_0^1 du_1du_2\int_0^\infty dT\frac{e^{-m^2T}e^{-\frac{x'^2}{4T}+ia\cdot x'}}{(4\pi T)^{D/2}}\\
&\quad\Big[\varepsilon_1\cdot x'\,\varepsilon_2\cdot x' - 2iT\varepsilon_1\cdot x'\,\varepsilon_2\cdot k_1\Delta^\bullet(u_1,u_2) - 2iT\varepsilon_2\cdot x'\,\varepsilon_1\cdot k_2{}^\bullet\Delta(u_1,u_2)\\
&\quad-2T(\varepsilon_1\cdot\varepsilon_2)^\bullet\Delta^\bullet(u_1,u_2) - 4T^2(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_1)^\bullet\Delta(u_1,u_2)\Delta^\bullet(u_1,u_2)\Big]e^{2T\Delta(u_1,u_2)k_1\cdot k_2}\,,\end{aligned} \tag{6.56}$$

where

$$a = \frac{p_2-p_1-k_1-k_2}{2} + k_1u_1 + k_2u_2 = p_2 + k_1u_1 + k_2u_2\,, \tag{6.57}$$

for all external momentum to be incoming . Now let's do the x'-integral which is Gaussian, for Gaussian we use

$$\int_{-\infty}^{+\infty} d^nxe^{-a\mathbf{x}^2+2\mathbf{b}\cdot\mathbf{x}} = (\frac{\pi}{a})^{\frac{n}{2}}e^{\frac{b_1^2}{a}}\cdots e^{\frac{b_n^2}{a}}\,. \tag{6.58}$$

So the x'-integral gives

$$\begin{aligned}\int d^D x' e^{-\frac{x'^2}{4T}+ia\cdot x'} &= (4\pi T)^{\frac{D}{2}} e^{-Ta^2}\,,\\ \frac{\partial^2}{\partial a^\mu \partial a^\nu} e^{-Ta^2} &= (-2T\delta_{\mu\nu}+4T^2 a_\mu a_\nu)e^{-Ta^2}\,.\end{aligned} \tag{6.59}$$

Now, we decompose our amplitude in (6.56) to some sub-amplitudes which $\Gamma = \sum_{i=1}^4 \Gamma_i$, where Γ_i after T-integral give:

$$\begin{aligned}\Gamma_1 &= 2e^2 \int_0^1 du_1 du_2 \Big\{ \frac{-\varepsilon_1\cdot\varepsilon_2}{[m^2+a^2-2\Delta(u_1,u_2)k_1\cdot k_2]^2} + \frac{4\varepsilon_1\cdot a\varepsilon_2\cdot a}{[m^2+a^2-2\Delta(u_1,u_2)k_1\cdot k_2]^3}\Big\}\,,\\ \Gamma_2 &= 2e^2 \int_0^1 du_1 du_2 \Big\{ \frac{-4\varepsilon_1\cdot a\varepsilon_2\cdot k_1 \Delta^\bullet(u_1,u_2)}{[m^2+a^2-2\Delta(u_1,u_2)k_1\cdot k_2]^3}\Big\}\,,\\ \Gamma_3 &= 2e^2 \int_0^1 du_1 du_2 \Big\{ \frac{-4\varepsilon_2\cdot a\varepsilon_1\cdot k_2 {}^\bullet\Delta(u_1,u_2)}{[m^2+a^2-2\Delta(u_1,u_2)k_1\cdot k_2]^3}\Big\}\,,\\ \Gamma_4 &= 2e^2 \int_0^1 du_1 du_2 \Big\{ \frac{\varepsilon_1\cdot\varepsilon_2 {}^\bullet\Delta^\bullet(u_1,u_2)}{[m^2+a^2-2\Delta(u_1,u_2)k_1\cdot k_2]^2}\Big\}\,,\\ \Gamma_5 &= 8e^2 \int_0^1 du_1 du_2 \Big\{ \frac{\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1 {}^\bullet\Delta(u_1,u_2)\Delta^\bullet(u_1,u_2)}{[m^2+a^2-2\Delta(u_1,u_2)k_1\cdot k_2]^3}\Big\}\,.\end{aligned} \tag{6.60}$$

Now at this step we do the u-integrals and to do so we should consider two different orderings which are $u_1 > u_2$ and $u_2 > u_1$. We first look at the denominators for these orderings:

$$\begin{aligned}& a^2 = p_2^2 + 2p_2\cdot(k_1u_1+k_2u_2) + 2k_1\cdot k_2 u_1 u_2\,,\\ & a^2+m^2-2\Delta(u_1,u_2)k_1\cdot k_2 = m^2+p_2^2+2(p_2\cdot k_2+k_1\cdot k_2)u_2 + 2p_2\cdot k_1 u_1 \quad , \quad u_1>u_2\\ & a^2+m^2-2\Delta(u_1,u_2)k_1\cdot k_2 = m^2+p_2^2+2(p_2\cdot k_1+k_1\cdot k_2)u_1 + 2p_2\cdot k_2 u_2 \quad , \quad u_2>u_1\\ & \varepsilon_1\cdot a = \varepsilon_1\cdot p_2 + \varepsilon_1\cdot k_2 u_2\,,\\ & \varepsilon_2\cdot a = \varepsilon_2\cdot p_2 + \varepsilon_2\cdot k_1 u_1\,,\end{aligned} \tag{6.61}$$

To simplify our calculations in the next Section we consider different orderings.

6.5.2 Two orderings

We define the following scalars as

$$\begin{aligned}b_1 &= 2p_2\cdot k_2 + 2k_1\cdot k_2\,,\\ c_1 &= 2p_2\cdot k_1\,,\\ q_i^2 &= m^2+p_i^2\,,\end{aligned} \tag{6.62}$$

for $u_1 > u_2$ ordering and a half of the delta function contributes here only, i.e.

$$
{}^\bullet\Delta^\bullet(u_1, u_2) = 1 - \frac{1}{2}\delta(u_1 - u_2)\,. \tag{6.63}
$$

The sub-amplitudes can be written as

$$
\begin{aligned}
\Gamma_1 &= -2e^2\varepsilon_1\cdot\varepsilon_2\int_0^1 du_1\int_0^{u_1} du_2\frac{1}{(q_2^2+b_1u_2+c_1u_1)^2}\\
&\quad +8e^2\int_0^1 du_1\int_0^{u_1} du_2\frac{\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2+\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1u_1+\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2u_2+\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1u_1u_2}{(q_2^2+b_1u_2+c_1u_1)^3}\,,\\
\Gamma_2 &= -8e^2\int_0^1 du_1\int_0^{u_1} du_2\frac{\varepsilon_2\cdot k_1\varepsilon_1\cdot k_2(u_1-1)+\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1u_2(u_1-1)}{(q_2^2+b_1u_2+c_1u_1)^3}\,,\\
\Gamma_3 &= -8e^2\int_0^1 du_1\int_0^{u_1} du_2\frac{\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2u_2+\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1u_2u_1}{(q_2^2+b_1u_2+c_1u_1)^3}\,,\\
\Gamma_4 &= 2e^2\varepsilon_1\cdot\varepsilon_2\int_0^1 du_1\int_0^{u_1} du_2\frac{1}{(q_2^2+b_1u_2+c_1u_1)^2}-e^2\varepsilon_1\cdot\varepsilon_2\int_0^1 du_1\int_0^{u_1} du_2\frac{\delta(u_1-u_2)}{(q_2^2+b_1u_2+c_1u_1)^2}\,,\\
\Gamma_5 &= +8e^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1\int_0^1 du_1\int_0^{u_1} du_2\frac{u_2(u_1-1)}{(q_2^2+b_1u_2+c_1u_1)^3}\,.
\end{aligned} \tag{6.64}
$$

One can already see some cancelations at this stage, the remaining terms are

$$
\begin{aligned}
\Gamma_1 &= +8e^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2\int_0^1 du_1\int_0^{u_1} du_2\frac{1}{(q_2^2+b_1u_2+c_1u_1)^3}\,,\\
\Gamma_2 &= +8e^2\varepsilon_2\cdot k_1\varepsilon_1\cdot k_2\int_0^1 du_1\int_0^{u_1} du_2\frac{1}{(q_2^2+b_1u_2+c_1u_1)^3}\\
&\quad -8e^2\varepsilon_2\cdot k_1\varepsilon_1\cdot k_2\int_0^1 du_1\int_0^{u_1} du_2\frac{u_2(u_1-1)}{(q_2^2+b_1u_2+c_1u_1)^3}\,,\\
\Gamma_4 &= -e^2\varepsilon_1\cdot\varepsilon_2\int_0^1 du_1\int_0^{u_1} du_2\frac{\delta(u_1-u_2)}{(q_2^2+b_1u_2+c_1u_1)^2}\,,\\
\Gamma_5 &= +8e^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1\int_0^1 du_1\int_0^{u_1} du_2\frac{u_2(u_1-1)}{(q_2^2+b_1u_2+c_1u_1)^3}\,.
\end{aligned} \tag{6.65}
$$

and after u-integrals we find a nice results as

$$
\begin{aligned}
\Gamma_1 &= 4e^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2\frac{1}{q_2^2\,(c_1+q_2^2)\,(b_1+c_1+q_2^2)}\,,\\
\Gamma_2 &= 4e^2\varepsilon_2\cdot k_1\varepsilon_1\cdot k_2\frac{1}{q_2^2\,(c_1+q_2^2)\,(b_1+c_1+q_2^2)}\,,\\
\Gamma_4 &= -e^2\frac{\varepsilon_1\cdot\varepsilon_2}{q_2^2(b_1+c_1+q_2^2)}\,.
\end{aligned} \tag{6.66}
$$

Since we have amputated the external propagators at the beginning of our calculation, we should recover them by multiply the final results with scalar propagators $(p_i^2 + m^2)$, so we get

$$q_1^2\Gamma_1 q_2^2 \quad = \quad 4e^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2\frac{q_1^2 q_2^2}{q_2^2\,(c_1+q_2^2)\,(b_1+c_1+q_2^2)} = 4e^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2\frac{1}{(c_1+q_2^2)}\,, \tag{6.67}$$

where we have used $q_1^2 = b_1 + c_1 + q_2^2$.
Now if we go on-shell for scalar external legs $(q_i^2 = p_i^2 + m^2 = -m^2 + m^2 = 0)$ we end up with

$$\begin{aligned} q_1^2\Gamma_1 q_2^2 &= 2\frac{e^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_2}{p_2\cdot k_1}\,, \\ q_1^2\Gamma_2 q_2^2 &= 2\frac{e^2\varepsilon_2\cdot k_1\varepsilon_1\cdot k_2}{p_2\cdot k_1}\,, \\ q_1^2\Gamma_4 q_2^2 &= -e^2\varepsilon_1\cdot\varepsilon_2\,, \end{aligned} \tag{6.68}$$

which is for $u_1 > u_2$ ordering.

Other ordering $u_2 > u_1$ in the same way produces

$$\begin{aligned} q_1^2(\Gamma_1+\Gamma_3)q_2^2 &= -2\frac{e^2\varepsilon_1\cdot k_1\varepsilon_2\cdot k_2}{p_2\cdot k_2}\,, \\ q_1^2\Gamma_4 q_2^2 &= -e^2\varepsilon_1\cdot\varepsilon_2\,. \end{aligned} \tag{6.69}$$

Now to get the final result for Compton scattering Fig. 6.8 we add (6.68) and (6.69),

$$A_{\text{Compton,worldline}}(\gamma s\to\gamma s) = -2e^2\Big(\varepsilon_1\cdot\varepsilon_2 + \frac{\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1}{p_1\cdot k_2} + \frac{\varepsilon_1\cdot k_1\varepsilon_2\cdot k_2}{p_2\cdot k_2}\Big)\,. \tag{6.70}$$

To compare with the result from [247], we need to switch the signs of two external legs (p_2 and k_2) to make them outgoing to see a perfect match between (6.70) and the one from [247].

b) Graviton photonproduction

As we have mentioned in the introduction, several graviton interaction processes have been studied previously. The gravitational Compton scattering ($ge \to ge$) and the graviton photon-production ($\gamma e \to \gamma g$) were considered in the lowest order Born approximation in Refs. [263,264]. The cross sections of two annihilation processes $e\bar{e} \to g\gamma$ and $e\bar{e} \to gg$ were calculated in [265]. Also, first-order cross sections for the processes such as $ge \to \gamma e$, bremsstrahlung, and $e\bar{e}$-pair production by graviton in the Coulomb field were calculated [266]. But all these calculations do not agree with the results of Voronov [267]. According to the statement of Weinberg that was mentioned in the introduction 6.1, it is crucial to maintain general covariance in the theory that one should introduce gravitational gauge invariance on determining the interaction Lagrangian. Following that point of view, Choi *et al.* in [247] utilize the same Lagrangian as in [267]. In their work it is found that the gravitational gauge invariance forces a graviton interaction with a charged fermion and photon to have its transition amplitude factorized into an energy-momentum-dependent part and a spin- or polarization-dependent part. On the other hand, similar factorization exist in the transition amplitudes of photon because of the $U(1)_{\rm EM}$ gauge invariance of the theory. In the following, we use the worldline formalism to calculate the $\gamma s \to gs$ (s stands for scalar) process, and we show this factorization for this amplitude which has been obtained before [246,247]. The Feynman diagrams for the processes under investigation can be represented as Fig. 6.9.

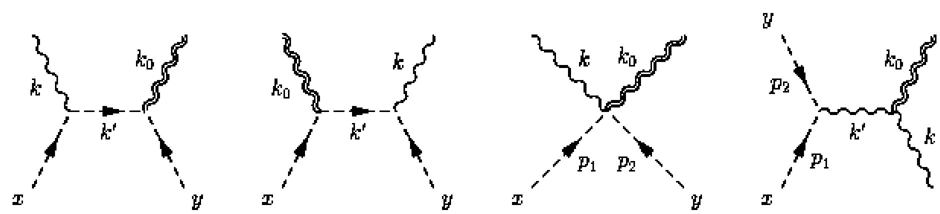


Figure 6.9: Irreducible and reducible diagrams for $s+\gamma \to s+g$ scattering, left to right: (a), (b), (c) and (d).

There is a contact interaction term 6.9(c) and a photon-graviton coupling term 6.9(d) in which the graviton can couple with a photon without mass but with its energy. Even without 6.9(d) which we call it γ-pole, the transition amplitude is $U(1)_{\rm EM}$ gauge invariant (as we have seen in electromagnetic Compton scattering Fig. 6.8, but this diagram should be included in order to ensure gravitational gauge invariance). Note that in the electromagnetic Compton scattering for QED process Fig. 6.8 ($s \to f$) there is no contact term, due to the Abelian property of the gauge group $U(1)_{\rm EM}$, but in the photonproduction case with fermion ($\gamma f \to gf$) we need the contact or seagull diagram. The existence of a seagull is required by the feature that the energy-momentum tensor is momentum-dependent and therefore yields contact interactions. In the following we analyze the scalar case of photonproduction process Fig. 6.9 by using worldline formalism. First we consider the irreducible part of these diagrams which corresponds to the first three diagrams (a), (b) and (c). From the worldline formalism for different ordering we can calculate these diagrams in one go, the amplitude can be written as

$$\Gamma[x,y;\varepsilon,\varepsilon_0]_{\rm irr} = i\kappa e \int_0^\infty dT e^{-m^2T} \int_{x(0)=x}^{x(T)=y} \mathcal{D}x(\tau) e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} \int_0^T d\tau_1 d\tau_2 \langle V^\gamma_{\rm scal}[k,\varepsilon] V^h_{\rm scal}[k_0,\varepsilon_0]\rangle \,, \tag{6.71}$$

where $V^\gamma_{\rm scal}[k,\varepsilon]$ and $V^h_{\rm scale}[k_0,\varepsilon_0]$ are photon and graviton vertex operators accordingly, coupled to scalar particles. By inserting these vertex operators we get

$$\begin{aligned}\Gamma[x,y;\varepsilon,\varepsilon_0]_{\rm irr} \quad = \quad & -i\frac{\kappa e}{4}\int_0^\infty dT e^{-m^2T} e^{-\frac{1}{4T}(x-y)^2} \int_{q(0)=q(T)=0} \mathcal{D}q(\tau) e^{-\frac{1}{4}\int_0^T d\tau \dot{q}^2} \\ & \int_0^T d\tau d\tau_0 e^{ik\cdot x_{\rm bg}(\tau)} e^{ik_0\cdot x_{\rm bg}(\tau_0)} \varepsilon_\alpha \varepsilon_{0\mu\nu} \Big\langle \Big[\frac{y-x}{T} + \dot{q}(\tau)\Big]^\alpha \\ & \times \Big\{ \Big[\frac{y-x}{T} + \dot{q}(\tau_0)\Big]^\mu \Big[\frac{y-x}{T} + \dot{q}(\tau_0)\Big]^\nu + a^\mu(\tau_0)a^\nu(\tau_0) + b^\mu(\tau_0)c^\nu(\tau_0) \Big\} e^{ik\cdot q(\tau)+ik_0\cdot q(\tau_0)} \Big\rangle \,.\end{aligned} \tag{6.72}$$

Note that on-shell, the ghost fields give no contributions to this amplitude so we can eliminate them from the beginning. The $\Big\langle \cdots \Big\rangle$ part, after all Wick contractions gives

$$
\begin{aligned}
\Big\langle \cdots \Big\rangle \;=\;& \Big[\frac{(y-x)^\alpha\,(y-x)^\mu(y-x)^\nu}{T^3}+\frac{(y-x)^\alpha\,(y-x)^\mu}{T^2}(-2ik^\nu\,{}^\bullet\Delta(\tau_0,\tau))\\
&+\frac{(y-x)^\alpha\,(y-x)^\nu}{T^2}(-2ik^\mu\,{}^\bullet\Delta(\tau_0,\tau))+\frac{(y-x)^\alpha}{T}(-4k^\mu k^\nu\,{}^\bullet\Delta(\tau_0,\tau)^2)\\
&+\frac{(y-x)^\mu\,(y-x)^\nu}{T^2}(-2ik_0^\alpha\,\Delta^\bullet(\tau_0,\tau))+\frac{(y-x)^\mu}{T}(-2\delta^{\nu\alpha}\,{}^\bullet\Delta^\bullet(\tau_0,\tau)\\
&-4k^\nu k_0^\alpha\,{}^\bullet\Delta(\tau_0,\tau)\Delta^\bullet(\tau_0,\tau))+\frac{(y-x)^\nu}{T}(-2\delta^{\mu\alpha}\,{}^\bullet\Delta^\bullet(\tau_0,\tau)\\
&-4k^\mu k_0^\alpha\,{}^\bullet\Delta(\tau_0,\tau)\Delta^\bullet(\tau_0,\tau))+4ik^\nu\delta^{\mu\alpha}\,{}^\bullet\Delta^\bullet(\tau_0,\tau)\,{}^\bullet\Delta(\tau_0,\tau)+4ik^\mu\delta^{\nu\alpha}\,{}^\bullet\Delta^\bullet(\tau_0,\tau)\,{}^\bullet\Delta(\tau_0,\tau)\\
&+8ik^\mu k^\nu k_0^\alpha\,{}^\bullet\Delta(\tau_0,\tau)^2\,\Delta^\bullet(\tau_0,\tau)\Big]e^{2\Delta(\tau_0,\tau)k\cdot k_0}\,.
\end{aligned}
\tag{6.73}
$$

