Universidad Michoacana de San Nicolás de Hidalgo

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Relatividad general en el formalismo de primer orden en espacios-tiempo con fronteras

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I would like to dedicate this thesis to Anahi, Ger y Nadia for always being there no matter what.

Also to The Walrus, he knows why...

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Contents

Resumen

Consideramos gravedad en el formalismo de primer orden en tres y cuatro dimensiones. En particular, consideramos formulaciones donde las variables fundamentales son una triada y tétrada e, y conexiones $\mathrm{SO}(2,1)$ and $\mathrm{SO}(3,1)$, ω, para tres y cuatro dimensiones respectivamente. Consideramos espacios tiempo que incluyen una frontera en infinito, que satisface condiciones de frontera asintóticamente planas y/o una frontera interna que satisface condiciones de frontera de horizontes aislados. Para nuestro análisis empleamos el formalismo hamiltoniano covariante donde el espacio fase Γ está dado por soluciones a las escuaciones de movimiento, y para el caso en tres dimensiones también utilizamos dos descomposiciones $2+1$. Proponemos una acción de Palatini bien definida y manifiestamente invariante de Lorentz bajo condiciones asintóticamente planas. Usando el formalismo covariante y canónico encontramos sus correspondientes expresiones para la energía. También estudiamos el principio de acción más general 4-dimensional compatible con la invarianza bajo difeomorfismos. Esto implica, en particular, considerar adrmás del término estándar de Einstein-Hilbert-Palatini, otros términos que o no modifican las ecuaciones de movimiento o son topológicos. Tener un principio de acción bien definido implica la adición de términos de frontera, cuya forma explícita puede depender de las condiciones de frontera en cuestión. Para cada uno de los posibles términos de la acción mostramos que la acción está bien definida, su finitud, su contribución a la estructura simpléctica, y las cargas hamilnonianas y de Noether. Más aún, mostramos que los términos de frontera y topológicos no contribuyen a la estructura simpléctica, ni tampoco a la carga hamiltoniana conservada. Las cargas conservadas de Noether, por el contrario, sí dependen de la adición de tales términos.

Palabras clave: Principio de acción - Formulación hamiltoniana - Términos de frontera - Configuraciones asintóticamente planas - Cantidades conservadas

Abstract

We consider first order gravity in three and four dimensions. In particular, we consider formulations where the fundamental variables are a triad and tetrad, e, and a $\mathrm{SO}(2,1)$ and $\mathrm{SO}(3,1)$ connections, ω, for the three and four dimensional case respectively. We consider spacetimes that include a boundary at infinity, satisfying asymptotically flat boundary conditions and/or an internal boundary satisfying isolated horizons boundary conditions. For our analysis we employ the covariant Hamiltonian formalism where the phase space Γ is given by solutions to the equation of motion, and for the three dimensional case we also employ two $2+1-$decompositions. We propose a three dimensional manifestly Lorentz invariant well posed Palatini action under asymptotically flat boundary conditions. By using a covariant analysis and two different $2+1-$decompositions we found the corresponding expressions for the energy. Also we study the most general four dimensional action principle compatible with diffeomorphism invariance. This implies, in particular, considering besides the standard Einstein-Hilbert-Palatini term, other terms that either do not change the equations of motion, or are topological in nature. Having a well defined action principle also implies adding additional boundary terms, whose detailed form may depend on the particular boundary conditions at hand. For each of the possible terms contributing to the action we study the well posedness of the action, its finiteness, the contribution to the symplectic structure, and the Hamiltonian and Noether charges. Furthermore, we show that the boundary and topological terms do not contribute to the symplectic structure, nor the Hamiltonian conserved charges. The Noether conserved charges, on the other hand, do depend on such additional terms.

Keywords: Action principle - Hamiltonian formulation - Boundary terms - Asymptotically flat configurations - Conserved charges

List of papers

The following papers are derived from the work during my PhD.

1. **Hamiltonian and Noether charges in first order gravity**
   Alejandro Corichi, Irais Rubalcava and Tatjana Vukašinac, Gen. Relativ. Gravit. **46**: 1813 (2014) "Editor's Choice".

2. **First order gravity: Actions, topological terms and boundaries**
   Alejandro Corichi, Irais Rubalcava and Tatjana Vukašinac, Preprint: arXiv:1312.7828. *To be submitted to International Journal of Modern Physics D.*

3. **Some remarks on energy in first order 2+1 gravity.**
   Alejandro Corichi and Irais Rubalcava, *In preparation, to be submitted to Physical Review D.*

List of symbols

$\mathcal{M}$ $n-$dimensional manifold.