After rescaling ($\tau_i = Tu_i$) and Fourier transform (6.72), it modifies to

$$
\begin{aligned}
\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\rm irr} \;=\;& -i\frac{\kappa e(p_1^2+m^2)(p_2^2+m^2)}{4}(2\pi)^D\delta^D(p_1+p_2+k+k_0)\\
&\int d^Dx' e^{ix'\cdot p_2}\int_0^\infty dT e^{-m^2T}e^{-\frac{x'^2}{4T}}(4\pi T)^{-\frac{D}{2}}\int_0^1 du du_0\\
&\Big\{\frac{(x'\cdot\varepsilon)(x'\varepsilon_0 x')}{T^3}-4i(\varepsilon\cdot x')(k\varepsilon_0 x')\,{}^\bullet\Delta(u_0,u)\\
&-4T(\varepsilon\cdot x')(k\varepsilon_0 k)\,{}^\bullet\Delta^2(u_0,u)-2i(\varepsilon\cdot k_0)(x'\varepsilon_0 x')\Delta^\bullet(u_0,u)-4(\varepsilon\varepsilon_0 x')\,{}^\bullet\Delta^\bullet(u_0,u)\\
&-8T(\varepsilon\cdot k_0)(k\varepsilon_0 x')\,{}^\bullet\Delta(u_0,u)\Delta^\bullet(u_0,u)+8iT(k\varepsilon_0\varepsilon)\,{}^\bullet\Delta^\bullet(u_0,u)\,{}^\bullet\Delta(u_0,u)\\
&+8iT^2(k\varepsilon_0 k)\,(k_0\cdot\varepsilon)\,{}^\bullet\Delta(u_0,u)^2\,\Delta^\bullet(u_0,u)\Big\}e^{2Tk_0\cdot k\Delta(u_0,u)+i(ku+k_0u_0)\cdot x'}\,.
\end{aligned}
\tag{6.74}
$$

By doing the x'-integral which again is Gaussian, (6.74) can be written as

$$
\begin{aligned}
\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\rm irr} \;=\;& -i\frac{\kappa e(p_1^2+m^2)(p_2^2+m^2)}{4}\int_0^\infty dT\int_0^1 du du_0\\
&\Big\{8i\Big[T(\varepsilon\varepsilon_0 a)-T^2(\varepsilon\cdot a)\,(a\varepsilon_0 a)\Big]+4i\Big[4T^2(\varepsilon\cdot a)\,(k\varepsilon_0 a)-2T(\varepsilon\varepsilon_0 k)\Big]{}^\bullet\Delta(u_0,u)\\
&-8iT^2(\varepsilon\cdot a)\,(k\varepsilon_0 k)\,{}^\bullet\Delta^2(u_0,u)+8iT^2(\varepsilon\cdot k_0)\,(a\varepsilon_0 a)\Delta^\bullet(u_0,u)\\
&-8iT(\varepsilon\varepsilon_0 a)\,{}^\bullet\Delta^\bullet(u_0,u)-16iT^2(\varepsilon\cdot k_0)\,(k\varepsilon_0 a)\,{}^\bullet\Delta(u_0,u)\Delta^\bullet(u_0,u)\\
&+8iT(\varepsilon\varepsilon_0 k)\,{}^\bullet\Delta^\bullet(u_0,u)\,{}^\bullet\Delta(u_0,u)+8iT^2(\varepsilon\cdot k_0)\,(k\varepsilon_0 k)\,{}^\bullet\Delta(u_0,u)^2\,\Delta^\bullet(u_0,u)\Big\}\\
&\qquad\times e^{-T(m^2+a^2-2k_0\cdot k\Delta(u_0,u))}\,,
\end{aligned}
\tag{6.75}
$$

where $a = p_2 + ku + k_0u_0$.

After some cancelation which happens by using the fact that ${}^\bullet\Delta^\bullet(u_0,u) = 1-\delta(u_0-u)$, the final results before T and u integrals simplifies as

$$\begin{aligned}
\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\rm irr} &= \kappa e(p_1^2+m^2)(p_2^2+m^2)\int_0^\infty dT\int_0^1 dudu_0 \\
&\times(-2T^2)\Big[(\varepsilon\cdot a)-(\varepsilon\cdot k_0\Delta^\bullet(u_0,u))\Big] \\
&\times\Big[-2(k\varepsilon_0 a)^\bullet\Delta(u_0,u)+(a\varepsilon_0 a)+(k\varepsilon_0 k)^\bullet\Delta^2(u_0,u)\Big]e^{-T(m^2+a^2-2k_0\cdot k\Delta(u_0,u))} \\
&-\kappa e(p_1^2+m^2)(p_2^2+m^2)\varepsilon\varepsilon_0(p_1-p_2)\int_0^\infty dTe^{-m^2T}T \\
&\times\int_0^1 due^{-T[m^2+p_2^2+2p_2\cdot(k+k_0)u+2k_0\cdot ku]}\,.
\end{aligned}\tag{6.76}$$

At this stage as electromagnetic Compton scattering we consider two different orderings, first $u_0>u$ which simplifies (6.76) considerably as

$$\begin{aligned}
\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\rm irr} &\overset{u_0\geq u}{=} -2\kappa e(p_1^2+m^2)(p_2^2+m^2)(\varepsilon\cdot p_2)\,(p_1\varepsilon_0 p_1) \\
&\int_0^\infty dTT^2\int_0^1 du\int_0^u du_0 e^{-T(m^2+p_2^2+2p_2\cdot ku-2p_1\cdot k_0u_0)} \\
&-\frac{\kappa e(p_1^2+m^2)(p_2^2+m^2)}{k_0\cdot k}\int_0^\infty dTe^{-m^2T}T\int_0^1 du\Big[\varepsilon\varepsilon_0(p_1-p_2)k_0\cdot k\Big] \\
&\times e^{-T[m^2+p_2^2+2p_2\cdot(k+k_0)u+2k_0\cdot ku]}\,.
\end{aligned}\tag{6.77}$$

Now one can perform the integrals, and finally the results would be

$$\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\rm irr} \overset{u_0\geq u}{=} -\frac{2\kappa e}{2p_2\cdot k}(\varepsilon\cdot p_2)\,(p_1\varepsilon_0 p_1)-\kappa e\varepsilon\varepsilon_0(p_1-p_2)\,,\tag{6.78}$$

and for $u>u_0$ ordering,

$$\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\rm irr} \overset{u\geq u_0}{=} \frac{2\kappa e}{2p_2\cdot k_0}(\varepsilon\cdot p_1)\,(p_2\varepsilon_0 p_2)\,.\tag{6.79}$$

And finally for first three diagrams of Fig. 6.9 (irreducible part), by adding (6.78) and (6.79) we get

$$\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\rm irr} = \frac{2\kappa e}{2p_2\cdot k_0}(\varepsilon\cdot p_1)\,(p_2\varepsilon_0 p_2)-\frac{2\kappa e}{2p_2\cdot k}(\varepsilon\cdot p_2)\,(p_1\varepsilon_0 p_1)-\kappa e\varepsilon\varepsilon_0(p_1-p_2)\,.\tag{6.80}$$

As we mentioned above we need the reducible or γ-pole diagram (Fig. 6.9(d)) to get gauge invariant result. If we decompose the reducible diagram to two parts as Fig. 6.10, we get two vertex, one is the coupling of two scalars with photon ($ss\gamma$) and the other part is the coupling of one-graviton with two photons ($g\gamma\gamma$). We know the Feynman rule for the first part and also according to the worldline formalism it is just a self contraction of a photon vertex operator but for the second part we should use energy-momentum tensor.

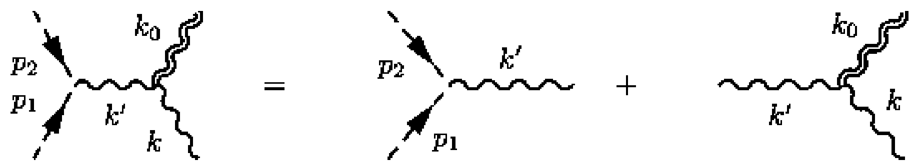


Figure 6.10: The reducible diagram can be decomposed to two parts, $ss\gamma$ and $g\gamma\gamma$.

The first part can be written as

$$\begin{aligned}
\Gamma_{ss\gamma} &= e\varepsilon'_\mu \int_0^1 du \int_0^\infty dT \Big[-2T(k'^\mu u + p_2^\mu) + 2Tk'^{\mu\bullet}\Delta(u,u)\Big] e^{-T(m^2+(k'u+p_2)^2-k'^2\Delta(u,u))} \\
&= -2e\int_0^1 du \int_0^\infty dTT\Big[\varepsilon'\cdot(p_2+\frac{1}{2}k')\Big]e^{-T[m^2+p_2^2+(2p_2\cdot k'+k'^2)u]} \\
&= e\int_0^1 du \int_0^\infty dTT\Big[\varepsilon'\cdot(p_1-p_2)\Big]e^{-T[m^2+p_2^2+(2p_2\cdot k'+k'^2)u]}\,.
\end{aligned} \tag{6.81}$$

Note that $k'^2 \neq 0$ because it is an off-shell photon.

And for the other part of this diagram, we use the following definition of energy-momentum tensor [256]

$$T^{\mu\nu} = F^{\mu\alpha}F_\alpha{}^\nu - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu}\,. \tag{6.82}$$

By coupling energy-momentum tensor ($T^{\mu\nu}$) to graviton $h_{\mu\nu}$ and using the fact that $h_\mu^{\ \mu} = 0$ (on-shell), we get

$$\Gamma_{g\gamma\gamma} = h_{\mu\nu}T^{\mu\nu} = -2\kappa\Big[(k\varepsilon_0k')\,(\varepsilon\cdot\epsilon') - (k\varepsilon_0\varepsilon')(\varepsilon\cdot k') - (\varepsilon\varepsilon_0k')(\varepsilon'\cdot k) + (\varepsilon\varepsilon_0\varepsilon')(k\cdot k')\Big]\,. \tag{6.83}$$

Now we have everything to calculate the γ-pole, by using the Feynman gauge ($\varepsilon'^\alpha\varepsilon'^\beta \to \frac{-\eta^{\alpha\beta}}{k'^2}$), we can put these two parts together again to get

$$\begin{aligned}
\Gamma_{\rm red} \;=\; & \frac{e\kappa(p_1^2+m^2)(p_2^2+m^2)}{2k_0\cdot k}\int_0^1 du\int_0^\infty dTTe^{-T[m^2+p_2^2+2p_2\cdot(k+k_0)u+2k_0\cdot ku]} \\
& \times\Big[(k\varepsilon_0k)\,\varepsilon\cdot(p_1-p_2) - k\varepsilon_0(p_1-p_2)(\varepsilon\cdot k_0) - (\varepsilon\varepsilon_0k)\,k\cdot(p_1-p_2) + \varepsilon\varepsilon_0(p_1-p_2)(k\cdot k_0)\Big]\,.
\end{aligned} \tag{6.84}$$

Now after performing the T and u-integrals, (6.84) can be written as

$$\Gamma_{\rm red} = \frac{e\kappa}{2k_0\cdot k}\Big[(k\varepsilon_0 k)\,\varepsilon\cdot(p_1-p_2) - k\varepsilon_0(p_1-p_2)(\varepsilon\cdot k_0) - (\varepsilon\varepsilon_0 k)\,k\cdot(p_1-p_2) + \varepsilon\varepsilon_0(p_1-p_2)(k\cdot k_0)\Big]\,. \tag{6.85}$$

Summary and comparison with previous studies

As a summary for this Section, we put the irreducible and reducible diagrams together to be able to compare with previous studies,

$$\begin{aligned}\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\gamma s\to gs} &= \Gamma_{\rm irr}+\Gamma_{\rm red}\\ &= \frac{\kappa e}{p_1\cdot k}(\varepsilon\cdot p_1)\,(p_2\varepsilon_0 p_2) - \frac{\kappa e}{p_1\cdot k_0}(\varepsilon\cdot p_2)\,(p_1\varepsilon_0 p_1) - \kappa e\varepsilon\varepsilon_0(p_1-p_2)\\ &\quad+\frac{e\kappa}{2k_0\cdot k}\Big[(k\varepsilon_0 k)\,\varepsilon\cdot(p_1-p_2) - k\varepsilon_0(p_1-p_2)(\varepsilon\cdot k_0)\\ &\qquad -(\varepsilon\varepsilon_0 k)(p_1-p_2)\cdot k + \varepsilon\varepsilon_0(p_1-p_2)\,(k_0\cdot k)\Big]\,.\end{aligned} \tag{6.86}$$

We are missing a factor of two, but presently are still looking for an explanation. Our result from the worldline formalism beside of this factor of two, and by considering the outgoing or ingoing particles (since in our calculation all particles are ingoing) we have a perfect match with previous studies [246].

Another nice property of this amplitude is the factorization property. By adding these four diagrams Fig. 6.9 we find (after considerable but simple algebra) a remarkably simple result

$$\Gamma[p_1,p_2;k,\varepsilon;k_0,\varepsilon_0]_{\gamma s\to gs} = 2H\,A_{\rm Compton,worldline}(\gamma s\to\gamma s)\,, \tag{6.87}$$

where $A_{\rm Compton,worldline}$ is what we have calculated in Section 6.5.1 which was

$$A_{\rm Compton,worldline}(\gamma s\to\gamma s) = -2e^2\Big(\varepsilon_1\cdot\varepsilon_2 + \frac{\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1}{p_1\cdot k_2} + \frac{\varepsilon_1\cdot k_1\varepsilon_2\cdot k_2}{p_2\cdot k_2}\Big)\,, \tag{6.88}$$

and

$$H = \frac{\kappa}{4e}\frac{(\varepsilon_0\cdot p_2)(k_0\cdot p_1) - (\varepsilon_0\cdot p_1)(k_0\cdot p_2)}{k\cdot k_0}\,, \tag{6.89}$$

which is compatible with previous studies related to the factorization property of this scattering amplitude [246, 247]. This worldline representation of Compton scattering is in progress, the final results will be presented elsewhere [240].

6.6 Conclusions

In this Chapter we have discussed the worldline formalism for a scalar particle coupled to a gravitational background. Using an UV regularization one can see how the results expected from QFT follow unambiguously. Our main goal was the KLT type relation for one-loop calculation in a mixed electromagnetic and gravitational background. In this way, we first presented the gravitational tadpole and self energy diagrams which have been obtained in [51]. Then we presented the photon-photon-graviton amplitude (on-shell) which has been calculated in [75], and after that we discussed our main object which is one-graviton four-photon amplitude at one-loop. This amplitude contains two diagrams, (Fig. 6.29), which from the worldline formalism and spinor helicity we know how deal with the irreducible one, but the reducible one needs more effort, and also we need to combine these two contributions in nice way through the worldline formalism, this part of our goal is in progress. After discussing the one-loop amplitude we discussed some tree-level calculations including graviton. At first to compare our method with conventional QFT calculations we have derived the electromagnetic Compton scattering ($\gamma + s \to \gamma + s$) for the scalar case. Our result is in agreement with the previous studies. And finally we calculated the photonproduction amplitude ($\gamma + s \to g + s$) which graviton is in final state, and again we have an agreement with previous results by other authors. The win reason to study the tree-level diagrams especially the photonproduction was to extract the factorization property of this diagrams from the worldline formalism because we are faced with the same situation but in one-loop order for our five-point calculation. As we have seen in the last part, by this factorization one can arrange the diagrams in a way to just forget about the reducible diagram which is very interesting property and might help us in our main project. The gravitational Compton scattering is also in progress, the irreducible part of this case contains two Born and one seagull diagrams which contains a vertex with two-graviton and its reducible part has a diagram with a coupling of three-gravitons (gravitonproduction). This case needs more work.

Appendix A

Conventions

We work with the $(-+++)$ metric. The non-abelian covariant derivative is $D_\mu \equiv \partial_\mu + igA^a_\mu T^a$, with $[T^a, T^b] = if^{abc}T^c$. The adjoint representation is given by $(T^a)^{bc} = -if^{abc}$. The normalization of the generators is $\mathrm{tr}(T^aT^b) = C(r)\delta^{ab}$, where for $SU(N)$ one has $C(N) = \frac{1}{2}$ for the fundamental and $C(G) = N$ for the adjoint representation. We also work with the natural units $\hbar = c = 1$.

Appendix B

Quantum mechanical path integral

In this appendix which is closely following the [268], we briefly discuss quantum mechanical path integral.

B.1 Path integral representation of quantum mechanical transition amplitude for non-relativistic bosonic particle

The quantum-mechanical transition amplitude for a time-dependent Hamiltonian operator is given by (in natural units where $\hbar = c = 1$) [1]

$$K(x, x'; t) = \langle x|e^{it\mathcal{H}}|x'\rangle = \langle x, t|x', 0\rangle \,, \tag{B.1}$$

$$K(x, x'; 0) = \delta(x - x') \,, \tag{B.2}$$

and describes the evaluation of the wave function from time 0 to time t

$$\psi(x, t) = \int dx' K(x, x'; t)\psi(x', 0) \,, \tag{B.3}$$

and satisfies the Schrödinger equation

$$i\partial_t K(x, x'; t) = HK(x, x'; t) \,, \tag{B.4}$$

with H being the Hamiltonian operator in coordinate representation; for a non-relativistic particle on a line where we have $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. In particular, for a free particle $V = 0$, it is easy to solve the Schrödinger equation (B.4) with boundary condition (B.2). One obtains

$$K_f(x, x'; t) = N_f(t)e^{iS_{\rm cl}(x,x';t)} \,, \tag{B.5}$$

[1] A generalization to time-dependent Hamiltonian $\mathbb{H}(t)$ can be obtained with the replacement $e^{-it\mathbb{H}} \to \mathcal{T}e^{-i\int_0^t d\tau \mathbb{H}(\tau)}$ and $\mathcal{T}e$ being a time-ordered exponential.

with

$$\begin{aligned} N_f(t) &= \sqrt{\frac{m}{2\pi i t}} \\ S_{\rm cl} &\equiv \frac{m(x-x')^2}{2t\,.} \end{aligned} \tag{B.6}$$

Now we are ready to derive the path integral representation of evolution kernel

$$K(x',x;t) = \langle x'|e^{-itH/\hbar}|x\rangle \tag{B.7}$$

In order to introduce the path integral we "slice" the evaluation operator in (B.7) by defining $\epsilon = t/N$ and insert $N-1$ decomposition of unity in terms of position eigenstates $\mathbb{1} = \int |x\rangle\langle x|$; namely

$$\begin{aligned} K(x,x';t) &= \langle x|e^{-i\epsilon HH}e^{-i\epsilon H}\cdots e^{-i\epsilon H}|x'\rangle \\ &= \int\Big(\prod_{i=1}^{N-1}dx_i\Big)\langle x_N|e^{-i\epsilon H}|x_{N-1}\rangle\prod_{j=1}^{N-1}\langle x_j|e^{-i\epsilon H}|x_{j-1}\rangle\,, \end{aligned} \tag{B.8}$$

with $x_N \equiv x$ and $x_0 \equiv x'$. We now insert N spectral decomposition of unity in term of momentum eigenstates $\mathbb{1} = \int \frac{dx}{2\pi}|p\rangle\langle p|$ to get

$$\begin{aligned} K(x,x';t) &= \int\Big(\prod_{i=1}^{N-1}dx_i\Big)\Big(\prod_{k=1}^{N}\frac{dp_k}{2\pi}\Big)\langle x_N|e^{i\epsilon H}|p_N\rangle\langle p_N|x_{N-1}\rangle \\ &\times \prod_{j=1}^{N-1}\langle x_j|e^{-i\epsilon H}|p_j\rangle\langle p_j|x_{j-1}\rangle\,. \end{aligned} \tag{B.9}$$

For large N, assuming $H = \mathbb{T}+\mathbb{V} = \frac{\mathbb{p}^2}{2m} + V(\mathbb{q})$, we can use the "Trotter formula" [269]

$$\Big(e^{-i\epsilon H}\Big)^N = \Big(e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}} + O(1/N^2)\Big)^N \approx \Big(e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}}\Big)^N\,, \tag{B.10}$$

that allows to replace $e^{-i\epsilon H}$ with $e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}}$ in (B.9), so that one gets $\langle x_j|e^{-i\epsilon H}|p_j\rangle \approx e^{(ix_jp_j - i\epsilon H(x_j,p_j))}$, with $\langle x|p\rangle = e^{ixp}$. Hence

$$K(x,x';t) = \int\Big(\prod_{i=1}^{N-1}dx_i\Big)\Big(\prod_{k=1}^{N}\frac{dp_k}{2\pi}\Big)\exp\Big[i\sum_{j=1}^{N}\epsilon\big(p_j\frac{x_j-x_{j-1}}{\epsilon} - H(x_j,p_j)\big)\Big] \,, \tag{B.11}$$

where

$$H(x_j,p_j) = \frac{p_j^2}{2m} + V(x_j)\,, \tag{B.12}$$

is the Hamiltonian function. In large N limit we can formally write the latter as

$$\begin{aligned} K(x,x';t) &= \int_{x(0)=x'}^{x(t)=x} \mathcal{D}x\mathcal{D}p \exp\Big[i\int_0^t d\tau\big(p\dot{x} - H(x,p)\big)\Big]\,, \\ \mathcal{D}x &\equiv \prod_{0<\tau<t} dx(\tau)\,, \\ \mathcal{D}p &\equiv \prod_{0<\tau<t} dp(\tau)\,, \end{aligned} \tag{B.13}$$

that is referred to as Feynman-Kac formula or "phase-space path integral". Alternatively, momenta can be integrated out in (B.9) as they are (analytic continuation of) gaussian integrals. Completing the square one get

$$K(x,x';t) = \int \Big(\prod_{i=1}^{N-1} dx_i\Big)\big(\frac{m}{2\pi i\epsilon}\big)^{\frac{N}{2}} \exp\Big[i\sum_{j=1}^{N} \epsilon\Big(\frac{m}{2}\big(\frac{x_j - x_{j-1}}{\epsilon}\big)^2 - V(x_j)\Big)\Big]\,, \tag{B.14}$$

which can be written as

$$K(x,x';t) = \int_{x(0)=x'}^{x(t)=x} \mathcal{D}x e^{iS[x(\tau)]}\,, \tag{B.15}$$

with

$$S[x(\tau)] = \int_0^t d\tau\Big(\frac{m}{2}\dot{x}^2 - V(x(\tau))\Big)\,. \tag{B.16}$$

Expression (B.15) is referred to as "configuration pace integral" and is interpreted as a functional integral over trajectories with boundary condition $x(0) = x'$ and $x(\tau) = x$.

B.1.1 Wick rotation to Euclidean time

As we already mentioned path integrals were born in statistical physics. In fact we can easily obtain the particle partition function from (B.15) by, (a) "Wick rotating" time to imaginary time, namely $it \equiv \beta = \frac{1}{k\theta}$ (where θ is the temperature) and (b) taking the trace

$$Z(\beta) \;=\; \mathrm{tr}e^{-\beta\mathcal{H}} = \int dx\langle x|e^{-\beta\mathcal{H}}|x\rangle = \int dx K(x,x;-i\beta) = \int_{\mathrm{PBC}} \mathcal{D}x\, e^{-S_E[x(\tau)]}\,, \tag{B.17}$$

where the Euclidean action

$$S_E(x(\tau)) = \int_0^\beta d\tau\big(\frac{m}{2}\dot{x}^2 + V(x(\tau))\big)\,, \tag{B.18}$$

has been obtained by Wick rotating the worldline time $i\tau \to \tau$, and PBC stands for periodic boundary conditions and means that the path integral is taken over all closed paths.

B.2 The free particle path integral and its partition function

We consider the path integral (B.15) for a special case of a free particle, i.e $V = 0$. For simplicity we consider a particle confined on a line and rescale the worldline time $\tau \to Tu$ in such a way the free action and boundary conditions turn into [268]

$$S_f[x(\tau)] = \frac{m}{2T}\int_0^1 du \dot{x}^2 \quad , \quad x(0) = x' \quad , \quad x(1) = x \, , \tag{B.19}$$

where x' and x are two points on the line and subscript f stands for free. We decompose $x(\tau)$ to the classical ($x_{\rm cl}(\tau)$) and quantum part ($q(\tau)$) with the following boundary condition

$$\begin{aligned} x(\tau) &= x_{\rm cl}(\tau) + q(\tau) \\ x_{\rm cl}(\tau) &= x' + (x - x')\tau = x + (x' - x)(1 - \tau) \quad , \quad q(0) = q(1) = 0 \, , \end{aligned} \tag{B.20}$$

and the above action reads as

$$\begin{aligned} S_f[x_{\rm cl} + q] &= \frac{m}{2T}\int_0^1 du(\dot{x}_{\rm cl}^2 + \dot{q}^2 + 2\dot{x}_{\rm cl}\dot{q}) = \frac{m}{2T}(x - x')^2 + \frac{m}{2T}\int_0^1 du\dot{q}^2 \, , \\ &= S_f[x_{\rm cl}] + S_f[q] \, , \end{aligned} \tag{B.21}$$

where the mixed term identically vanishes due to equation of motion and boundary conditions. The path integral can thus be written as

$$K_f(x, x'; T) = e^{S_f[x_{\rm cl}]}\int_{q(0)=0}^{q(1)=0} \mathcal{D}q \, e^{iS_f[q(u)]} = e^{i\frac{m}{2T}(x-x')^2}\int_{q(0)=0}^{q(1)=0} \mathcal{D}q \, e^{i\frac{m}{2T}\int_0^1 du\dot{q}^2} \, . \tag{B.22}$$

By comparison to (B.5) and (B.6), for free particle one obtains

$$\int_{q(0)=0}^{q(1)=0} \mathcal{D}q \, e^{i\frac{m}{2T}\int_0^1 du\dot{q}^2} = \sqrt{\frac{m}{2\pi i T}} \, . \tag{B.23}$$

Now one can expand $q(u)$ on a basis of a function which satisfies Dirichlet boundary condition,

$$q(u) = \sum_{n=1}^{\infty} q_n \sin(n\pi u) \, , \tag{B.24}$$

where q_n are arbitrary real coefficients. We can now define the measure as

$$\mathcal{D}q \equiv A\prod_{n=1}^{\infty} dq_n \sqrt{\frac{mn^2\pi}{4T}} \, , \tag{B.25}$$

with A an unknown coefficient

$$\int_{q(0)=0}^{q(1)=0} \mathcal{D}q\, e^{i\frac{m}{2T}\int_0^1 du \dot{q}^2} = A\int \prod_{n=1}^{\infty} dq_n \sqrt{\frac{mn^2\pi}{4iT}}\, e^{i\frac{m}{4T}\sum_n (\pi n q_n)^2} = A\,, \tag{B.26}$$

which by comparison with (B.23), A reads

$$A = \sqrt{\frac{m}{2\pi i T}}\,. \tag{B.27}$$

This result can easily be generalized to D space dimensions where

$$A = \int_{q(0)=0}^{q(1)=0} \mathcal{D}q\, e^{i\frac{m}{2T}\int_0^1 du \dot{q}^2} = (\frac{m}{2\pi i T})^{\frac{D}{2}}\,. \tag{B.28}$$

Equation (B.25) allows to use the so-called mode regularization to commute more generic particle path integrals where interaction terms may introduce computational ambiguity. Namely: whenever an ambiguity appears one can always rely on the mode expansion truncated at a finite mode M and then take the large limit at the very end. Other regularization schemes that have been adopted to such purpose are: time slicing that rely on the well-defined expression for the path integral as multiple slices (what we have done in equation B.14), and dimensional regularization that regulates ambiguities by dimensionally extending the worldline, for a review on three methods one can see [52]. However dimensional regularization is a regularization that only works in the perturbative approach to the path integral, by regulating single Feynman diagrams [268].
Let us compute the propagator of a free particle in D-dimensional space

$$Z_f(\beta) = \int d^D x K(x,x;-i\beta) = (\frac{m}{2\pi\beta})^{\frac{D}{2}}\int d^D x = \mathcal{V}(\frac{m}{2\pi\beta})^{\frac{D}{2}} \tag{B.29}$$

with $\mathcal{V}$ being the spatial volume. One can check if this gives a correct result by considering the partition function for a free particle which reads as

$$Z_f(\beta) = \sum_p e^{-\beta\frac{p^2}{2m}} = \frac{\mathcal{V}}{(2\pi)^D}\int d^D p\, e^{\beta\frac{p^2}{2m}} = \mathcal{V}(\frac{m}{2\pi\beta})^{\frac{D}{2}} \tag{B.30}$$

where $\frac{\mathcal{V}}{(2\pi)^D}$ is the "density of state".

B.2.1 Perturbation theory about free particle solution: Feynman diagrams

We now turn our attention to the interacting quantum systems. In the presence of an interaction or an arbitrary potential the path integral for the transition amplitude is not exactly solvable but if the potential is small compared to the kinetic term one can us perturbation theory about the free particle solution that seen above.

In the following we obtain a perturbative expansion for the transition amplitude (B.15) with action (B.16). As before, we split the arbitrary path in terms of the classical path (with respect to the free action) and deviation $q(\tau)$, after rescaling the proper time, we can rewrite the amplitude as

$$K(x,x';T)=N_f(T)e^{i\frac{m}{2T}(x-x')^2}\,\frac{\int_{q(0)=0}^{q(1)=0}\mathcal{D}q\,e^{i\int_0^1 du\left(\frac{m}{2T}\dot{q}^2-TV(x_{\rm cl}+q)\right)}}{\int_{q(0)=1}^{q(1)=0}\mathcal{D}q e^{i\frac{m}{2T}\int_0^1 du\dot{q}^2}}\,. \tag{B.31}$$

We then Taylor expand the potential in the exponent about the classical free solution: this gives rise to a infinite set of interaction terms

Figure B.1: Feynman diagrams which represent (B.32).

$$S_{\rm int}=-T\int_0^1 du\Big(V(x_{\rm cl})+V^{(1)}(x_{\rm cl})q+\frac{1}{2!}V^{(2)}(x_{\rm cl})q^2+\frac{1}{3!}V^{(3)}(x_{\rm cl})q^3+\cdots\Big)\,, \tag{B.32}$$

which term by term it corresponds to the Fig. B.1. Next if we expand the exponential $e^{S_{\rm int}}$, what we get are only polynomials in q to be integrated with the path integral weight, in other words we need to compute expressions like

$$\frac{\int_{q(0)=0}^{q(1)=0}\mathcal{D}q e^{i\frac{m}{2T}\int_0^1 du\dot{q}^2}q(u_1)q(u_2)\cdots q(u_n)}{\int_{q(0)=0}^{q(1)=1}\mathcal{D}q e^{i\frac{m}{2T}\int_0^1 du\dot{q}^2}}\equiv\Big\langle q(u_1)q(u_2)\cdots q(u_n)\Big\rangle\,, \tag{B.33}$$

and the full perturbative path integral can be written as

$$K(x,x';T)=N_f(T)e^{i\frac{m}{2T}(x-x')^2}\Big\langle e^{-iT\int_0^1 du V(x_{\rm cl}+q)}\Big\rangle\,, \tag{B.34}$$

where the expression $\Big\langle f(q)\Big\rangle$ are referred to as "correlation function". In order to compute the above correlations functions we define and compute the so-called "generating functional".
The N-point function of a scalar field theory (where T denotes the time-ordering operator)

$$\langle 0|T\phi(x_1)\cdots\phi(x_N)|0\rangle\,, \tag{B.35}$$

can be computed by the generating functional

$$\mathcal{Z}[j]=\langle 0|Te^{i\int d^Dx j(x)\phi(x)}|0\rangle \tag{B.36}$$

that is given by the path integral

$$\mathcal{Z}[j]=\int\mathcal{D}\phi\,e^{iS[\phi]+i\int d^Dx j(x)\phi(x)} \tag{B.37}$$

where the action $S[\phi]$ is the action for a relativistic scalar field. So the N-point function (B.35) is obtained by functional differentiation,i.e.

$$\langle 0|T\phi(x_1)\cdots\phi(x_N)|0\rangle=(-i)^N\frac{1}{\mathcal{Z}[j]}\frac{\delta^N\mathcal{Z}[j]}{\delta j(x_1)\delta j(x_2)\cdots\delta j(x_N)}\Big|_{j=0} \tag{B.38}$$

Thus, all we need to find is to compute $\mathcal{Z}[j]$. So we have the following generating functional

$$\mathcal{Z}[j] = \int_{q(0)=0}^{q(1)=0} \mathcal{D}q e^{i\frac{m}{2T}\int_0^1 du\dot{q}^2 + i\int_0^1 duqj} = N_f(T)\Big\langle e^{i\int_0^1 duqj}\Big\rangle\,, \tag{B.39}$$

in term of this generating functional the correlation function reads

$$\Big\langle q(u_1)q(u_2)\cdots q(u_n)\Big\rangle = (i)^n \frac{1}{\mathcal{Z}[0]}\frac{\delta^n \mathcal{Z}[j]}{\delta j(u_1)\delta j(u_2)\cdots\delta j(u_n)}\Big|_{j=0}\,. \tag{B.40}$$

By partially integrating the kinetic term and completing the square we get

$$\mathcal{Z}[j] = e^{\frac{i}{2}\int\int jD^{-1}j}\int_{q(0)=0}^{q(1)=0}\mathcal{D}q\, e^{i\frac{m}{2T}\int_0^1 du\dot{\tilde{q}}^2}\,, \tag{B.41}$$

where $D^{-1}(u,u')$, the "propagator ", is the inverse kinetic operator, with $D \equiv \frac{m}{T}\partial_u^2$, such that

$$D(u,u')D^{-1}(u,u') = \delta(u-u')\,, \tag{B.42}$$

in the basis of functions with Dirichlet boundary condition, and

$$\tilde{q}(u) \equiv q(u) - \int_0^1 j(u')D^{-1}(u,u')\,. \tag{B.43}$$

Now, by defining $D^{-1}(u,u') = \frac{T}{m}\Delta(u,u')$, from (B.42) we get

$$^{\bullet\bullet}\Delta(u,u') \;=\; \delta(u-u')\,, \tag{B.44}$$

where "dots" on the left (right) means derivatives with respect to u (u') and the propagator is defined as

$$\Delta(u,u') = \theta(u-u')(u-1)u' + \theta(u'-u)(u'-1)u\,, \tag{B.45}$$

which satisfies the following properties

$$\begin{aligned} \Delta(u,u') &= \Delta(u',u)\,, \\ \Delta(u,0) &= \Delta(u,1) = 0\,. \end{aligned} \tag{B.46}$$

from which $\tilde{q}(0) = \tilde{q}(1) = 0$. Therefore we can shift the integration variable in (B.41) from q to $\tilde{q}$ and get

$$\mathcal{Z}[j] = N_f(T)e^{i\frac{T}{2m}\int\int j\Delta j}\,, \tag{B.47}$$

and finally obtain

$$\Big\langle q(u_1)q(u_2)\cdots q(u_n)\Big\rangle = (i)^n \frac{\delta^n}{\delta j(u_1)\delta j(u_2)\cdots\delta j(u_n)} e^{i\frac{T}{2m}\int\int j\Delta j}\Big|_{j=0}\,. \tag{B.48}$$

In particular:

- correlation functions of an odd number of "fields" vanish.

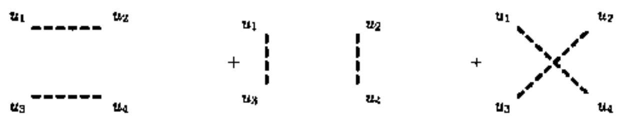


Figure B.2: Diagrammatic representation of four fields Wick contraction: $\langle q_1 q_2 q_3 q_4 \rangle$.

- the two-point function is nothing but the propagator $\langle q(u_1) q(u_2) \rangle = -\frac{T}{m} \Delta(u_1, u_2)$.
- correlation functions of an even number of fields are obtained by all possible contractions of pairs of fields. For example, for $n = 4$ we have $\langle q_1 q_2 q_3 q_4 \rangle = (-i\frac{T}{m})^2 (\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23})$ where an obvious shortcut notation has been used. The latter statement is known as the "Wick theorem", Fig. B.2.