M $(n-1)-$dimensional hypersurface.

ι embeding $\iota : M \to \mathcal{M}$.

$\tilde{x}^a$ coordinates on $\mathcal{M}$.

x^a coordinates on M.

$a, b, c, \ldots$ spacetime indices $0, 1, 2, \ldots, n$.

$\bar{a}, \bar{b}, \bar{c}, \ldots$ ‘spacelike’ indices $1, 2, \ldots, n-1$.

e^a_I, e^I_a In 4D tetrad and co-tetrad, in 3D triad, co-triad.

$T_p\mathcal{M}$ Tangent space of $\mathcal{M}$ at point p.

$T^\star_p\mathcal{M}$ Cotangent space of $\mathcal{M}$ at point p.

g_{ab} Spacetime metric.

q_{ab} Induced metric on M.

n^a Normal to the hypersurface M.

P^a_b Projector to M.

$\pounds_v$ Lie derivative along v.

∇_a Any torsion free covariant derivative.

N Laps function.

N^a Shift function.

$\mathcal{D}$ Covariant derivative with respect of the internal indices.

$\mathring{\mathcal{D}}$	Any flat internal connection such that $\mathcal{D} = \mathring{\mathcal{D}} + \omega$.
$\mathcal{L}$	Lagrangian density.
Θ	Symplectic potential.
J	Symplectic current.
$\bar{\Omega}$	Pre-symplectic structure.
Ω	Symplectic structure.
J_N	Noether current.

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Chapter 1

Introduction

"One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them."
—Heinrich Hertz writing about Maxwell's equations.

Our quest to understand nature may have begun hundreds of thousands years ago with the first humans, when they imagine divine or magic explanations for what happen around them, with time and observation they could predict the seasons and then they were able to implement agriculture and thus the beginnings of civilization as we know it today. Although it was until some few hundred years ago that we begin to use mathematics to describe nature. It was until Sir Isaac Newton (1643-1727) in his *Principia*[1], where he formulated the laws of motion and universal gravitation, that for the first time in history, someone was able to unify our view that the motion of objects on Earth (described so far by Galileo's mechanics) and of celestial bodies (Kepler laws) could be described by the same principles. Newton was only the beginning of a long path in trying to unify our understanding of the laws of nature. In 1865 James Clerk Maxwell (1831-1879) published his equations, that show that electricity, magnetism and optics were manifestations of the same phenomena, the electromagnetic field. Unifying, in this way, branches of physics that seemed unrelated until that moment. One of the predictions of his equations was that the electromagnetic field could be propagated in wave form with constant velocity, in fact equal to the light velocity measured up to that time. This would be explained later by Albert Einstein (1879-1955) that in 1905 formulated his special theory of relativity, that could unify Galileo's relativity with classical

[1] PhilosophiæNaturalis Principia Mathematica, Latin for "Mathematical Principles of Natural Philosophy", often referred to as simply the Principia. we can consult Newton's annotated and personal first edition at http://cudl.lib.cam.ac.uk/view/PR-ADV-B-00039-00001/1

electrodynamics. Almost a decade later, in 1916, Einstein published his general theory of relativity, which could unify his special theory of relativity with Newton's universal law of gravitation. With all this success in unifying our understanding of the laws of nature, Einstein embarked himself in a monumental quest: Trying to unify his general theory of relativity with the electrodynamics. His idea was trying to express electrodynamics in the same geometric footing of general relativity, unfortunately he died in 1955 without seen his dream come true. Some decades later, Eintein's dream would be partially realized, when in 1986 Abhay Ashtekar showed that general relativity could be formulated as a $\mathrm{SU}(2)$ connection theory [4].

In this quest of trying to unify our understanding of nature, the natural next big challenge is to try to put together the two pillars of modern physics, say quantum mechanics and general relativity. Both of them shocked and reshape our vision of the world we had at the beginning of the past century, but they describe nature at very different scales and are founded on seemingly contradictory postulates. We have not been able to fully bring them together to describe phenomena that include both, as in the origins of the universe or black holes for instance.

Although we do not have a complete quantum description of the gravitational field we have some very promising candidates as loop quantum gravity (LQG). Ashtekar's ideas of rewriting canonical general relativity in terms of $\mathrm{SU}(2)$ connections, led to the loop representation of quantum general relativity and then into loop quantum gravity [61]. This missing complete quantum description is beyond the scope of this thesis, though it serves as motivation for the many issues studied here. This thesis is focused in the classical aspects of general relativity in the first order formulation in spacetimes with boundaries.