Noting that each vertex and each propagator carry a power of T/m we can write the perturbative expansion as a short-time expansion (or inverse-mass expansion). It is thus not difficult to convince oneself that the expansion reorganizes as

$$K(x, x'; T) = N_f(T) e^{i \frac{m}{2T}(x-x')^2} \exp\Big(\text{connected diagrams}\Big) , \tag{B.49}$$

where the diagrammatic expansion in the exponent (that is ordered by increasing powers of T/m) only involves connected diagrams, i.e. diagrams whose vertices are connected by at least one propagator. We recall that $N_f(T) = \sqrt{\frac{m}{2\pi i T}}$ and we can also give yet another representation for the transition amplitude, the so-called "heat-kernel expansion"

$$K(x, x'; T) = \sqrt{\frac{m}{2\pi i T}} e^{i \frac{m}{2T}(x-x')^2} \sum_{n=0}^{\infty} a_n(x, x') T^n , \tag{B.50}$$

where the terms $a_n(x, x')$ are known as Seeley-DeWitt coefficients. We can thus express such coefficients in terms of Feynman diagrams and get

$$\begin{aligned} a_0(x, x') &= 1 \\ a_1(x, x') &= -i \int_0^1 du V(x_{\rm cl}(u)) , \\ a_2(x, x') &= \frac{1}{2!}\Big(-i \int_0^1 du V(x_{\rm cl}(u)) \Big)^2 - \frac{1}{2! m} \int_0^1 du V^{(2)}(x_{\rm cl}(u)) u(u-1) . \end{aligned} \tag{B.51}$$

Let us now comment on the validity of the expansion: each propagator inserts a power T/m, therefore for a fixed potential V, the larger the mass, the larger the time for which the expansion is accurate. In other words for a very massive particle it results quite costy to move away from the classical path [268].

Appendix C

Mathematica program for Wick contractions

Here we present our alternative program for the Wick contractions.

```
G[i_, j_]/;i > j:=G[j, i]
Gp[i_, j_]/;i > j:= - Gp[j, i]
Gpp[i_, j_]/;i > j:=Gpp[j, i]
kron[μ[i_], μ[j_]]/;i > j:=kron[μ[j], μ[i]]

perm1 = Permutations[{1, 2, 3, 4, 5, 6, 7, 8}];

permm = Sum[x[perm1[[n]][[1]]].x[perm1[[n]][[2]]].x[perm1[[n]][[3]]].x[perm1[[n]][[4]]].
x[perm1[[n]][[5]]].x[perm1[[n]][[6]]].x[perm1[[n]][[7]]].x[perm1[[n]][[8]]], {n, 1, 8!}];

contt = x[i_].x[j_] → -G[i, j]

x[i_].x[j_] → -G[i, j]

permm1 = permm/.contt;

permm2 = permm//.contt;

remdot = a__.b__ → a * b;

permm3 = permm2//.remdot;

putkron = G[i_, j_] → g[i, j] * kron[ν[i], ν[j]]

G[i_, j_] → g[i, j]kron[ν[i], ν[j]]
```

GtoGpp = G[i_, j_]kron[ν[i_], ν[j_]]/;$i > 4 \wedge j > 4 \to -$Gpp[i, j]kron[μ[i], μ[j]]

GtoGp = {G[i_, j_]kron[ν[i_], ν[j_]]/;$i > 4 \wedge j < 5 \to -$Gp[i, j]kron[ν[i], μ[j]],

G[i_, j_]kron[ν[i_], ν[j_]]/;$i < 5 \wedge j > 4 \to$ Gp[i, j]kron[ν[i], μ[j]]}

G[i_, j_]kron[ν[i_], ν[j_]]/;$i > 4$&&$j > 4 \to -$Gpp[i, j]kron[μ[i], μ[j]]

{G[i_, j_]kron[ν[i_], ν[j_]]/;$i > 4$&&$j < 5 \to -$Gp[i, j]kron[ν[i], μ[j]], G[i_, j_]kron[ν[i_], ν[j_]]/;$i < 5$&&$j > 4 \to$ Gp[i, j]kron[ν[i], μ[j]]}

permm6 = permm5//.GtoGpp//.GtoGp;

G[i_, i_] = 0

Gp[i_, i_] = 0

Gpp[i_, i_] = 0

0

0

0

permm7 = permm6/.5 → 1/.6 → 2/.7 → 3/.8 → 4/.kron[μ[i_], μ[j_]]->KroneckerDelta[μ[i], μ[j]]/.kron[ν[i_], μ[j_]]->KroneckerDelta[ν[i], μ[j]]/.kron[ν[i_], ν[j_]]->KroneckerDelta[ν[i], ν[j]];

Length[permm7]

60

AS = permm7 * F[μ[1], ν[1]] * F[μ[2], ν[2]] * F[μ[3], ν[3]] * F[μ[4], ν[4]]//Expand;

T = 1

1

G[i_, j_] = TAbs[u[i] − u[j]] − T(u[i] − u[j])^2

Abs[$u[i] - u[j]$] $- (u[i] - u[j])^2$

Gp[i_, j_] = Sign[u[i] − u[j]] − 2(u[i] − u[j])

$\mathrm{Sign}[u[i]-u[j]]-2(u[i]-u[j])$

Gpp[i_, j_] = (2/T)DiracDelta[u[i] − u[j]] − 2/T

$-2+2\mathrm{DiracDelta}[u[i]-u[j]]$

Integrate[AS, {u[1], 0, 1}, {u[2], 0, 1}, {u[3], 0, 1}, {u[4], 0, 1}];

AAA = $\frac{128}{15}F[\mu[1],\nu[1]]F[\mu[2],\nu[2]]F[\mu[3],\nu[3]]F[\mu[4],\nu[4]]$
(KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], ν[3]]KroneckerDelta[μ[3], ν[4]]
KroneckerDelta[μ[4], ν[1]] + KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], ν[4]]
KroneckerDelta[μ[3], ν[1]]KroneckerDelta[μ[4], ν[3]]+
5KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], ν[1]]KroneckerDelta[μ[3], ν[4]]
KroneckerDelta[μ[4], ν[3]] − KroneckerDelta[μ[1], μ[4]]KroneckerDelta[μ[2], ν[3]]
KroneckerDelta[μ[3], ν[4]]KroneckerDelta[ν[1], ν[2]]−
KroneckerDelta[μ[1], μ[3]]KroneckerDelta[μ[2], ν[4]]KroneckerDelta[μ[4], ν[3]]
KroneckerDelta[ν[1], ν[2]] − 5KroneckerDelta[μ[1], μ[2]]KroneckerDelta[μ[3], ν[4]]
KroneckerDelta[μ[4], ν[3]]KroneckerDelta[ν[1], ν[2]]−
KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], ν[4]]KroneckerDelta[μ[3], μ[4]]
KroneckerDelta[ν[1], ν[3]] − KroneckerDelta[μ[1], μ[4]]KroneckerDelta[μ[2], ν[4]]
KroneckerDelta[μ[3], ν[2]]KroneckerDelta[ν[1], ν[3]]+
KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], μ[4]]KroneckerDelta[μ[3], ν[4]]
KroneckerDelta[ν[1], ν[3]] − 5KroneckerDelta[μ[1], μ[3]]KroneckerDelta[μ[2], ν[4]]
KroneckerDelta[μ[4], ν[2]]KroneckerDelta[ν[1], ν[3]]−
KroneckerDelta[μ[1], μ[2]]KroneckerDelta[μ[3], ν[4]]KroneckerDelta[μ[4], ν[2]]
KroneckerDelta[ν[1], ν[3]] − KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], ν[3]]
KroneckerDelta[μ[3], μ[4]]KroneckerDelta[ν[1], ν[4]]−
5KroneckerDelta[μ[1], μ[4]]KroneckerDelta[μ[2], ν[3]]KroneckerDelta[μ[3], ν[2]]
KroneckerDelta[ν[1], ν[4]] − KroneckerDelta[μ[1], μ[3]]KroneckerDelta[μ[2], ν[3]]
KroneckerDelta[μ[4], ν[2]]KroneckerDelta[ν[1], ν[4]]+
KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], μ[3]]KroneckerDelta[μ[4], ν[3]]
KroneckerDelta[ν[1], ν[4]] − KroneckerDelta[μ[1], μ[2]]KroneckerDelta[μ[3], ν[2]]
KroneckerDelta[μ[4], ν[3]]KroneckerDelta[ν[1], ν[4]]+
KroneckerDelta[μ[1], μ[4]]KroneckerDelta[μ[2], ν[4]]KroneckerDelta[μ[3], ν[1]]

$$\begin{aligned}
&\mathbf{KroneckerDelta}[\nu[2],\nu[3]]+\mathbf{KroneckerDelta}[\mu[1],\mu[4]]\mathbf{KroneckerDelta}[\mu[2],\nu[1]]\\
&\mathbf{KroneckerDelta}[\mu[3],\nu[4]]\mathbf{KroneckerDelta}[\nu[2],\nu[3]]-\\
&\mathbf{KroneckerDelta}[\mu[1],\mu[3]]\mathbf{KroneckerDelta}[\mu[2],\nu[4]]\mathbf{KroneckerDelta}[\mu[4],\nu[1]]\\
&\mathbf{KroneckerDelta}[\nu[2],\nu[3]]-\mathbf{KroneckerDelta}[\mu[1],\mu[2]]\mathbf{KroneckerDelta}[\mu[3],\nu[4]]\\
&\mathbf{KroneckerDelta}[\mu[4],\nu[1]]\mathbf{KroneckerDelta}[\nu[2],\nu[3]]+\\
&\mathbf{5KroneckerDelta}[\mu[1],\mu[4]]\mathbf{KroneckerDelta}[\mu[2],\mu[3]]\mathbf{KroneckerDelta}[\nu[1],\nu[4]]\\
&\mathbf{KroneckerDelta}[\nu[2],\nu[3]]+\mathbf{KroneckerDelta}[\mu[1],\mu[3]]\mathbf{KroneckerDelta}[\mu[2],\mu[4]]\\
&\mathbf{KroneckerDelta}[\nu[1],\nu[4]]\mathbf{KroneckerDelta}[\nu[2],\nu[3]]+\\
&\mathbf{KroneckerDelta}[\mu[1],\mu[2]]\mathbf{KroneckerDelta}[\mu[3],\mu[4]]\mathbf{KroneckerDelta}[\nu[1],\nu[4]]\\
&\mathbf{KroneckerDelta}[\nu[2],\nu[3]]+\\
&\mathbf{KroneckerDelta}[\mu[1],\nu[4]]\\
&(-\mathbf{KroneckerDelta}[\mu[2],\mu[3]]\mathbf{KroneckerDelta}[\mu[4],\nu[3]]\mathbf{KroneckerDelta}[\nu[1],\nu[2]]+\\
&\mathbf{KroneckerDelta}[\mu[2],\nu[3]]\\
&(\mathbf{5KroneckerDelta}[\mu[3],\nu[2]]\mathbf{KroneckerDelta}[\mu[4],\nu[1]]+\\
&\mathbf{KroneckerDelta}[\mu[3],\nu[1]]\mathbf{KroneckerDelta}[\mu[4],\nu[2]]+\\
&\mathbf{KroneckerDelta}[\mu[3],\mu[4]]\mathbf{KroneckerDelta}[\nu[1],\nu[2]])+\\
&\mathbf{KroneckerDelta}[\mu[2],\mu[4]]\mathbf{KroneckerDelta}[\mu[3],\nu[2]]\mathbf{KroneckerDelta}[\nu[1],\nu[3]]-\\
&\mathbf{KroneckerDelta}[\mu[2],\mu[3]]\mathbf{KroneckerDelta}[\mu[4],\nu[2]]\mathbf{KroneckerDelta}[\nu[1],\nu[3]]-\\
&\mathbf{KroneckerDelta}[\mu[2],\mu[4]]\mathbf{KroneckerDelta}[\mu[3],\nu[1]]\mathbf{KroneckerDelta}[\nu[2],\nu[3]]-\\
&\mathbf{5KroneckerDelta}[\mu[2],\mu[3]]\mathbf{KroneckerDelta}[\mu[4],\nu[1]]\mathbf{KroneckerDelta}[\nu[2],\nu[3]]+\\
&\mathbf{KroneckerDelta}[\mu[2],\nu[1]]\\
&(\mathbf{KroneckerDelta}[\mu[3],\nu[2]]\mathbf{KroneckerDelta}[\mu[4],\nu[3]]-\\
&\mathbf{KroneckerDelta}[\mu[3],\mu[4]]\mathbf{KroneckerDelta}[\nu[2],\nu[3]]))-\\
&\mathbf{KroneckerDelta}[\mu[1],\mu[4]]\mathbf{KroneckerDelta}[\mu[2],\nu[3]]\mathbf{KroneckerDelta}[\mu[3],\nu[1]]\\
&\mathbf{KroneckerDelta}[\nu[2],\nu[4]]+\mathbf{KroneckerDelta}[\mu[1],\mu[3]]\mathbf{KroneckerDelta}[\mu[2],\nu[3]]\\
&\mathbf{KroneckerDelta}[\mu[4],\nu[1]]\mathbf{KroneckerDelta}[\nu[2],\nu[4]]+\\
&\mathbf{KroneckerDelta}[\mu[1],\mu[3]]\mathbf{KroneckerDelta}[\mu[2],\nu[1]]\mathbf{KroneckerDelta}[\mu[4],\nu[3]]\\
&\mathbf{KroneckerDelta}[\nu[2],\nu[4]]-\mathbf{KroneckerDelta}[\mu[1],\mu[2]]\mathbf{KroneckerDelta}[\mu[3],\nu[1]]\\
&\mathbf{KroneckerDelta}[\mu[4],\nu[3]]\mathbf{KroneckerDelta}[\nu[2],\nu[4]]+\\
&\mathbf{KroneckerDelta}[\mu[1],\mu[4]]\mathbf{KroneckerDelta}[\mu[2],\mu[3]]\mathbf{KroneckerDelta}[\nu[1],\nu[3]]\\
&\mathbf{KroneckerDelta}[\nu[2],\nu[4]]+\mathbf{5KroneckerDelta}[\mu[1],\mu[3]]\mathbf{KroneckerDelta}[\mu[2],\mu[4]]\\
&\mathbf{KroneckerDelta}[\nu[1],\nu[3]]\mathbf{KroneckerDelta}[\nu[2],\nu[4]]+
\end{aligned}$$

```
KroneckerDelta[μ[1], μ[2]]KroneckerDelta[μ[3], μ[4]]KroneckerDelta[ν[1], ν[3]]
KroneckerDelta[ν[2], ν[4]]+
KroneckerDelta[μ[1], ν[3]]
(−KroneckerDelta[μ[2], μ[4]]KroneckerDelta[μ[3], ν[4]]KroneckerDelta[ν[1], ν[2]]+
KroneckerDelta[μ[2], ν[4]]
(KroneckerDelta[μ[3], ν[2]]KroneckerDelta[μ[4], ν[1]]+
5KroneckerDelta[μ[3], ν[1]]KroneckerDelta[μ[4], ν[2]]+
KroneckerDelta[μ[3], μ[4]]KroneckerDelta[ν[1], ν[2]])−
KroneckerDelta[μ[2], μ[4]]KroneckerDelta[μ[3], ν[2]]KroneckerDelta[ν[1], ν[4]]+
KroneckerDelta[μ[2], μ[3]]KroneckerDelta[μ[4], ν[2]]KroneckerDelta[ν[1], ν[4]]−
5KroneckerDelta[μ[2], μ[4]]KroneckerDelta[μ[3], ν[1]]KroneckerDelta[ν[2], ν[4]]−
KroneckerDelta[μ[2], μ[3]]KroneckerDelta[μ[4], ν[1]]KroneckerDelta[ν[2], ν[4]]+
KroneckerDelta[μ[2], ν[1]]
(KroneckerDelta[μ[3], ν[4]]KroneckerDelta[μ[4], ν[2]]−
KroneckerDelta[μ[3], μ[4]]KroneckerDelta[ν[2], ν[4]]))−
5KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], ν[1]]KroneckerDelta[μ[3], μ[4]]
KroneckerDelta[ν[3], ν[4]] − KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], μ[4]]
KroneckerDelta[μ[3], ν[1]]KroneckerDelta[ν[3], ν[4]]−
KroneckerDelta[μ[1], μ[4]]KroneckerDelta[μ[2], ν[1]]KroneckerDelta[μ[3], ν[2]]
KroneckerDelta[ν[3], ν[4]] − KroneckerDelta[μ[1], ν[2]]KroneckerDelta[μ[2], μ[3]]
KroneckerDelta[μ[4], ν[1]]KroneckerDelta[ν[3], ν[4]]+
KroneckerDelta[μ[1], μ[2]]KroneckerDelta[μ[3], ν[2]]KroneckerDelta[μ[4], ν[1]]
KroneckerDelta[ν[3], ν[4]] − KroneckerDelta[μ[1], μ[3]]KroneckerDelta[μ[2], ν[1]]
KroneckerDelta[μ[4], ν[2]]KroneckerDelta[ν[3], ν[4]]+
KroneckerDelta[μ[1], μ[2]]KroneckerDelta[μ[3], ν[1]]KroneckerDelta[μ[4], ν[2]]
KroneckerDelta[ν[3], ν[4]] + KroneckerDelta[μ[1], μ[4]]KroneckerDelta[μ[2], μ[3]]
KroneckerDelta[ν[1], ν[2]]KroneckerDelta[ν[3], ν[4]]+
KroneckerDelta[μ[1], μ[3]]KroneckerDelta[μ[2], μ[4]]KroneckerDelta[ν[1], ν[2]]
KroneckerDelta[ν[3], ν[4]] + 5KroneckerDelta[μ[1], μ[2]]KroneckerDelta[μ[3], μ[4]]
KroneckerDelta[ν[1], ν[2]]KroneckerDelta[ν[3], ν[4]]);

AAA1 = AAA//Expand;
```

```
dott =
{F[μ[i_], ν[i_]]F[μ[j_], ν[j_]]F[μ[k_], ν[k_]]F[μ[l_], ν[l_]]KroneckerDelta[μ[i_], ν[l_]]
KroneckerDelta[μ[j_], ν[k_]]KroneckerDelta[μ[k_], ν[j_]]KroneckerDelta[μ[l_], ν[i_]] → (tr F^2)^2
F[μ[i_], ν[i_]]F[μ[j_], ν[j_]]F[μ[k_], ν[k_]]F[μ[l_], ν[l_]]KroneckerDelta[μ[i_], ν[k_]]
KroneckerDelta[μ[j_], ν[l_]]KroneckerDelta[μ[k_], ν[j_]]KroneckerDelta[μ[l_], ν[i_]] → (tr F^4)
F[μ[i_], ν[i_]]F[μ[j_], ν[j_]]F[μ[k_], ν[k_]]F[μ[l_], ν[l_]]KroneckerDelta[μ[i_], ν[l_]]
KroneckerDelta[μ[j_], ν[k_]]KroneckerDelta[μ[k_], μ[l_]]KroneckerDelta[ν[i_], ν[j_]] → (tr F^4)
F[μ[i_], ν[i_]]F[μ[j_], ν[j_]]F[μ[k_], ν[k_]]F[μ[l_], ν[l_]]KroneckerDelta[μ[i_], ν[k_]]
KroneckerDelta[μ[j_], μ[l_]]KroneckerDelta[μ[k_], ν[l_]]KroneckerDelta[ν[i_], ν[j_]] → −(tr F^4)
F[μ[i_], ν[i_]]F[μ[j_], ν[j_]]F[μ[k_], ν[k_]]F[μ[l_], ν[l_]]KroneckerDelta[μ[i_], μ[j_]]
KroneckerDelta[μ[k_], ν[l_]]KroneckerDelta[μ[l_], ν[k_]]KroneckerDelta[ν[i_], ν[j_]] → −(tr F^2)^2
F[μ[i_], ν[i_]]F[μ[j_], ν[j_]]F[μ[k_], ν[k_]]F[μ[l_], ν[l_]]KroneckerDelta[μ[i_], μ[l_]]
KroneckerDelta[μ[j_], μ[k_]]KroneckerDelta[ν[i_], ν[l_]]KroneckerDelta[ν[j_], ν[k_]] → (tr F^2)^2
F[μ[i_], ν[i_]]F[μ[j_], ν[j_]]F[μ[k_], ν[k_]]F[μ[l_], ν[l_]]KroneckerDelta[μ[i_], μ[k_]]
KroneckerDelta[μ[j_], μ[l_]]KroneckerDelta[ν[i_], ν[l_]]KroneckerDelta[ν[j_], ν[k_]] → (tr F^4)

AAA2 = AAA1//.dott
```

$512\,(\mathrm{tr\ F}^2)^2 + \frac{2048(\mathrm{tr\ F}^4)}{5}$

$\left(512\,(\mathrm{tr\ F}^2)^2 + \frac{2048(\mathrm{tr\ F}^4)}{5}\right) \Big/ (384\text{^}2)//\mathrm{Expand}$

$\frac{(\mathrm{tr\ F}^2)^2}{288} + \frac{(\mathrm{tr\ F}^4)}{360}$

This expression is what one needs to get the right coefficients, one of 384's comes from $(384 = 4! \times 16)$ in (2.67) and the other one is the symmetry factor coming from our permutations.

Appendix D

Mathematica program for calculating the leading terms c_2 and c_4 in section 2.4

(calculation of the coefficients c_{scal, spin})

(definitions)

```
M = {{0,0,0,1},{0,0,-1,0},{0,1,0,0},{-1,0,0,0}}
```

$\{\{0,0,0,1\},\{0,0,-1,0\},\{0,1,0,0\},\{-1,0,0,0\}\}$

```
M.M//MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
X = Array[x,4]
```

$\{x[1],x[2],x[3],x[4]\}$

```
A = Exp[-a * X.X]/(X.X + 1) * M.X;

F = Table[D[A[[j]],x[i]] - D[A[[i]],x[j]],{i,4},{j,4}]//FullSimplify;

dF = Table[D[F[[i,j]],x[l]],{i,1,4},{j,1,4},{l,1,4}]//FullSimplify;
```

```
ddF = Table[D[dF[[i, j, k]], x[l]], {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}]//Simplify;

rule1 = x[1]^2 + x[2]^2 + x[3]^2 + x[4]^2 -> Dn - 1
rule2 = x[4]^2 -> Dn - 1 - x[2]^2 - x[3]^2 - x[1]^2
```

$x[1]^2 + x[2]^2 + x[3]^2 + x[4]^2 \to -1 + \mathrm{Dn}$

$x[4]^2 \to -1 + \mathrm{Dn} - x[1]^2 - x[2]^2 - x[3]^2$

```
F = F/.rule1/.rule2//FullSimplify;
dF = dF/.rule1/.rule2//FullSimplify;
ddF = ddF/.rule1/.rule2//Simplify;

( calculating the four invariants )

Inv1 = Sum[dF[[i, j, k]]^2, {i, 1, 4}, {j, 1, 4}, {k, 1, 4}]//Expand;
Inv1 = %/.x[4] -> Sqrt[Dn - 1 - x[1]^2 - x[2]^2 - x[3]^2]//Expand;
Inv1 = %//Simplify;
Inv1 = %//FullSimplify;
Inv1 = Inv1//Expand;

rule3 = Dn -> 1 + r^2
```

$\mathrm{Dn} \to 1 + r^2$

```
Inv1 = Inv1/.rule3//Expand;

res1 = 2Pi^2Integrate[r^3 * Inv1, {r, 0, Infinity}, Assumptions -> a > 0];

Inv2 = Sum[ddF[[i, j, k, l]]^2, {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}]//Expand;
Inv2 = %/.x[4] -> Sqrt[Dn - 1 - x[1]^2 - x[2]^2 - x[3]^2]//Expand;
Inv2 = %//Simplify;
Inv2 = %//FullSimplify;
Inv2 = Inv2/.rule3//Expand;

res2 = 2Pi^2Integrate[r^3 * Inv2, {r, 0, Infinity}, Assumptions -> a > 0];

Inv3 = Tr[F.F.F.F]//Expand;
Inv3 = %/.x[4] -> Sqrt[Dn - 1 - x[1]^2 - x[2]^2 - x[3]^2]//Expand;
```

Inv3 = %//Simplify;

Inv3 = Inv3/.rule3//Expand;

res3 = 2Pi^2Integrate[r^3 * Inv3, {r, 0, Infinity}, Assumptions → a > 0];

Inv4 = Tr[F.F]^2//Expand;

Inv4 = %/.x[4] → Sqrt[Dn − 1 − x[1]^2 − x[2]^2 − x[3]^2]//Expand;

Inv4 = %//Simplify

$$\frac{64\left(1-2a(-1+\mathrm{Dn})\mathrm{Dn}+\mathrm{Dn}^2+a^2(-1+\mathrm{Dn})^2\mathrm{Dn}^2\right)^2 e^{-4a(-1+\mathrm{Dn})}}{\mathrm{Dn}^8}$$

Inv4 = Inv4/.rule3//Expand;

res4 = 2Pi^2Integrate[r^3 * Inv4, {r, 0, Infinity}, Assumptions → a > 0];

(plugging into the equations from section 2.4)

scal2 = −1/1920/Pi^2 * Normal[Series[res1, {a, 0, 3}]]//Expand//FullSimplify//Expand

$$-\frac{1}{75}+\frac{23a}{1200}-\frac{31a^2}{600}-\frac{19a^3}{600}-\frac{2a^3\mathrm{EulerGamma}}{15}-\frac{1}{15}a^3\mathrm{Log}[4]-\frac{2}{15}a^3\mathrm{Log}[a]$$

spin2 = 8scal2

$$8\left(-\frac{1}{75}+\frac{23a}{1200}-\frac{31a^2}{600}-\frac{19a^3}{600}-\frac{2a^3\mathrm{EulerGamma}}{15}-\frac{1}{15}a^3\mathrm{Log}[4]-\frac{2}{15}a^3\mathrm{Log}[a]\right)$$

scal4 = Normal[Series[1/26880/Pi^2 * res2 − 1/16/Pi^2(res3/360 + res4/288), {a, 0, 3}]]//Expand

$$-\frac{107}{105840}+\frac{2941a}{66150}+\frac{32663a^2}{4233600}-\frac{60601a^3}{79380}+\frac{11a^2\mathrm{EulerGamma}}{60}+\frac{203a^3\mathrm{EulerGamma}}{270}+\frac{11}{60}a^2\mathrm{Log}[4]+\frac{203}{270}a^3\mathrm{Log}[4]+\frac{11}{60}a^2\mathrm{Log}[a]+\frac{203}{270}a^3\mathrm{Log}[a]$$

spin4 = Normal[Series[1/2240/Pi^2 * res2 − 1/16/Pi^2(7res3/90 − res4/36), {a, 0, 3}]]//Expand

$$\frac{683}{13230}+\frac{3149a}{33075}+\frac{8819a^2}{264600}+\frac{7946a^3}{19845}+\frac{2a^2\mathrm{EulerGamma}}{15}+\frac{232a^3\mathrm{EulerGamma}}{135}+\frac{2}{15}a^2\mathrm{Log}[4]+\frac{232}{135}a^3\mathrm{Log}[4]+\frac{2}{15}a^2\mathrm{Log}[a]+\frac{232}{135}a^3\mathrm{Log}[a]$$

(calculating the coefficients for $\alpha = 1/120$ and $\alpha = 1/100$)

c2sc = scal2/.a → 1/120//N

c4sc = scal4/.a → 1/120//N

c2sp = spin2/.a → 1/100//N

c4sp = spin4/.a → 1/100//N

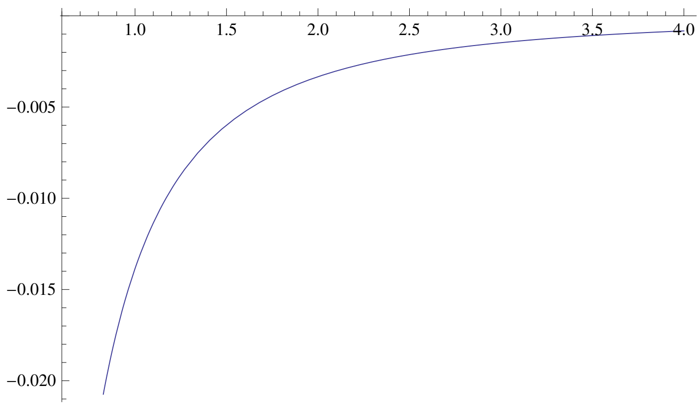
-0.0131769

-0.000677552

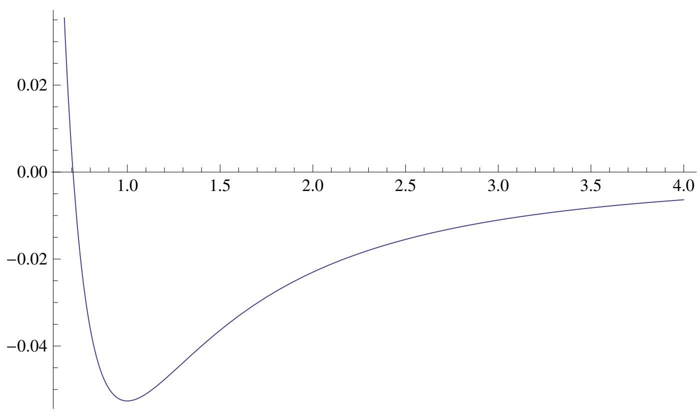
-0.105171

0.0525411

Plot[c2sc/m^2 + c4sc/m^4, {m, 0.66, 4}]



Plot[c2sp/m^2 + c4sp/m^4, {m, 0.66, 4}]



(verifying that the expansion to cubic order works fine for such small α)

scal2exact = −1/1920/Pi^2 * res1/.a → 1/120//N

-0.0131769

calculating the values for $m = 4$ (the end point of our graphs)

−0.01317694601628/x^2 − 0.000677551799286/x^4/.x → 4

-0.000826206

$-0.1051713628721701/x\wedge 2 + 0.0525411445726166/x\wedge 4/.x \to 4$

-0.00636797

Appendix E

Two point function for $O(2) \times O(3)$ background

E.1 Two point amplitude for scalar loop case

Here we will find the two-point function,

$$\Gamma^{(2)}_{\text{scal}} = e^2 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^T d\tau_1 d\tau_2 \int \mathcal{D}x \prod_{i=1}^{2} \dot{x}_i \cdot A^\alpha(x_i) \exp\Big[- \int_0^T d\tau \frac{\dot{x}^2}{4} \Big] . \tag{E.1}$$

Fourier transform of $A^\alpha(x_i)$ gives

$$A^\alpha_\mu(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x} \bar{A}^\alpha_\mu(k) , \tag{E.2}$$

where

$$\bar{A}^\alpha_\mu(k) = -i M_{\mu\nu} k^\nu \bar{b}(k^2) , \tag{E.3}$$

where $\bar{b}(k^2)$ will be found in the following.
By substituting (E.2) in (E.1) one gets

$$\Gamma^{(2)}_{\text{scal}} = \prod_{i=1}^{2} \int \frac{d^D k_i}{(2\pi)^D} \bar{b}(k_i^2) (2\pi)^D \delta(k_1 + k_2) \Gamma^{(2)}_{\text{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] , \tag{E.4}$$

where

$$\begin{aligned} \Gamma^{(2)}_{\text{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] &= -e^2 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^T d\tau_1 d\tau_2 \int Dy \prod_{i=1}^{2} \varepsilon_i \cdot \dot{y}_i \, e^{ik_i\cdot y_i} e^{-\int_0^T d\tau \frac{\dot{y}^2}{4}} , \\ \varepsilon_{i\mu} &:= M_{\mu\nu} k_i^\nu \quad , \quad i = 1, 2 . \end{aligned} \tag{E.5}$$

By performing all possible Wick contractions, one gets

$$\Gamma^{(2)}_{\text{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = e^2 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^T d\tau_1 d\tau_2 (4\pi T)^{-\frac{D}{2}} P_2 e^{G_{B12} k_1 \cdot k_2} , \tag{E.6}$$

where P_2 is a polynomial in $\dot{G}_B$ and $\ddot{G}_B$,

$$P_2 = \dot{G}_{B12}\varepsilon_1 \cdot k_2 \, \dot{G}_{B21}\varepsilon_2 \cdot k_1 - \ddot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \,. \tag{E.7}$$

After doing IBP one can replace P_2 by Q_2 which reads as

$$Q_2 = \dot{G}_{B12}\dot{G}_{B21}(\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_1 - \varepsilon_1 \cdot \varepsilon_2 k_1 \cdot k_2) = \frac{1}{2}\dot{G}_{B12}\dot{G}_{B21}\mathrm{tr}(f_1 f_2) = \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\,, \tag{E.8}$$

where

$$Z_2(12) = \frac{1}{2}\mathrm{tr}(f_1 f_2)\,,$$

(E.6) can be written as

$$\Gamma^{(2)}_{\mathrm{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = e^2 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^T d\tau_1 d\tau_2 (4\pi T)^{-\frac{D}{2}} Q_2 e^{G_{B12} k_1 \cdot k_2} \,. \tag{E.9}$$

Now if we rescale the τ variables as $\tau_i = T u_i$ and perform the T-integral

$$\Gamma^{(2)}_{\mathrm{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = \frac{e^2}{(4\pi)^{\frac{D}{2}}}\Gamma(2 - \frac{D}{2}) \int_0^1 du_1 du_2 \frac{Q_2}{[m^2 - G_{B12} k_1 \cdot k_2]^{2-\frac{D}{2}}} \,. \tag{E.10}$$

Now we put $D = 4 - 2\epsilon$.

If we do the epsilon-expansion,

$$\Gamma(2 - \frac{D}{2}) = \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma\,,$$

$$\frac{1}{(4\pi)^{\frac{D}{2}}} = \frac{1}{(4\pi)^{2-\epsilon}} = \frac{1}{(4\pi)^2}[1 + \epsilon \mathrm{Log}(4\pi)]\,,$$

then we change the coupling constant as

$$e \to e_4 \mu^\epsilon\,,$$

where e is the coupling constant in D-dimension which is dimensionful and e_4 is the four-dimensional coupling constant which is dimensionless. μ is an arbitrary parameter which has a mass dimension. Now if we put $e_4 \to 1$ as our convention we have the following replacement

$$e \to \mu^\epsilon\,.$$

Now we have

$$\Gamma^{(2)}_{\mathrm{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = \mu^{2\varepsilon} \times \left\{ \frac{1}{(4\pi)^2}(1 + \epsilon \mathrm{Log}\, 4\pi)\,(\frac{1}{\epsilon} - \gamma) \int_0^1 du \frac{Q_2|_{(u_1=u, u_2=0)}}{(m^2 - G_{B12} k_1 \cdot k_2)^\epsilon} \right\}, \tag{E.11}$$

where

$$Q_2|_{(u_1=u,u_2=0)} = -\dot{G}^2_{B12} Z_2(12) = (4G_{B12} - 1)\, Z_2(12) \qquad , \qquad G_{B12} = u(1-u)\,.$$

Now, if we expand the denominator

$$\frac{1}{(m^2 - G_{B12} k_1 \cdot k_2)^\epsilon} = 1 - \epsilon \mathrm{Log}(m^2 - G_{B12} k_1 \cdot k_2) = 1 - \epsilon \mathrm{Log} m^2 - \epsilon \mathrm{Log}(1 - G_{B12} k_1 \cdot k_2 / m^2)\,.$$

and

$$\mu^{2\epsilon} = 1 + \epsilon \mathrm{Log}(\mu^2)\,.$$

By multiplying all these terms, one gets

$$\begin{aligned}&(1+\epsilon\mathrm{Log}\mu^2)\,(1+\epsilon\mathrm{Log}4\pi)\,(\frac{1}{\epsilon}-\gamma)\Big[1-\epsilon\mathrm{Log}m^2-\epsilon\mathrm{Log}\Delta\Big]\\&=\frac{1}{\epsilon}-\gamma-\mathrm{Log}m^2-\mathrm{Log}\Delta+\mathrm{Log}4\pi+\mathrm{Log}\mu^2\\&=\frac{1}{\epsilon}-\gamma+\mathrm{Log}4\pi-\mathrm{Log}\frac{m^2}{\mu^2}-\mathrm{Log}\Delta\,,\end{aligned}\tag{E.12}$$

where

$$\Delta=1-G_{B12}k_1\cdot k_2/m^2=1+G_{B12}k^2/m^2\quad,\quad k_1=-k_2=k\,.$$

Then

$$\begin{aligned}&\int_0^1du\,Q_2|_{(u1=u,u_2=0)}\Big\{\frac{1}{\epsilon}-\gamma+\mathrm{Log}4\pi-\mathrm{Log}\frac{m^2}{\mu^2}-\mathrm{Log}\Delta\Big\}\\&=-\frac{Z_2(12)}{3}\Big\{\frac{1}{\epsilon}-\gamma+\mathrm{Log}4\pi-\mathrm{Log}\frac{m^2}{\mu^2}\Big\}-Z_2(12)\int_0^1du\,(4G_{B12}-1)\mathrm{Log}[1+G_{B12}k^2/m^2]\,.\end{aligned}\tag{E.13}$$

By doing the u-integral

$$\begin{aligned}\int_0^1du\,(4G_{B12}-1)\mathrm{Log}[1+G_{B12}k^2/m^2]&=\frac{8(k^3+3km^2)-3(k^2+4m^2)^{3/2}\mathrm{ArcTanh}[\frac{k\sqrt{k^2+4m^2}}{k^2+2m^2}]}{9k^3}\\&=\frac{8}{9}(1+\frac{3m^2}{k^2})-\frac{1}{3}(1+\frac{4m^2}{k^2})^{\frac{3}{2}}\,\mathrm{ArcTanh}\left[\frac{\sqrt{1+\frac{4m^2}{k^2}}}{1+\frac{2m^2}{k^2}}\right].\end{aligned}\tag{E.14}$$

So from (E.11)

$$\begin{aligned}\Gamma^{(2)}_{\mathrm{scal}}[k_1,\varepsilon_1;k_2,\varepsilon_2]&=-\frac{Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big\{\frac{1}{\epsilon}-\gamma+\mathrm{Log}4\pi-\mathrm{Log}\frac{m^2}{\mu^2}\Big\}\\&\quad-\frac{Z_2(12)|_{(k_1=-k_2=k)}}{(4\pi)^2}\left\{\frac{8}{9}(1+\frac{3m^2}{k^2})-\frac{1}{3}(1+\frac{4m^2}{k^2})^{\frac{3}{2}}\,\mathrm{ArcTanh}\Big[\frac{\sqrt{1+\frac{4m^2}{k^2}}}{1+\frac{2m^2}{k^2}}\Big]\right\}.\end{aligned}\tag{E.15}$$

In the following we use the $\overline{MS}$ renormalization scheme. The effective action for two-point function for scalar loop is written as

$$\Gamma^{(2)}_{\mathrm{scal,ren}}[m,\mu]=\int\frac{d^4k}{(2\pi)^{2D}}\bar{b}^2(k^2)(2\pi)^D\Gamma^{(2)}_{\mathrm{scal,ren}}[m,\mu]\,,\tag{E.16}$$

where

$$\Gamma^{(2)}_{\mathrm{scal,ren}}[m,\mu]=\Gamma^{(2)}_{\mathrm{scal}}[m,\mu]+\frac{Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big\{\frac{1}{\epsilon}-\gamma+\mathrm{Log}4\pi\Big\}\,.\tag{E.17}$$

By our definition it is dimensionally renormalized effective action for scalar loop.