One of the main lessons from the general theory of relativity is that one can formulate theories that, in their Lagrangian description, are diffeomorphism invariant. This means that one can perform generic diffeomorphism on the spacetime manifold and the theory remains invariant. In most instances diffeo invariance is achieved by formulating the theory as an action principle where the Lagrangian density is defined without the use of background structures; it is only the dynamical fields that appear in the action. In this manner one incorporates the 'stage', the gravitational field, as one of the dynamical fields that one can describe. The fact that one can write a term that captures the dynamics of the gravitational field is interesting by itself. It is then worth exploring all the freedom available in the definition of an action principle for general relativity. This is one of the main tasks that we shall undertake in this thesis. Note that we shall restrict ourselves to general relativity and shall not consider generalizations such as scalar-tensor theories nor massive gravity in our analysis.

The first question that we shall address is that of having a well posed variational principle. This particularly 'tame' requirement seems, however, to be sometimes overlooked in the literature. It is natural to ask why we need to have a well posed action principle if, at the end of the day, we already ‘know’ what the field equations are. While this is certainly true, one should not forget that the classical theory is only a (very useful indeed!) approximation to a deeper underlying theory that must be quantum in nature. If, for instance, we think of a quantum theory defined by some path integral, in order for this to be well defined, we need to be able to write a meaningful action for the whole space of histories, and not only for the space of classical solutions. This observation becomes particularly vexing when the physical situation under consideration involves a spacetime region with boundaries. One must be particularly careful to extend the formalism in order to incorporate such boundary terms.

In order to explore some properties of the theories defined by an action principle, the covariant Hamiltonian formalism seems to be particularly appropriate (See, e.g. [6], [29] and [42]). In this formalism, one can introduce the standard Hamiltonian structures such as a phase space, symplectic structure, canonical transformations etc, without the need of a $3+1$ decomposition of the theory. All the relevant objects are *covariant*. The most attractive feature of this formalism is that one can find all these structures in a unique fashion given the action principle. Furthermore one can, in a ‘canonical’ way, find conserved quantities. On the one hand one can derive Hamiltonian generators of canonical transformations and, on the other hand, Noetherian conserved quantities associated to symmetries. One important and interesting issue is to understand the precise relation between these two sets of quantities.

The study of field theories with boundaries in the Hamiltonian approach has received certain attention in the literature. Most of these studies have focused on the standard formalism where a decomposition is involved and constraints are present. One recent example is [14], that considers linear gauge systems in the presence of boundaries, both in the Hamiltonian and covariant Hamiltonian frameworks, with an emphasis on the geometric approach and the functional analytic aspects of the problem (see the references there for previous studies). However, a detail study of a diffeomorphism invariant theory from this perspective is, in our opinion, still lacking.

Another equally important issue in the definition of a physical theory is that of the choice of fundamental variables, specially when gauge symmetries are present. Again, even when the space of solutions might coincide for two formulations, the corresponding actions will in general be different and that will certainly have an effect in the path integral formulation of the quantum theory. In the case of general relativity, the better known formulation is of course in terms of a metric tensor g_{ab}, satisfying second order (Einstein) equations. But there are other choices of variables that yield alternative descriptions. Here we shall consider one of those possibilities. In particular, the choice we shall make is motivated by writing the theory as a *local* gauge theory under the Lorentz group.

It is well known that one can either obtain Einstein equations of motion by means of the Einstein Hilbert action or in terms of the so called Palatini action, a first order action in terms of tetrads e_a^I and a connection ω_a^I valued on a Lie Algebra of $\mathrm{SO}(3,1)$ in the four dimensional case or $\mathrm{SO}(2,1)$ in the three dimensional one (see. e.g. [60] and [56])[1].

Once we choose our fundamental variables, in this case we choose to work in the so called first order general relativity, where the fundamental variables are the tetrad of triad and connection valued in the $\mathrm{SO}(3,1)$ or $\mathrm{SO}(2,1)$ respectively if we are in four or three dimensions. We are ready to explain, which are the questions addressed in this thesis and how we cope with them. In order to understand first order gravity with boundaries in the Hamiltonian approach, we shall restrict ourselves to the three and four dimensional cases, as well as asymptotically flat and isolated horizons boundary conditions.