As a final result for scalar-loop case for two point function, we write

$$\begin{aligned}
\Gamma^{(2)}_{\text{scal,ren}}[m,\mu] &= \frac{1}{3\pi^2(4\pi)^4}\text{Log}(\frac{m^2}{\mu^2})\int d^4k\bar{b}^2(k^2)Z_2(12)|_{(k_1=-k_2=k)} \\
&- \frac{1}{\pi^2(4\pi)^4}\int d^4k\bar{b}^2(k^2)\Big\{\frac{8}{9}(1+\frac{3m^2}{k^2})-\frac{1}{3}(1+\frac{4m^2}{k^2})^{\frac{3}{2}}\,\text{ArcTanh}\Big[\frac{\sqrt{1+\frac{4m^2}{k^2}}}{1+\frac{2m^2}{k^2}}\Big]\Big\} \\
&\times \;\; Z_2(12)|_{(k_1=-k_2=k)}\,.
\end{aligned} \tag{E.18}$$

E.2 Spinor loop

Now in this Section we study the spinor case, which we apply the Bern-Kosower replacement rule to the scalar case two-point function,

$$\dot{G}_{B12}\dot{G}_{B21}\to\dot{G}_{B12}\dot{G}_{B21}-G_{F12}G_{F21}\,,$$

and from statistics we have a -2 pre-factor for (E.1)

$$\Gamma^{(2)}=\prod_{i=1}^{2}\int\frac{d^Dk_i}{(2\pi)^D}\bar{b}(k_i^2)(2\pi)^D\delta(k_1+k_2)\Gamma^{(2)}[k_1,\varepsilon_1;k_2,\varepsilon_2]\,, \tag{E.19}$$

where

$$\Gamma^{(2)}[k_1,\varepsilon_1;k_2,\varepsilon_2]=-2\mu^{2\varepsilon}\times\left\{\frac{1}{(4\pi)^2}(1+\epsilon\text{Log}\,4\pi)\,(\frac{1}{\epsilon}-\gamma)\int_0^1du\frac{(Q_2+Z_2(12))|_{(u_1=u,u_2=0)}}{(m^2-G_{B12}k_1\cdot k_2)^\epsilon}\right\}. \tag{E.20}$$

Here our renormalization scheme is the same as scalar case, so let us just show the final result, then we need to find

$$Z_2(12)\int_0^1du4G_{B12}\Big\{1-\epsilon\text{Log}[1+G_{B12}\frac{k^2}{m^2}]\Big\}\,, \tag{E.21}$$

where

$$\begin{aligned}
&\int_0^1du4u(1-u)\Big\{1-\epsilon\text{Log}[1+u(1-u)\frac{k^2}{m^2}]\Big\}= \\
&\frac{2}{3}-\epsilon\Big\{-\frac{2}{9}(5-12\frac{m^2}{k^2})+\frac{4(1+2\frac{m^2}{k^2}-8\frac{m^4}{k^4})}{3\sqrt{1+\frac{4m^2}{k^2}}}\text{ArcTanh}\Big[\frac{1}{\sqrt{1+4\frac{m^2}{k^2}}}\Big]\Big\}\,,
\end{aligned} \tag{E.22}$$

so, spinor case reads

$$\begin{aligned}
\Gamma^{(2)}[m,\mu,k] &= -\frac{2Z_2(12)|_{(k_1=-k_2=k)}}{(4\pi)^2}(1-\epsilon\text{Log}\frac{m^2}{\mu^2})(\frac{1}{\epsilon}-\gamma+\text{Log}4\pi) \\
&\times \;\; \Big\{\frac{2}{3}-\epsilon\Big\{-\frac{2}{9}(5-12\frac{m^2}{k^2})+\frac{4(1+\frac{2m^2}{k^2}-\frac{8m^4}{k^4})}{3\sqrt{1+\frac{4m^2}{k^2}}}\text{ArcTanh}\Big[\frac{1}{\sqrt{1+4\frac{m^2}{k^2}}}\Big]\Big\}\Big\}\,,
\end{aligned} \tag{E.23}$$

then

$$\Gamma^{(2)}[m,\mu,k] = - \frac{4Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big\{\frac{1}{\epsilon}-\gamma+\mathrm{Log}4\pi-\mathrm{Log}\frac{m^2}{\mu^2}\Big\}$$
$$- \frac{2Z_2(12)|_{(k_1=-k_2=k)}}{(4\pi)^2}\Bigg\{\frac{10}{9}-\frac{8}{3}\frac{m^2}{k^2}-\frac{4}{3}\frac{(1+\frac{2m^2}{k^2}-\frac{8m^4}{k^4})}{\sqrt{1+\frac{4m^2}{k^2}}}\mathrm{ArcTanh}\Big[\frac{1}{\sqrt{1+4\frac{m^2}{k^2}}}\Big]\Bigg\}. \tag{E.24}$$

Now we can define our renormalized effective action as scalar case

$$\begin{aligned}\Gamma^{(2)}_{\mathrm{ren}}[m,\mu,k] &= \Gamma^{(2)}[m,\mu,k]+\frac{4Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big\{\frac{1}{\epsilon}-\gamma+\mathrm{Log}4\pi\Big\}\\ &= \frac{4Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\mathrm{Log}[\frac{m^2}{\mu^2}]\\ &- \frac{2Z_2(12)|_{(k_1=-k_2=k)}}{(4\pi)^2}\Bigg\{\frac{10}{9}-\frac{8}{3}\frac{m^2}{k^2}-\frac{4}{3}\frac{(1+\frac{2m^2}{k^2}-\frac{8m^4}{k^4})}{\sqrt{1+\frac{4m^2}{k^2}}}\mathrm{ArcTanh}\Big[\frac{1}{\sqrt{1+4\frac{m^2}{k^2}}}\Big]\Bigg\}.\end{aligned} \tag{E.25}$$

The effective action can be written as

$$\Gamma^{(2)}_{\mathrm{ren}}[m,\mu] = \int\frac{d^4k}{(2\pi)^{2D}}\bar{b}^2(k^2)(2\pi)^D\Gamma^{(2)}_{\mathrm{ren}}[m,\mu,k]\,. \tag{E.26}$$

So our final result for spinor loop from (E.26) would be

$$\begin{aligned}\Gamma^{(2)}_{\mathrm{ren}}[m,\mu] = &+ \frac{4}{3\pi^2(4\pi)^4}\mathrm{Log}(\frac{m^2}{\mu^2})\int d^4k\bar{b}^2(k^2)Z_2(12)|_{(k_1=-k_2=k)}\\ &- \frac{2}{\pi^2(4\pi)^4}\int d^4k\bar{b}^2(k^2)Z_2(12)|_{(k_1=-k_2=k)}\Bigg\{\frac{10}{9}-\frac{8}{3}\frac{m^2}{k^2}-\frac{4}{3}\frac{(1+\frac{2m^2}{k^2}-\frac{8m^4}{k^4})}{\sqrt{1+\frac{4m^2}{k^2}}}\\ &\times\ \mathrm{ArcTanh}\Big[\frac{1}{\sqrt{1+4\frac{m^2}{k^2}}}\Big]\Bigg\}.\end{aligned} \tag{E.27}$$

E.3 Fourier transformation of the vector field

Now at this stage we find $\bar{A}^\alpha_\mu(k)$,

$$\bar{A}^\alpha_\mu(k) = \int d^4x e^{-ik\cdot x}A^\alpha_\mu(x)\,, \tag{E.28}$$

where

$$A^\alpha_\mu(x) = \frac{M_{\mu\nu}x^\nu}{x^2+\rho^2}e^{-\alpha x^2}\,.$$

By plugging $A^\alpha_\mu(x)$ to (E.28) we get

$$\bar{A}^\alpha_\mu(k) = \int d^4x e^{-ik\cdot x}\frac{M_{\mu\nu}x^\nu}{x^2+\rho^2}e^{-\alpha x^2}\,. \tag{E.29}$$

If we do the following replacements

$$\begin{aligned} x^\nu &= i\frac{\partial}{\partial k_\nu}e^{-ik\cdot x}\,, \\ \frac{1}{x^2+\rho^2} &= \int_0^\infty d\beta e^{-\beta(x^2+\rho^2)}\,, \end{aligned} \tag{E.30}$$

then

$$\bar{A}^\alpha_\mu(k) = i\frac{\partial}{\partial k_\nu}M_{\mu\nu}\int_0^\infty d\beta e^{-\beta\rho^2}\int d^4x e^{-ik\cdot x - x^2(\alpha+\beta)}\,. \tag{E.31}$$

The x-integral is Gaussian, then

$$\int d^4x e^{-ik\cdot x-(\alpha+\beta)x^2} = \frac{\pi^2}{(\alpha+\beta)^2}\,e^{-\frac{k^2}{4(\alpha+\beta)}}\,.$$

Fom (E.31) we have

$$\bar{A}^\alpha_\mu(k) = i\pi^2 M_{\mu\nu}\frac{\partial}{\partial k_\nu}\int_0^\infty d\beta\frac{1}{(\alpha+\beta)^2}\,e^{-\beta\rho^2-\frac{k^2}{4(\alpha+\beta)}}\,. \tag{E.32}$$

Now our issue is the β-integral. We can change the β variable to a new one as

$$\beta+\alpha = t \to d\beta = dt\,. \tag{E.33}$$

Then (E.31) can be written as

$$\bar{A}^\alpha_\mu(k) = ie^{\alpha\rho^2}\pi^2 M_{\mu\nu}\frac{\partial}{\partial k_\nu}\int_\alpha^\infty dt\frac{1}{t^2}\,e^{-t\rho^2-\frac{k^2}{4t}}\,. \tag{E.34}$$

The form of this integral called *generalized incomplete gamma function* which by definition is

$$\Gamma(a,x;b) = \int_x^\infty dz\, z^{a-1}e^{-z-bz^{-1}}\,. \tag{E.35}$$

By this definition we arrive to

$$\int_\alpha^\infty dt\frac{1}{t^2}\,e^{-t\rho^2-\frac{k^2}{4t}} \equiv \rho^2\int_{\alpha\rho^2}^\infty dy\frac{1}{y^2}\,e^{-y-\frac{k^2\rho^2}{4y}} = \rho^2\Gamma(-1,\alpha\rho^2;\frac{k^2\rho^2}{4})\,, \tag{E.36}$$

where $y = t\rho^2$.

The Fourier transformed of our vector field is

$$\bar{A}^\alpha_\mu(k) = i\pi^2\rho^2 e^{\alpha\rho^2}M_{\mu\nu}\frac{\partial}{\partial k_\nu}\Gamma(-1,\alpha\rho^2;\frac{k^2\rho^2}{4})\,. \tag{E.37}$$

The derivative of the *generalized incomplete gamma function* respect to b-variable is known to be

$$\frac{\partial}{\partial b}\Gamma(a,x;b) = -\Gamma(a-1,x;b)\,, \tag{E.38}$$

then

$$\begin{aligned} \frac{\partial}{\partial k_\nu}\Gamma(-1,\alpha\rho^2;\frac{k^2\rho^2}{4}) &= -2\frac{\rho^2}{4}k^\nu\Gamma(-2,\alpha\rho^2;\frac{k^2\rho^2}{4}) \\ &= -\frac{\rho^2}{2}k^\nu\Gamma(-2,\alpha\rho^2;\frac{k^2\rho^2}{4})\,. \end{aligned} \tag{E.39}$$

So from (E.37) we have

$$\bar{A}^{\alpha}_{\mu}(k) = -\frac{i\pi^2\rho^4 e^{\alpha\rho^2}}{2} M_{\mu\nu}k^{\nu}\Gamma(-2,\alpha\rho^2;\frac{k^2\rho^2}{4})\,. \tag{E.40}$$

At the beginning we have defined

$$\bar{A}^{\alpha}_{\mu}(k) = -iM_{\mu\nu}k^{\nu}\bar{b}(k^2)\,. \tag{E.41}$$

Then $\bar{b}(k^2)$ is just

$$\bar{b}(k^2) = \frac{\pi^2\rho^4 e^{\alpha\rho^2}}{2}\Gamma(-2,\alpha\rho^2;\frac{k^2\rho^2}{4})\,, \tag{E.42}$$

in term of *generalized incomplete gamma function*. Now if set $\rho = 1$ we get what we were looking for as

$$\bar{b}(k^2) = \frac{\pi^2 e^{\alpha}}{2}\Gamma(-2,\alpha;\frac{k^2}{4})\,. \tag{E.43}$$

E.4 Conclusion

Here comes our final results for both cases (scalar and spinor loop), just by substituting (E.43) into (E.18) and (E.27).

$$Z_2(12)|_{(k_1=-k_2=k)} = \frac{1}{2}\mathrm{tr}(f_1 f_2)|_{(k_1=-k_2=k)} = +k^2(\varepsilon_1\cdot\varepsilon_2) - (\varepsilon_1\cdot k)(\varepsilon_2\cdot k) \qquad , \tag{E.44}$$

and since $\epsilon_{\mu i} := M_{\mu\nu}k^{\nu}_i$ where M is an antisymmetric matrix, i.e. $M^2 = -1$, $M^T = -M$ we have

$$\begin{aligned} \varepsilon_1\cdot\varepsilon_2 &= M^T\cdot Mk^2 = -M^2k^2 = k^2\,, \\ \varepsilon_1\cdot k\,\varepsilon_2\cdot k &= 0\,, \end{aligned} \tag{E.45}$$

so

$$Z_2(12)|_{(k_1=-k_2=k)} = -k^4\,. \tag{E.46}$$

Our final results can be presented as

$$\begin{aligned} \Gamma^2_{\mathrm{scal,ren}}[m,\mu] &= -\frac{e^{2\alpha}}{3072\pi^2}\mathrm{Log}(\frac{m^2}{\mu^2})\int d^4k\,k^4\Gamma^2(-2,\alpha;\frac{k^2}{4}) \\ &\quad +\frac{e^{2\alpha}}{1024\pi^2}\int d^4k\,k^4\Gamma^2(-2,\alpha;\frac{k^2}{4}) \\ &\quad \times\left\{\frac{8}{9}(1+\frac{3m^2}{k^2}) - \frac{1}{3}(1+\frac{4m^2}{k^2})^{\frac{3}{2}}\,\mathrm{ArcTanh}\Big[\frac{\sqrt{1+\frac{4m^2}{k^2}}}{1+\frac{2m^2}{k^2}}\Big]\right\}, \end{aligned} \tag{E.47}$$

and

$$\begin{aligned} \Gamma^{(2)}_{\mathrm{ren}}[m,\mu] = &- \frac{e^{2\alpha}}{768\pi^2}\mathrm{Log}(\frac{m^2}{\mu^2})\int d^4k\,k^4\Gamma^2(-2,\alpha;\frac{k^2}{4}) \\ &+ \frac{e^{2\alpha}}{256\pi^2}\int d^4k\,k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\left\{\frac{5}{9} - \frac{4}{3}\frac{m^2}{k^2} - \frac{2}{3}\frac{(1+\frac{2m^2}{k^2}-\frac{8m^4}{k^4})}{\sqrt{1+\frac{4m^2}{k^2}}}\mathrm{ArcTanh}\Big[\frac{1}{\sqrt{1+4\frac{m^2}{k^2}}}\Big]\right\}. \end{aligned} \tag{E.48}$$

E.5 Two-point function in massless limit

E.5.1 Scalar loop

Here we consider massless limit of our calculations, so just by considering (E.10) and putting $m = 0$ one gets

$$\Gamma^{(2)}_{\text{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = \frac{e^2}{(4\pi)^{\frac{D}{2}}}\Gamma(2 - \frac{D}{2})\int_0^1 du_1 du_2 \frac{Q_2}{[-G_{B12}k_1 \cdot k_2]^{2-\frac{D}{2}}}\,, \tag{E.49}$$

where

$$Q_2 = \dot{G}_{B12}\dot{G}_{B21}Z_2(12) = -\dot{G}^2_{B12}Z_2(12) = (4G_{B12} - 1)Z_2(12)\,, \tag{E.50}$$

then

$$\Gamma^{(2)}_{\text{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = \frac{\mu^{2\epsilon}}{(4\pi)^{\frac{D}{2}}}\Gamma(2 - \frac{D}{2})\int_0^1 du \frac{Q_2|_{(u_1=u, u_2=0)}}{(G_{B12}k^2)^{\epsilon}}\,, \tag{E.51}$$

where

$$\begin{aligned} \int_0^1 du \frac{4G_{B12} - 1}{G^{\epsilon}_{B12k^2}} &= \int_0^1 du(4G_{B12} - 1)\Big\{1 - \epsilon\text{Log}(G_{B12}k^2)\Big\} \\ &= -\frac{1}{3} - \epsilon(\frac{8}{9} - \frac{1}{3}\text{Log}k^2)\,. \end{aligned} \tag{E.52}$$

From (E.51) we have

$$\begin{aligned} \Gamma^{(2)}_{\text{scal}}[\mu, k] = &- \frac{Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big[\frac{1}{\epsilon} - \gamma + \text{Log}\mu^2 + \text{Log}4\pi\Big] \\ &- \frac{Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}(\frac{8}{3} - \text{Log}k^2)\,. \end{aligned} \tag{E.53}$$

We define our renormalized $\Gamma^{(2)}$ as

$$\begin{aligned} \Gamma^{(2)}_{\text{scal,ren}}[\mu, k] &= \Gamma_{\text{scal}}[\mu, k] + \frac{Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}(\frac{1}{\epsilon} - \gamma + \text{Log}4\pi) \\ &= -\frac{Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big\{\text{Log}(\frac{\mu^2}{k^2}) + \frac{8}{3}\Big\}\,. \end{aligned} \tag{E.54}$$

The effective action for scalar loop

$$\Gamma^{(2)}_{\text{scal,ren}}[\mu] = \prod_{i=1}^{2}\int \frac{d^D k_i}{(2\pi)^D}\bar{b}^2(k^2)(2\pi)^D\delta(k_1 + k_2)\Gamma^{(2)}_{\text{scal,ren}}[\mu, k]\,, \tag{E.55}$$

by inserting $\bar{b}(k^2)$ we have

$$\Gamma^{(2)}_{\text{scal,ren}}[\mu] = -\frac{e^{2\alpha}}{3072\pi^2}\int d^4k\Gamma^2(-2, \alpha; \frac{k^2}{4})\Big\{\text{Log}(\frac{\mu^2}{k^2}) + \frac{8}{3}\Big\}Z_2(12)|_{(k_1=-k_2=k)}\,, \tag{E.56}$$

where

$$Z_2(12)|_{(k_1=-k_2=k)} = -k^4\,,$$

so

$$\Gamma^{(2)}_{\text{scal,ren}}[\mu] = \frac{e^{2\alpha}}{3072\pi^2}\int d^4k\, k^4\Gamma^2(-2, \alpha; \frac{k^2}{4})\Big\{\text{Log}(\frac{\mu^2}{k^2}) + \frac{8}{3}\Big\}\,. \tag{E.57}$$

E.6 Spinor case

We recall from above, our formula for spinor loop, we just need to put $m=0$ and $k_1=-k_2=k$

$$\Gamma^{(2)}[k_1,\varepsilon_1;k_2,\varepsilon_2] = -2\mu^{2\varepsilon}\Bigg\{\frac{1}{(4\pi)^2}(1+\epsilon\text{Log}\,4\pi)\,(\frac{1}{\epsilon}-\gamma)\int_0^1 du\frac{(Q_2+Z_2(12))|_{(u_1=u,u_2=0)}}{(G_{B12}k^2)^{\epsilon}}\Bigg\}, \quad \text{(E.58)}$$

since

$$Q_2|(k_1=-k_2=k)+Z_2(12) = 4G_{B12}\,Z_2(12)\,, \quad \text{(E.59)}$$

then we do the u-integral

$$\begin{aligned}\int_0^1 du\,\frac{4G_{B12}}{G^{\epsilon}_{B12}} &= \int_0^1 4G_{B12}\Big\{1-\epsilon\text{Log}[G_{B12}k^2]\Big\}\\ &= \frac{2}{3}-\epsilon\Big\{-\frac{10}{9}+\frac{2}{3}\text{Log}k^2\Big\}\,. \end{aligned} \quad \text{(E.60)}$$

One gets

$$\begin{aligned}\Gamma^{(2)}[\mu,k] = & -\frac{2Z_2(12)|_{(k_1=-k_2=k)}}{(4\pi)^2}(1+\epsilon\text{Log}4\pi)(1+\epsilon\text{Log}\mu^2)(\frac{1}{\epsilon}-\gamma)\\ & \times\Big\{\frac{2}{3}-\epsilon(-\frac{10}{9}+\frac{2}{3}\text{Log}k^2)\Big\}\\ = & -\frac{4Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big[\frac{1}{\epsilon}-\gamma+\text{Log}4\pi+\text{Log}\frac{\mu^2}{k^2}+\frac{5}{3}\Big]\,,\end{aligned} \quad \text{(E.61)}$$

again we define the renormalized one as

$$\begin{aligned}\Gamma^{(2)}_{\text{ren}}[\mu,k] &= \Gamma^{(2)}[\mu,k]+\frac{4Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big[\frac{1}{\epsilon}-\gamma+\text{Log}4\pi\Big]\\ &= -\frac{4Z_2(12)|_{(k_1=-k_2=k)}}{3(4\pi)^2}\Big[\text{Log}(\frac{\mu^2}{k^2})+\frac{5}{3}\Big]\,.\end{aligned} \quad \text{(E.62)}$$

Our final effective action by inserting the $\bar{b}(k^2)$ for spinor loop is

$$\Gamma^{(2)}_{\text{ren}}[\mu] = \prod_{i=1}^{2}\int\frac{d^Dk_i}{(2\pi)^D}\bar{b}^2(k^2)(2\pi)^D\delta(k_1+k_2)\Gamma^{(2)}_{\text{ren}}[\mu,k]\,, \quad \text{(E.63)}$$

by plugging $\bar{b}(k^2)$ and $\Gamma_{\text{ren}}[\mu,k]$ we have the effective action for spinor loop as

$$\Gamma^{(2)}_{\text{ren}}[\mu] = -\frac{e^{2\alpha}}{768\pi^2}\int d^4k\Gamma^2(-2,\alpha;\frac{k^2}{4})\Big[\text{Log}(\frac{\mu^2}{k^2})+\frac{5}{3}\Big]\,Z_2(12)|_{(k_1=-k_2=k)}\,, \quad \text{(E.64)}$$

where

$$Z_2(12)|_{(k_1=-k_2=k)} = -k^4\,.$$

And finally we have

$$\Gamma^{(2)}_{\text{ren}}[\mu] = \frac{e^{2\alpha}}{768\pi^2}\int d^4k\,k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Big[\text{Log}(\frac{\mu^2}{k^2})+\frac{5}{3}\Big]\,. \quad \text{(E.65)}$$

Let us to conclude our final effective actions for scalar and spinor loop in massless limit as

$$\begin{aligned}\Gamma^{(2)}_{\rm scal,ren}[\mu] &= \frac{e^{2\alpha}}{3072\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Big[\mathrm{Log}(\frac{\mu^2}{k^2})+\frac{8}{3}\Big]\,,\\ \Gamma^{(2)}_{\rm ren}[\mu] &= \frac{e^{2\alpha}}{768\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Big[\mathrm{Log}(\frac{\mu^2}{k^2})+\frac{5}{3}\Big]\,.\end{aligned}\tag{E.66}$$

To compare our results we need to check the massless limit of our massive results, then we need the series expansion of our functions,

$$\begin{aligned}&\text{series}\quad \Bigg\{\Big\{\frac{8}{9}(1+\frac{3m^2}{k^2})-\frac{1}{3}(1+\frac{4m^2}{k^2})^{\frac{3}{2}}\,\mathrm{ArcTanh}\Big[\frac{\sqrt{1+\frac{4m^2}{k^2}}}{1+\frac{2m^2}{k^2}}\Big]\Big\},\{m,0,1\}\Bigg\}\\ &= \frac{1}{18}(16-3\mathrm{Log}[2]+3\mathrm{Log}[\frac{2}{k^4}]+12\mathrm{Log}[m])+O[m]^2=\frac{8}{9}+\frac{1}{3}\mathrm{Log}(\frac{m^2}{k^2})=\frac{1}{3}(\frac{8}{3}+\mathrm{Log}\frac{m^2}{k^2})\,.\end{aligned}\tag{E.67}$$

By substituting (E.67) into (E.47) we have

$$\begin{aligned}\Gamma^{(2)}_{\rm scal,ren}[m,\mu] = &- \frac{e^{2\alpha}}{3072\pi^2}\mathrm{Log}(\frac{m^2}{\mu^2})\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\\ &+ \frac{e^{2\alpha}}{1024\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Bigg\{\frac{1}{3}(\frac{8}{3}+\mathrm{Log}\frac{m^2}{k^2})\Bigg\}\\ = &- \frac{e^{2\alpha}}{3072\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Bigg\{-\frac{8}{3}-\mathrm{Log}\mu^2+\mathrm{Log}m^2-\mathrm{Log}m^2+\mathrm{Log}k^2\Bigg\}\end{aligned}\tag{E.68}$$

So

$$\Gamma^{(2)}_{\rm scal,ren}[\mu] = \frac{e^{2\alpha}}{3072\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Bigg\{\frac{8}{3}+\mathrm{Log}\frac{\mu^2}{k^2}\Bigg\}\,,\tag{E.69}$$

which is exactly our final result for the massless scalar loop in (F.30)

And for spinor case one gets

$$\begin{aligned}&\text{series}\quad \Bigg\{\Big\{\frac{5}{9}-\frac{4}{3}\frac{m^2}{k^2}-\frac{2}{3}\frac{(1+\frac{2m^2}{k^2}-\frac{8m^4}{k^4})}{\sqrt{1+\frac{4m^2}{k^2}}}\mathrm{ArcTanh}\Big[\frac{1}{\sqrt{1+4\frac{m^2}{k^2}}}\Big]\Big\},\{m,0,1\}\Bigg\}\\ &= \frac{5}{9}-\frac{1}{3}\mathrm{Log}[2]+\frac{1}{3}\mathrm{Log}[\frac{2}{k^2}]+\frac{1}{3}\mathrm{Log}[m^2]=\frac{5}{9}+\frac{1}{3}\mathrm{Log}[\frac{m^2}{k^2}]=\frac{1}{3}(\frac{5}{3}+\mathrm{Log}\frac{m^2}{k^2})\,.\end{aligned}\tag{E.70}$$

By substituting (E.67) into (E.48) we have

$$\begin{aligned}\Gamma^{(2)}_{\rm ren}[m,\mu] = &- \frac{e^{2\alpha}}{768\pi^2}\mathrm{Log}(\frac{m^2}{\mu^2})\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\\ &+ \frac{e^{2\alpha}}{256\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Bigg\{\frac{1}{3}(\frac{5}{3}+\mathrm{Log}\frac{m^2}{k^2})\Bigg\}\\ = &+ \frac{e^{2\alpha}}{768\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Bigg\{-\mathrm{Log}(\frac{m^2}{\mu^2})+\frac{5}{3}+\mathrm{Log}(\frac{m^2}{k^2})\Bigg\}\,.\end{aligned}\tag{E.71}$$

And finally

$$\Gamma^2_{\rm ren}[\mu] = \frac{e^{2\alpha}}{768\pi^2}\int d^4k\, k^4\Gamma^2(-2,\alpha;\frac{k^2}{4})\Big[\mathrm{Log}(\frac{\mu^2}{k^2})+\frac{5}{3}\Big]\,, \tag{E.72}$$

which is exactly our final result for the massless limit of spinor loop in (F.30).

Appendix F

Calculation of the three-gluon vertex using Feynman rules

F.1 Three-gluon vertex calculation using Feynman rules

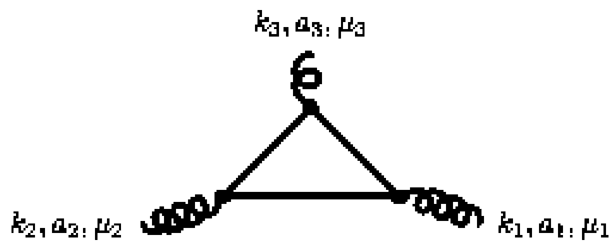


Figure F.1: Three-gluon vertex for scalar loop particle.

We recalculate the spinor loop contribution to the three-gluon-vertex using Feynman diagrams, we follow the conventions of Srednicki [270]. It is given by Fig. F.1 plus the one with the opposite orientation of the spinor line. Thus we write

$$V = \Gamma_A + \Gamma_B \tag{F.1}$$

This diagram can be written as (we call them diagram A and B in the following)

$$\Gamma_A^{a_1 a_2 a_3} = -(ig)^3(-i)^3 \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}) \int \frac{d^D q}{(2\pi)^D} \frac{\mathrm{tr}[\not{\varepsilon}_3(m - \not{q} - \not{k}_1 - \not{k}_2)\not{\varepsilon}_2(m - \not{q} - \not{k}_1)\not{\varepsilon}_1(m - \not{q})]}{[m^2 + (q + k_1 + k_2)^2][m^2 + (k_1 + q)^2][m^2 + q^2]}\,, \tag{F.2}$$

and diagram B differs from this only by an interchange of $T^{a_2} \leftrightarrow T^{a_3}$. Thus we can use

$$\mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}]) = iC(r)f^{a_1 a_2 a_3}\,, \tag{F.3}$$

to write

$$V^{a_1a_2a_3} = \mathcal{N} I\,, \tag{F.4}$$

where

$$\mathcal{N} = -if^{a_1a_2a_3} 4C(r)\frac{g^3}{(4\pi)^{\frac{D}{2}}}\,, \tag{F.5}$$

and

$$I = \frac{(4\pi)^{\frac{D}{2}}}{4}\int \frac{d^Dq}{(2\pi)^D}\frac{\mathrm{tr}[\not{\varepsilon}_3(m-\not{q}-\not{k}_1-\not{k}_2)\not{\varepsilon}_2(m-\not{q}-\not{k}_1)\not{\varepsilon}_1(m-\not{q})]}{[m^2+(q+k_1+k_2)^2][m^2+(k_1+q)^2][m^2+q^2]}\,. \tag{F.6}$$

Our strategy will be the following:

1. Using the fact that all the invariants $A-H$ involve at least one $epsepsij$, and the permutation invariance, we can restrict ourselves to compute the terms with an $epseps12$.
2. Since we want only a sign check, we can, after doing the loop momentum integral, perform a low momentum (= large mass) expansion, and take only the first nonvanishing terms for each invariant.

Doing the trace in the numerator and keeping only those terms which either contain a $\varepsilon_1 \cdot \varepsilon_2$, or could develop one through the q integration, we get (note that $\{\gamma^\mu, \gamma^\nu\} = -2eta^{\mu\nu}$ in Srednichki's conventions)

$$\begin{aligned} N \quad equiv \quad & \frac{1}{4}\,\mathrm{tr}\big[\not{\varepsilon}_3(m-\not{q}-\not{k}_1-\not{k}_2)\not{\varepsilon}_2(m-\not{q}-\not{k}_1)\not{\varepsilon}_1(m-\not{q})\big] \\ = \quad & -m^2\varepsilon_1\cdot\varepsilon_2\,\varepsilon_3\cdot(q+k_2) + \varepsilon_1\cdot q\,\varepsilon_2\cdot q\,\varepsilon_3\cdot(4q+2k_1+2k_2) \\ & +\varepsilon_1\cdot\varepsilon_2\Big[-\varepsilon_3\cdot q\,k_1\cdot(q+k_1+k_2) + \varepsilon_3\cdot k_1\,q\cdot(q+k_1+k_2) - \varepsilon_3\cdot(q+k_1+k_2)q\cdot(q+k_1)\Big]\,. \end{aligned} \tag{F.7}$$

Next, we exponentiate the denominators:

$$\frac{1}{\big[m^2+(q+k_1+k_2)^2\big]\big[m^2+(k_1+q)^2\big]\big[m^2+q^2\big]} = \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 e^{-\alpha_1\left[m^2+(k_1+k_2+q)^2\right]-\alpha_2\left[m^2+(q+k_1)^2\right]-\alpha_3\left[m^2+q^2\right]}\,. \tag{F.8}$$

We introduce the global propertime $T=\sum\alpha_i$, using

$$\begin{aligned} \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \quad &= \quad \int_0^\infty dT\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\delta\left(T-\sum_{i=1}^3\alpha_i\right) \\ &= \quad \int_0^\infty \frac{dT}{T}\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\delta\left(1-\sum_{i=1}^3\frac{\alpha_i}{T}\right) \\ &= \quad \int_0^\infty dTT^2\int_0^1 da_1 da_2 da_3\delta\left(1-\sum_{i=1}^3 a_i\right)\,, \end{aligned} \tag{F.9}$$

where now $a_i = \alpha_i/T$. Thus we have now

$$I = (4\pi)^{\frac{D}{2}} \int_0^\infty dT T^2 \, e^{-m^2 T} \int_0^1 da_1 da_2 da_3 \delta\left(1 - \sum_{i=1}^{3} a_i\right) \int \frac{d^D q}{(2\pi)^D} N \, e^{-T(q^2+2b\cdot q+c)} \,, \tag{F.10}$$

where

$$\begin{aligned} b &= (a_1+a_2)k_1 + a_1 k_2 \,, \\ c &= a_1(k_1+k_2)^2 + a_2 k_1^2 \,. \end{aligned} \tag{F.11}$$

We now do the gaussian q integrals, using

$$\begin{aligned}
\int \frac{d^D q}{(2\pi)^D} \, e^{-T[q^2+2b\cdot q]} &= \frac{e^{Tb^2}}{(4\pi T)^{\frac{D}{2}}} \,, \\
\int \frac{d^D q}{(2\pi)^D} q^\mu \, e^{-T[q^2+2b\cdot q]} &= -b^\mu \frac{e^{Tb^2}}{(4\pi T)^{\frac{D}{2}}} \,, \\
\int \frac{d^D q}{(2\pi)^D} q^\mu q^\nu \, e^{-T[q^2+2b\cdot q]} &= (b^\mu b^\nu + \frac{1}{2T} eta^{\mu\nu}) \frac{e^{Tb^2}}{(4\pi T)^{\frac{D}{2}}} \,, \\
\int \frac{d^D q}{(2\pi)^D} q^\mu q^\nu q^\lambda \, e^{-T[q^2+2b\cdot q]} &= -\Big(b^\mu b^\nu b^\lambda + \frac{1}{2T}(eta^{\mu\nu} b^\lambda + eta^{\mu\lambda} b^\nu + eta^{\nu\lambda} b^\mu)\Big) \frac{e^{Tb^2}}{(4\pi T)^{\frac{D}{2}}} \,.
\end{aligned} \tag{F.12}$$

This gives (keeping again only terms containing $\varepsilon_1 \cdot \varepsilon_2$)

$$\begin{aligned}
I &= \varepsilon_1 \cdot \varepsilon_2 \int_0^\infty dT T^{2-\frac{D}{2}} \, e^{-m^2 T} \int_0^1 da_1 da_2 da_3 \delta\left(1 - \sum_{i=1}^{3} a_i\right) \, e^{-(c-b^2)T} \\
&\quad \times \Big\{ \varepsilon_3 \cdot b \, k_1 \cdot (k_1+k_2-b) + \varepsilon_3 \cdot k_1 \, b \cdot (b-k_1-k_2) + \varepsilon_3 \cdot (k_1+k_2-b) b \cdot (k_1 - b) \\
&\qquad + \Big(m^2 + \frac{\frac{D}{2}-1}{T}\Big) \varepsilon_3 \cdot (b - k_2) \Big\} \,,
\end{aligned} \tag{F.13}$$

where

$$c - b^2 = a_2 a_3 k_1^2 + a_1 a_2 k_2^2 + a_1 a_3 k_3^2 = a_1(a_1-1) k_2 \cdot k_3 + a_2(a_2-1) k_1 \cdot k_2 + a_3(a_3-1) k_1 \cdot k_3 \,. \tag{F.14}$$