In four dimensions is known that we can have a generalization of this action by adding a term, the Holst term, that still gives us the same equations of motion and also allows us to express the theory in terms of real SU(2) connections in its canonical description (see. e.g. [35] and [16]). This action, known as the Holst action, is the starting point of loop quantum gravity and some spin foam models. In the same first order scheme one can look for the most general diffeomorphism invariant first order action that classically describes general relativity, which can be written as the Palatini action (including the Holst term) plus topological contributions, namely, the Pontryagin, Euler and Nieh-Yan terms (see for instance [30] for early references). Furthermore, if the spacetime region we are considering possesses boundaries one might have to add extra terms (apart from the topological terms that can also be seen as boundary terms) to the action principle[2].

Thus, the most general first order action for gravity has the form,

$$S[e,\omega] = S_{\text{Palatini}} + S_{\text{HolstTerm}} + S_{\text{Pontryagin}} + S_{\text{Euler}} + S_{\text{Nieh-Yan}} + S_{\text{Boundary}}. \tag{1.1}$$

It is noteworthy to emphasize that in the standard textbook treatment of Hamiltonian systems one usually considers compact spaces without boundary, so there is no need to worry about the boundary terms that come from the integration by parts in the variational principle. But if one is interested in spacetimes with boundaries we can no longer neglect these boundary terms and it is mandatory to analyze them carefully. In order to properly study this action in the whole spacetime with boundaries, we need the action principle to be well posed, i.e. we want the action to be *differentiable and finite* under the appropriate boundary conditions, and under the most general variations compatible with the boundary conditions.

[1] One should recall that the original Palatini action was written in terms of the metric g_{ab} and an affine connection $\Gamma^a{}_{bc}$ [1; 55]. The action we are considering here, in the so called "vielbein" formalism, was developed in [40; 62; 64] and in [31] in the canonical formulation.

[2] We should clarify our use of the name 'topological term'. For us a term is topological if it can be written as a total derivative. This in turn implies that it does not contribute to the equations of motion. There are other possible terms that do not contribute to the equations of motion but that can not be written as a total derivative (such as the so called Holst term). For us, this term is not topological.

It has been shown that under appropriate boundary conditions[1], the Palatini action plus a boundary term provides a well posed action principle, that is, it is differentiable and finite. Furthermore, in [25] the analysis for asymptotically flat boundary conditions was extended to include the Holst term. Here we will refer to this well posed Holst action as the generalized Holst action (GHA).

This thesis is structured as follows:

In chapter 2 we give a brief review of the key basic material needed to dive in the rest of the thesis. Although it is a review, in sections 2.4 and 2.5 we discuss some aspects there are no so widely known, and that they have not been presented altogether in a coherent and systematic way in the literature. The chapter is structured as follows: In section 2.1 we introduce the geometrical concepts to make a $n+1$ decomposition in the cases when we have and have not defined a metric. We discuss the definition of hypersurface, globally hyperbolic spacetimes, foliations, projectors, time evolution and adapted coordinates. This concepts are independent of the theory to which are applied. In section 2.2 we review, in the four dimensional case, the fundamental variables of the first order formulations, that is, we introduced and give the geometrical interpretation of tetrads and connections. In Section 2.4 we review what it means for an action principle to be well posed, which is when it is finite and differentiable. In Section 2.5 we use some results discussed in the previous section, to review the covariant Hamiltonian formalism taking enough care in the cases when the spacetime has boundaries. We begin by defining the covariant phase space and its relation with the canonical phase space. Then we introduce the symplectic structure with its ambiguities and its dependence on boundary terms in the action. Finally we define the symplectic current structure, and the Hamiltonian and Noether charges.

In chapter 3 we begin with a simpler yet interesting enough model. That is, we shall consider three dimensional first order gravity only with an outer boundary corresponding to asymptotically flat configurations that are known for a while in the metric variables [11; 47].

For this example, in section 3.1 we derive the asymptotically flat conditions for the first order variables. Then in section 3.2 we prove that the 3-dimensional Palatini action with boundary term[2], which give us the same equations of motion that the 3-dimensional Einstein-Hilbert action, has a well posed action principle, is finite and differentiable under the asymptotically flat boundary conditions. Moreover if we introduce an additional boundary term to the action to make it explicitly Lorentz invariant we find that the resulting action is equivalent to the Einstein-Hilbert action with Gibbons-Hawking term. In section 3.3 we prove that the energy is bounded from below and above, through the covariant hamiltonian formalism (CHF) of first order gravity with this this fall-off conditions. Agreeing with previous results in the metric variables via Regge-Teitelboim methods

[1] See e.g. [7], [8], [9] and references therein for the asymptotically flat, isolated horizons and asymptotically AdS spacetimes respectively

[2] In analogy to the four dimensional case [7] and as previously introduced in the literature.

[11]. Although CHF provides an elegant and short derivation for the energy (and other relevant symmetries as discussed in [27]), this quantity is determined up to a constant, that shifts the region in which the energy is bounded. We also prove in section 3.4 that the energy is bounded bounded from below and above with the Canonical formalism (following two different $2+1$ decompositions), but in contrast with the CHF, here there is no ambiguity in the election of the constant, the energy is given directly from the hamiltonian. Our results agree with those of [47].And finally in section 3.5 propose a Chern-Simons action with boundary term valued on the Lie algebra of $\mathrm{ISO}(2,1)$ that lead us to the well posed manifestly Lorentz invariant Palatini action previously introduced. And at each stage we prove that we obtain all the same relevant quantities, in particular the energy. This action may served to further study some topological aspects of the theory. But we shall leave it to forthcoming works.

In chapter 4 we have three main goals. The first one is to explore the well-posedness of the action principle with generic boundary terms. For that we shall study two sets of boundary conditions that are physically interesting; as outer boundary we shall consider configurations that are asymptotically flat, and in an inner boundary, those histories that satisfy isolated horizon boundary conditions. The second objective is to explore the most basic structures in the covariant phase space formulation. More precisely, we shall study the existence of the symplectic structure as a finite quantity and its dependence on the various topological and boundary terms. Finally, the last goal is to explore the different conserved quantities that can be defined. Concretely, we shall consider Hamiltonian conserved charges both at infinity and at the horizon. Finally we shall construct the associated Noetherian conserved current and charges. In both cases we shall study in detail how these quantities depend on the existence of the boundary terms that make the action well defined. As we shall show, while the Hamiltonian charges are insensitive to those quantities, the Noether charges do depend on the form of the boundary terms added. While some of these results are not new and have appeared somewhere else, we have several new results and clarifications of several issues. Equally important is the fact that in no publication have all the results available been put in a coherent and systematic fashion.

The structure of the chapter is as follows: In Section 4.1 we use the covariant Hamiltonian formalism to study the action of Eq. (1.1). We find the generic boundary terms that appear when we vary the different components of the action. In Section 4.2 we consider particular choices of boundary conditions in the action principle. In particular we study spacetimes with boundaries: Asymptotically flatness at the outer boundaries, and an isolated horizon as an internal one. In Section 4.3 we study symmetries and their generators for both sets of boundary conditions. In particular we first compute the Hamiltonian conserved charges, and in the second part, the corresponding Noetherian quantities are found. We comment on the difference between them. We summarize and provide some discussion in the final Section 4.4.

I want to apologize the experienced reader for some ‘very basic’ comments here and there. I write them for ‘my younger self’.

Chapter 2

Preliminaries

"It is important to do everything with enthusiasm, it embellishes life enormously."
—Lev Landau.

In this chapter we briefly review some of the 'basic concepts' used along this thesis. It is based on several readings and notes[1] taken during my PhD years. I hope to give proper credit to all the sources.

2.1 $n+1$ decomposition

We refer to a $3+1$ decomposition by a way of slicing an n dimensional manifold $\mathcal{M}$ into 3 dimensional surfaces (hypersurfaces), but we do not necessarily assume the existence of a metric. In contrast to the standard $3+1$ formalism where we ask these hypersurfaces to be spacelike, so that the metric induced on them by the Lorentzian spacetime metric [signature$(-,+,\ldots,+)$] is Riemannian [signature$(+,\ldots+)$]. So a $n+1$ decomposition is a generalization of the $n+1$ formalism that includes the cases where we have and do not have a metric defined on the manifold, as we shall see in the $2+1$ and $3+1$ examples studied on this thesis.

All the machinery reviewed in this section was originaly developed in the context of the $3+1$ formulation, which in the late 1950's and 1960's received impulse by providing the foundation of Hamiltonian formulations of general relativity by Paul A.M. Dirac [32; 33], and Richard Arnowitt, Stanley Deser and Charles W. Misner (ADM) [1]. It was also during this time that John A. Wheeler put forward the concept of geometrodynamics and coined the names of lapse and shift [66].