The later form seems preferable, since our invariants do not involve k_i^2's.
We wish to compare with (5.11), (5.12) in [132]. In the formulas there we always take the lowest terms in the low energy expansion. Those are given by replacing each of the integrals in (5.2) in [132] by its

value at $k_1^2=0$, except for $I^D_{\rm bt,B}$ and $I^D_{\rm bt,F}$, where we need to include also the second terms in their Taylor expansion. The integrals then are trivial, and give

$$\begin{aligned}
I^D_{3,B}(0,0,0) &= -\frac{1}{15}\frac{1}{m^{6-D}}\,,\\
I^D_{3,F}(0,0,0) &= -\frac{1}{2}\frac{1}{m^{6-D}}\,,\\
I^D_{2,B}(0,0,0) &= \frac{1}{30}\frac{1}{m^{6-D}}\,,\\
I^D_{2,F}(0,0,0) &= \frac{1}{6}\frac{1}{m^{6-D}}\,,\\
I^D_{\rm bt,B}(k^2) &= \frac{1}{3}\frac{1}{m^{4-D}}-\frac{1}{30}(2-\frac{D}{2})k^2\frac{1}{m^{6-D}}\,,\\
I^D_{\rm bt,F}(k^2) &= \frac{1}{m^{4-D}}-\frac{1}{6}(2-\frac{D}{2})k^2\frac{1}{m^{6-D}}\,,
\end{aligned}\tag{F.15}$$

Using these results, we can then write the low energy approximation to (5.11) of [132] in the form

$$\Gamma^{a_1a_2a_3}_{\frac12}=\mathcal{N}(\gamma^3_{\frac12}+\gamma^2_{\frac12}+\gamma^{\rm bt}_{\frac12})\,,\tag{F.16}$$

where

$$\begin{aligned}
\gamma^3_{\frac12} &= -\frac{13}{60}\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}{\rm tr}(f_1f_2f_3)\\
\gamma^2_{\frac12} &= -\frac{1}{30}\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}\Big[{\rm tr}(f_1f_2)\varepsilon_3\cdot(k_1-k_2)+{\rm tr}(f_2f_3)\varepsilon_3\cdot(k_2-k_3)\\
&+ \qquad {\rm tr}(f_3f_1)\varepsilon_2\cdot(k_3-k_1)\Big]\\
\gamma^{\rm bt}_{\frac12} &= -\Gamma\left(2-\frac{D}{2}\right)\Big[\varepsilon_3\cdot f_1\cdot\varepsilon_2(-\frac{1}{3}\frac{1}{m^{4-D}}+\frac{1}{15}k_1^2(2-\frac{D}{2})\frac{1}{m^{6-D}})+2\,{\rm perm.}\Big]\,.
\end{aligned}\tag{F.17}$$

Keeping only terms involving $\varepsilon_1\cdot\varepsilon_2$ and by using energy-momentum conservation to put $k_1^2=-k_1\cdot(k_2+k_3)$ and $k_2^2=-k_2\cdot(k_1+k_3)$, this reduces (using also $\Gamma(2-\frac{D}{2})(2-\frac{D}{2})=\Gamma(3-\frac{D}{2})$) to

$$\begin{aligned}
\gamma^3_{\frac12} &= +\frac{13}{60}\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}\varepsilon_1\cdot\varepsilon_2(\varepsilon_3\cdot k_1\,k_2\cdot k_3-\varepsilon_3\cdot k_2\,k_1\cdot k_3)\,,\\
\gamma^2_{\frac12} &= +\frac{1}{15}\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}\varepsilon_3\cdot\varepsilon_2\,k_1\cdot k_2\,\varepsilon_3\cdot(k_1-k_2)\,,\\
\gamma^{\rm bt}_{\frac12} &= +\frac{1}{3}\Gamma\left(2-\frac{D}{2}\right)\frac{1}{m^{4-D}}\,\varepsilon_1\cdot\varepsilon_2\,\varepsilon_3\cdot(k_1-k_2)\\
&\quad+\frac{1}{15}\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}\varepsilon_1\cdot\varepsilon_2\,k_1\cdot k_2\,\varepsilon_3\cdot(k_1-k_2)\\
&\quad+\frac{1}{15}\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}\varepsilon_1\cdot\varepsilon_2\big[\varepsilon_3\cdot k_1\,k_1\cdot k_3-\varepsilon_3\cdot k_2\,k_2\cdot k_3\big]\,.
\end{aligned}\tag{F.18}$$

We return to the Feynman diagram calculation, splitting I in (F.13) into four parts, $I=I_1+I_2+I_3+I_4$, where

$$I_1 = (\varepsilon_1 \cdot \varepsilon_2) \int_0^\infty dT\, T^{2-\frac{D}{2}} e^{-m^2 T} \int_0^1 da_1\, da_2\, da_3 \delta \left(1 - \sum_{i=1}^{3} a_i\right) \left\{ \varepsilon_3 \cdot b\, k_1 \cdot (k_1 + k_2 - b) \right\} e^{-T(c-b^2)}\,, \tag{F.19}$$

Let us do the a integrals. For $I_{1,2,3}$ in the low energy limit we can replace $e^{-T(c-b^2)}$ by 1 and find (using energy-momentum conservation)

$$\begin{aligned}&\int_0^1 da_1\, da_2\, da_3 \delta \left(1 - \sum_{i=1}^{3} a_i\right) \Big[\varepsilon_3 \cdot b\, k_1 \cdot (k_1 + k_2 - b) \Big] = \\ &\varepsilon_3 \cdot k_1 (\frac{1}{8} k_1 \cdot k_2 - \frac{1}{12} k_1 \cdot k_3) + \varepsilon_3 \cdot k_2 (\frac{1}{24} k_1 \cdot k_2 - \frac{1}{24} k_1 \cdot k_3)\,.\end{aligned} \tag{F.20}$$

The second integral is

$$I_2 = (\varepsilon_1 \cdot \varepsilon_2) \int_0^\infty dT\, T^{2-\frac{D}{2}} e^{-m^2 T} \int_0^1 da_1\, da_2\, da_3 \delta \left(1 - \sum_{i=1}^{3} a_i\right) \left\{ \varepsilon_3 \cdot k_1\, b \cdot (b - k_1 - k_2) \right\} e^{-T(c-b^2)}\,, \tag{F.21}$$

since

$$\int_0^1 da_1\, da_2\, da_3 \delta \left(1 - \sum_{i=1}^{3} a_i\right) \left\{ \varepsilon_3 \cdot k_1\, b \cdot (b - k_1 - k_2) \right\} = -\frac{1}{12} \varepsilon_3 \cdot k_1 (k_1 \cdot k_2 - k_1 \cdot k_3 - k_2 \cdot k_3)\,. \tag{F.22}$$

The third integral

$$I_3 = (\varepsilon_1 \cdot \varepsilon_2) \int_0^\infty dT\, T^{2-\frac{D}{2}} e^{-m^2 T} \int_0^1 da_1\, da_2\, da_3 \delta \left(1 - \sum_{i=1}^{3} a_i\right) \left\{ \varepsilon_3 \cdot (k_1 + k_2 - b) b \cdot (k_1 - b) \right\} e^{-T(c-b^2)}\,, \tag{F.23}$$

since

$$\begin{aligned}&\int_0^1 da_1\, da_2\, da_3 \delta \left(1 - \sum_{i=1}^{3} a_i\right) \left\{ \varepsilon_3 \cdot (k_1 + k_2 - b) b \cdot (k_1 - b) \right\} = \\ &-\frac{1}{120} \Big[\varepsilon_3 \cdot k_1 (3 k_1 \cdot k_2 + 4 k_1 \cdot k_3 - 2 k_2 \cdot k_3) + \varepsilon_3 \cdot k_2 (7 k_1 \cdot k_2 + 7 k_1 \cdot k_3 - 4 k_2 \cdot k_3) \Big]\,.\end{aligned} \tag{F.24}$$

By defining A as a sum of (F.20), (F.22) and (F.24) we have

$$\begin{aligned} A &= \frac{1}{60} \Big[\varepsilon_3 \cdot k_1 \,(k_1 \cdot k_2 - 2 k_1 \cdot k_3 + 6 k_2 \cdot k_3) - \varepsilon_3 \cdot k_2 \,(k_1 \cdot k_2 + 6 k_1 \cdot k_3 - 2 k_2 \cdot k_3) \Big] \\ &= \frac{1}{60} k_1 \cdot k_2\, \varepsilon_3 \cdot (k_1 - k_2) - \frac{1}{30} (\varepsilon_3 \cdot k_1\, k_1 \cdot k_3 - \varepsilon_3 \cdot k_2\, k_2 \cdot k_3) + \frac{1}{10} (\varepsilon_3 \cdot k_1\, k_2 \cdot k_3 - \varepsilon_3 \cdot k_2\, k_1 \cdot k_3)\,. \end{aligned} \tag{F.25}$$

Now, by performing the T-integral for the first three terms of (F.13) one gets

$$I_1 + I_2 + I_3 = \varepsilon_1 \cdot \varepsilon_2 \Gamma \left(3 - \frac{D}{2}\right) \frac{1}{m^{6-D}}\, A\,. \tag{F.26}$$

The last integral is

$$
\begin{aligned}
I_4 &= (\varepsilon_1\cdot\varepsilon_2)\int_0^\infty dT\,T^{2-\frac{D}{2}}e^{-m^2T}\int_0^1 da_1\,da_2\,da_3\delta\left(1-\sum_{i=1}^3 a_i\right)\\
&\times\ e^{-T(c-b^2)}\Big(m^2+\frac{\frac{D}{2}-1}{T}\Big)\varepsilon_3\cdot(b-k_2)\,.
\end{aligned}
\tag{F.27}
$$

Here we expand the exponential to get

$$
\begin{aligned}
e^{-T(c-b^2)} &= 1-T(c-b^2)\\
&= 1-T\{a_1(a_1-1)k_2\cdot k_3+a_2(a_2-1)k_1\cdot k_2+a_3(a_3-1)k_1\cdot k_3\}
\end{aligned}
\tag{F.28}
$$

By doing the above integrals one gets

$$
\begin{aligned}
I_4 &= \varepsilon_1\cdot\varepsilon_2\Bigg\{\left[\Gamma\left(3-\frac{D}{2}\right)+(\frac{D}{2}-1)\Gamma\left(2-\frac{D}{2}\right)\right]\frac{1}{m^{4-D}}\frac{1}{3}\varepsilon_3\cdot(k_1-k_2)\\
&+\left[\Gamma\left(4-\frac{D}{2}\right)+(\frac{D}{2}-1)\Gamma\left(3-\frac{D}{2}\right)\right]\\
&\times\frac{1}{m^{6-D}}\frac{1}{120}\Big[\varepsilon_3\cdot k_1(7k_1\cdot k_2+7k_2\cdot k_3+6k_3\cdot k_1)-(1\leftrightarrow 2)\Big]\Bigg\}\\
&= \varepsilon_1\cdot\varepsilon_2\Bigg\{\Gamma\left(2-\frac{D}{2}\right)\frac{1}{m^{4-D}}\frac{1}{3}\varepsilon_3\cdot(k_1-k_2)\\
&+2\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}\frac{1}{120}\Big[\varepsilon_3\cdot k_1(7k_1\cdot k_2+7k_2\cdot k_3+6k_3\cdot k_1)-(1\leftrightarrow 2)\Big]\Bigg\}\,.
\end{aligned}
\tag{F.29}
$$

Comparing with (F.18), we see that the first term in I_4 corresponds to the first one in $\gamma^{\mathrm{bt}}_{\frac{1}{2}}$. The second term of I_4 combines with $I_1+I_2+I_3$ to make

$$
\begin{aligned}
&\frac{2}{15}\varepsilon_1\cdot\varepsilon_2\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}\,k_1\cdot k_2\,\varepsilon_3\cdot(k_1-k_2)\\
&+\frac{1}{15}\varepsilon_1\cdot\varepsilon_2\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}(\varepsilon_3\cdot k_1\,k_1\cdot k_3-\varepsilon_3\cdot k_2\,k_2\cdot k_3)\\
&+\frac{13}{60}\varepsilon_1\cdot\varepsilon_2\Gamma\left(3-\frac{D}{2}\right)\frac{1}{m^{6-D}}(\varepsilon_3\cdot k_1\,k_2\cdot k_3-\varepsilon_3\cdot k_2\,k_1\cdot k_3)\,,
\end{aligned}
\tag{F.30}
$$

which gives a perfect match with the remaining terms in (F.18).

Appendix G

Integrals related to the graviton self-energy calculation

In Section 6.2 of Chapter 6 we need to calculate the following integrals

$$
\begin{aligned}
I_1 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\ddot{G}_B(u_1,u_1) - G_{\rm gh}(u_1,u_1)\Big)\Big(\ddot{G}_B(u_2,u_2) - G_{\rm gh}(u_2,u_2)\Big) e^{-TG_B(u_1,u_2)k^2}\,,\\
I_2 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\ddot{G}_B^2(u_1,u_2) - G_{\rm gh}^2(u_1,u_2)\Big) e^{-TG_B(u_1,u_2)k^2}\,,\\
I_3 &= \int_0^1 du_1 \int_0^1 du_2 \dot{G}_B(u_2,u_1)\ddot{G}_B(u_1,u_2)\dot{G}_B(u_1,u_2) e^{-TG_B(u_1,u_2)k^2}\,,\\
I_4 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\ddot{G}_B(u_1,u_1) - G_{\rm gh}(u_1,u_1)\Big)\Big(\dot{G}_B(u_1,u_2)\Big)^2 e^{-TG_B(u_1,u_2)k^2}\,,\\
I_5 &= \int_0^1 du_1 \int_0^1 du_2 \Big(\dot{G}_B(u_1,u_2)\Big)^4 e^{-TG_B(u_1,u_2)k^2}\,.
\end{aligned}
\tag{G.1}
$$

In this appendix we show how to do these integrals since they are a bit tricky,

$$
I_1 = 4\int_0^1 du_1 \int_0^1 du_2 e^{-TG_B(u_1,u_2)k^2}\,,
\tag{G.2}
$$

where we used the fact that ($G_B(u_1,u_2) \equiv G_{B12}$)

$$
\ddot{G}_{B12} - G_{\rm gh12} = -2\,.
\tag{G.3}
$$

For start let us do I_4 first, we fixe $u_2 = 0$ and $u_1 = u$, then I_4 is written as

$$
I_4 = -2\int_0^1 du(1-2u)^2\, e^{-Tk^2(u-u^2)}\,.
\tag{G.4}
$$

A first derivative of the exponential is

$$
\partial_u^2\, e^{-Tk^2(u-u^2)} = -Tk^2\Big(-2 - Tk^2(1-2u)^2\Big)e^{-Tk^2(u-u^2)}\,,
\tag{G.5}
$$

and from here we can write

$$
\int_0^1 du(1-2u)^2 e^{-Tk^2(u-u^2)} = -\frac{1}{Tk^2}\int_0^1 du\Big(-\frac{\partial_u^2}{Tk^2} + 2\Big)e^{-Tk^2(u-u^2)}\,,
\tag{G.6}
$$

and

$$\int_0^1 du \partial_u^2 e^{-Tk^2(u-u^2)} = \partial_u e^{-Tk^2(u-u^2)}\Big|_{u=0}^{u=1} = 2Tk^2 \,. \tag{G.7}$$

Then we get the following result for I_4

$$I_4 = \frac{1}{kT^2}(I_1 - 4) \,. \tag{G.8}$$

Now for I_5

$$I_5 = \int_0^1 du(1-2u)^4 e^{-k^2T(u-u^2)} \,. \tag{G.9}$$

Again we have

$$\partial_u^4 e^{-k^2T(u-u^2)} = k^4T^2\Big[12 + 12k^2T(1-2u)^2 + k^4T^2(1-2u)^4\Big]e^{-k^2T(u-u^2)} \,, \tag{G.10}$$

which after integrating both sides we have

$$\int_0^1 du\partial_u^4 e^{-k^2T(u-u^2)} = k^4T^2\int_0^1 du\Big[12 + 12k^2T(1-2u)^2 + k^4T^2(1-2u)^2\Big]e^{-k^2T(u-u^2)} \,, \tag{G.11}$$

which leads to

$$2k^4T^2(6 + k^2T) = k^4T^2\Big(3I - 6k^2TI_4 + k^4T^2I_5\Big) \,. \tag{G.12}$$

Therefore I_5 can be written as

$$I_5 = \frac{2}{k^2T} + \frac{3}{k^4T^2}(I_1 - 4) \,. \tag{G.13}$$

Now let us look at the I_3 integral

$$I_3 = -\int_0^1 du_1du_2\dot{G}_{B12}^2\ddot{G}_{B12}e^{-k^2T(u-u^2)} \,, \tag{G.14}$$

where

$$\ddot{G}_{B12} = 2\delta(u_1 - u_2) - 2 \,. \tag{G.15}$$

By fixing $u_2 = 0$ and $u_1 = u$ we have

$$I_3 = -\int_0^1 du(2\delta(u) - 2)(1-2u)^2 e^{-k^2T(u-u^2)} = -2\Theta(0) - I_4 \,, \tag{G.16}$$

where Θ is the Heaviside-theta function, this leads to

$$I_3 = -1 - \frac{1}{k^2T}(I_1 - 4) \,, \tag{G.17}$$

where we used the fact that $\Theta(0) = \frac{1}{2}$.

Now the tricky one I_2, for this integral one needs to distinguish between the derivative respect to u_1 and u_2, so our integral changes to

$$\begin{aligned}
I_2 &= \int_0^1 du_1du_2\Big({}^{\bullet}G^{\bullet\,2}_{B12} - G^2_{\mathrm{gh}12}\Big)e^{-k^2TG_{B12}} \\
&= \int_0^1 du_1du_2\Big(Tk^2\,{}^{\bullet}G_{B12}G^{\bullet}_{B12}\,{}^{\bullet}G^{\bullet}_{B12} - G^{\bullet}_{B12}\,{}^{\bullet\bullet}G^{\bullet}_{B12} + G^2_{\mathrm{gh}12}\Big)e^{-k^2TG_{B12}} \\
&= k^2TI_3 + \int_0^1 du_1du_2\Big(G^{\bullet\bullet}_{B12}\,{}^{\bullet\bullet}G_{B12} - k^2TG^{\bullet}_{B12}G^{\bullet}_{B12}\,{}^{\bullet\bullet}G_{B12} - G^2_{\mathrm{gh}12}\Big)e^{-k^2TG_{B12}} \,,
\end{aligned} \tag{G.18}$$

where we used some IBP and also the left bullet represents derivative respect to u_1 and the right bullet respect to u_2. Now if we use the following relation

$$G^{\bullet\bullet}_{B12} = {}^{\bullet\bullet}G_{B12} = G_{\mathrm{gh}12} - 2\,, \tag{G.19}$$

to cancel the $G^2_{\mathrm{gh}12}$ (actually by this cancellation we regularized the integral) and we have

$$\begin{aligned} I_2 &= 2k^2TI_3 + \int_0^1 du_1du_2\Big(4 - 4G^2_{\mathrm{gh}12}\Big)e^{-k^2TG_{B12}} \\ &= 2k^2TI_3 + I_1 - 4\,, \end{aligned} \tag{G.20}$$

which finally leads to

$$I_2 = -2k^2T - I_1 + 4\,. \tag{G.21}$$

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