We want to emphasize that the concepts presented in this section are independent of the dy-

[1] Specially the ones given by Alejandro Corichi and Juan Daniel Reyes.

namics, that is, they are valid independently of the theory to which they are aplied, and whether this theory depends on a metric or not.

Some of the material presented in this section is based on [12; 19; 34; 57].

2.1.1 Hypersurfaces

2.1.1.1 Definition

A *hypersurface* M of $\mathcal{M}$ is the image of a $n-1$ dimensional manifold $\hat{M}$ by an embeding[1] $\iota : \hat{M} \rightarrow \mathcal{M}$ (see Fig.): $M = \iota(\hat{M})$.

Locally we can define a hypersurface as the set of points for which a global function $\hat{t}$ is constant, that is,

$$\forall p \in \mathcal{M}, \quad p \in M \Leftrightarrow \hat{t}(p) = t, \quad \text{with} \quad t = const. \tag{2.1}$$

We can introduce locally a coordinate system of $\mathcal{M}$, $\tilde{x}^a = (t, x^{\bar{a}})$ with $a = (0, 1, ..., n)$ and $\bar{a} = (0, 1, ..., n-1)$, such that t spans $\mathbb{R}$ and $x^{\bar{a}}$ are Cartesian coordinates spanning $\mathbb{R}^{n-1}$. M is defined by the condition $\hat{t} = t$. So the embeding takes the form,

$$\begin{array}{cccc} \iota & : & \hat{M} & \longrightarrow & \mathcal{M} \\ & & (x^{\bar{a}}) & \longmapsto & (t_0, x^{\bar{a}}). \end{array} \tag{2.2}$$

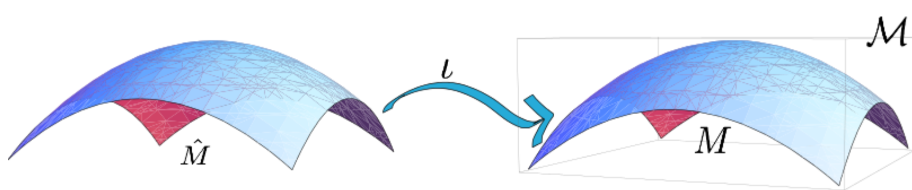

Figure 2.1: Embedding ι of the $n-1$ dimensional manifold $\hat{M}$ into the n dimensional manifold $\mathcal{M}$, defining the hypersurface $M = \iota(\hat{M})$.

Note that the embeding ι maps curves in $\hat{M}$ to curves in $\mathcal{M}$. Also it maps vectors on $T_p\hat{M}$ to vectors on $T_p\mathcal{M}$ by the *push-forward mapping*, ι_*. Conversely the embedding ι induces the *pull-back mapping*, ι^*, that maps linear forms on $T_p\hat{M}$ to linear forms on $T_p\mathcal{M}$ in the following way,

$$\tag{2.3}$$

From now we identify $\hat{M}$ and M since they are related by $M = \iota(\hat{M})$. So we shall refer to a hypersurface on $\mathcal{M}$ by M.

[1] An embeding $\iota : \hat{M} \rightarrow \mathcal{M}$ is a isomorphism, a one-to-one mapping such that both ι and ι^{-1} are continuous, which guarantees that M does not 'intersect itself'.

A very important application of the pull-back operation is that by knowing the bilinear form g (the spacetime metric), we can define another bilinear form, q, the induced metric on M by,

$$q := \iota^* g. \tag{2.4}$$

2.1.1.2 Normal vector

Given a global function t on $\mathcal{M}$, such that the hypersurface M is defined as a level surface of t, the gradient 1-form dt is normal to M, in the sense that for every vector v tangent to M, $< \mathrm{d}t, v >= 0$.

The metric dual to dt, i.e. the vector $\vec{\nabla}t$ (whose components are $\nabla^a t = g^{ab}\nabla_b t = g^{ab}(\mathrm{d}t)_b$) is a vector normal to M and satisfies that,

- $\vec{\nabla}t$ is timelike iff M is spacelike.
- $\vec{\nabla}t$ is spacelike iff M is timelike.
- $\vec{\nabla}t$ is null iff M is null.

The vector $\vec{\nabla}t$ defines the unique direction normal to M (see Fig. (2.1)). When M is not null we normalize $\vec{\nabla}t$ to make it a unit vector,

$$\mathbf{n} := (\pm\vec{\nabla}t \cdot \vec{\nabla}t)^{-1/2}\vec{\nabla}t, \tag{2.5}$$

such that,

$$\begin{aligned} \mathbf{n}\cdot\mathbf{n} &= -1 \qquad \text{if } M \text{ is spacelike,} \\ \mathbf{n}\cdot\mathbf{n} &= 1 \qquad \text{if } M \text{ is timelike.} \end{aligned} \tag{2.6}$$

2.1.2 Globally hyperbolic spacetimes and foliations

In the previous subsections we have studied some aspects of the geometry of a hypersurface M embedded in the manifold $\mathcal{M}$. We have not assumed the existence of a metric neither a particular topology of $\mathcal{M}$. Now we shall restrict ourselves to a wide class of spacetimes so-called globally hyperbolic spacetimes, where we can consider a continuous set of hypersurfaces $(M_t)_{t\in\mathbb{R}}$ that covers the manifold $\mathcal{M}$. This type of spacetimes cover most of the astrophysical and cosmological cases of interest.

2.1.2.1 Cauchy slice and globally hyperbolic spacetime

A *Cauchy surface* is a spacelike hyper surface M in $\mathcal{M}$ such that each causal (i.e. timelike or null) curve without end point intersects M once and only once []. In other words M is a Cauchy hyper surface iff its domain of dependence is the whole space-time $\mathcal{M}$.

A space-time $(\mathcal{M}, g_{ab})$ that admits a Cauchy surface M is said to be *globally hyperbolic*.

The topology of a globally hyperbolic space-time $\mathcal{M}$ is necessarily $M \times \mathbb{R}$, where M is the Cauchy surface entering in the definition of global hyperbolicity.

2.1.2.2 Foliations

Assume a globally hyperbolic spacetime $\mathcal{M}$, by the definition given in the previous section, it can be splitted of foliated by three dimensional spacelike hypersurfaces $M_t \approx M$ parametrized by a global function $\hat{t} : \mathcal{M} \to \mathbb{R}$, that is they are level surfaces[1] of $\hat{t}$,

$$M_t = \{p \in \mathcal{M} | \hat{t}(p) = t,\ t = const\}. \tag{2.7}$$

We consider $\hat{t}$ regular, so the hypersurfaces M_t are non-intersecting:

$$M_t \cap M_{t'} = \emptyset \ \ for \ \ t \neq t'. \tag{2.8}$$

We shall call each hyper surface M_t a *leaf* or a *slice* of the foliation. We assume that all M_t's are spacelike and that the foliation covers $\mathcal{M}$,

$$\mathcal{M} = \bigcup_{t \in \mathbb{R}} M_t. \tag{2.9}$$

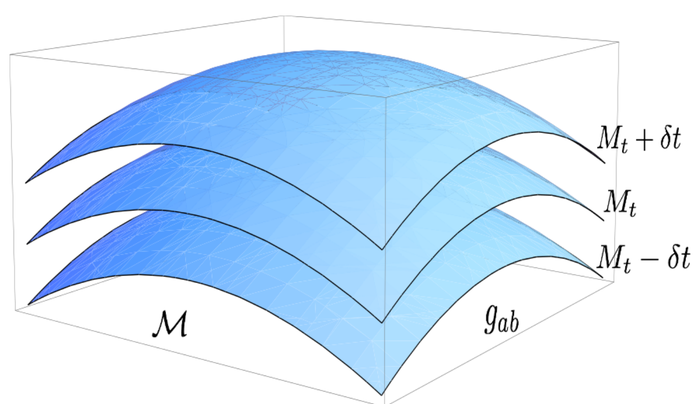

Figure 2.2: Foliation of spacetime $\mathcal{M}$, by a family of hypersurfaces.

For general coordinates $\tilde{x}^a$ in $\mathcal{M}$, $a = 0,1,2,3$, each M_t may be characterized by $\hat{t}(\tilde{x}^a) = t$, but also the hypersurface M with coordinates $x^{\bar{a}}$, $\bar{a} = 1,2,3$, may be thought as embeded in $\mathcal{M}$, $\tilde{x}^a = \tilde{x}^a(x^{\bar{a}})$.

[1] In what follows we shall not make any distinction between t and $\hat{t}$, we shall skip the 'hat' in the global functions.

As an example[1] consider spheres of radious r in 3 dimensional Euclidean space $\mathbb{R}^3$ embeded as

$$\phi(\tilde{x}^a) = (\tilde{x}^1)^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2 = r^2 = const \tag{2.10}$$

then $\tilde{x}^a = \tilde{x}^a(x^{\bar{a}})$, with $x^{\bar{a}} = (\theta, \varphi)$, is given by,

$$\begin{aligned} \tilde{x}^1 &= r\cos\theta\sin\varphi \\ \tilde{x}^2 &= r\cos\theta\cos\varphi \\ \tilde{x}^3 &= r\sin\varphi. \end{aligned}$$

Assuming there is a metric g_{ab} defined on $\mathcal{M}$, M_t has a normal vector, n^a (as defined in 2.1.1.2), (future pointing) given by[2]:

$$g^{ab}(\mathrm{d}t)_b = g^{ab}\partial_b t. \tag{2.11}$$

Here we are assuming a regular foliation, global function $\hat{t}$ is such that $\mathrm{d}t \neq 0$.

In the case the normal is timelike, (for a spacelike surface) we normalize such that $n^a n_a = g_{ab}n^a n^b = -1$.

M_t inherit an "induced euclidean metric" q_{ab} (the pull-back of g_{ab}, $q := \imath^* g$), which also can be expressed in coordinates as,

$$q_{\bar{a}\bar{b}} = g_{ab}\frac{\partial \tilde{x}^a}{\partial x^{\bar{a}}}\frac{\partial \tilde{x}^b}{\partial x^{\bar{b}}}. \tag{2.12}$$

This coincides on each M_t with the induced metric:

$$q_{ab} := g_{ab} + n_a n_b, \tag{2.13}$$

characterized by, $q_{ab}n^b = 0$, and when S^b is tangent to M_t ($S^b n_b = 0$) then $q_{ab}S^b = g_{ab}S^b$.

Indeed the tangent space $T_p\mathcal{M}$ can be decomposed into subspaces tangent and normal to M_t

$$T_p\mathcal{M} = T_pM_t \oplus \mathrm{Span}\{n^a\}. \tag{2.14}$$

Vectors $e^a_{\bar{a}} := \frac{\partial \tilde{x}^a}{\partial x^{\bar{a}}}$ give a basis for T_pM (they are orthogonal to n^a, that is, $e^a_{\bar{a}} n_a = 0$).

Although the induced metric q_{ab} on M_t plays a fundamental role in canonical gravity. We define it through the projection of a 4-tensor to M_t, $a, b = 0, 1, 2, 3$ still are spacetime indices. But we shall say a bit more about the projection in next subsection.

[1]Since for pedagogical reasons we are not foliating spacetime, just the Euclidean space into surfaces, we change $t \to \phi$.

[2]Obtained by 'rising the index of the one-form gradient $(\mathrm{d}t)_a$.

2.1.2.3 The projector

At this point we will assume that M is embedded in $\mathcal{M}$. $T_p\mathcal{M}$ can be decomposed at each point p in $\mathcal{M}$ into subspaces tangent and *normal* to M. Here we will refer to normal a vector m^a, a vector that is not tangent to M, only in the case when there is a metric defined on $\mathcal{M}$ we can chose m^a to be arbitrary[1] or to coincide with n^a the normal vector to M as defined in subsection (2.1.1.2).

We may decompose any vector v^a into its 'spatial' (tangent to M) and normal part.

$$v^a = v_{normal}m^a + v^{\bar{a}}e^a_{\bar{a}} \tag{2.15}$$

where $m_a v^a =: -v_{normal}$ is the projection of v^a along m^a, which implies

$$v^a = -(v^a m_a)m^a + v^{\bar{a}}e_{\bar{a}} \tag{2.16}$$

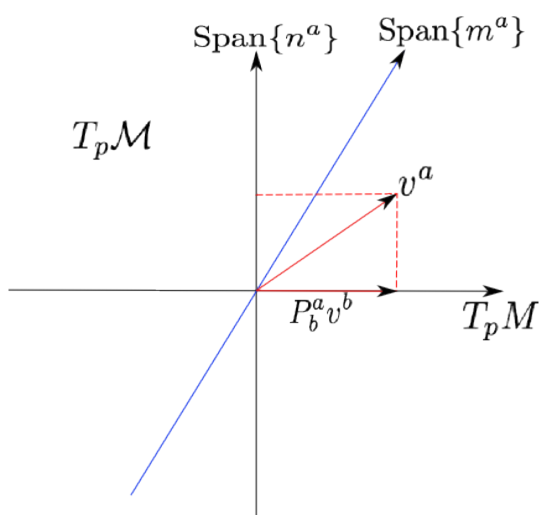

Figure 2.3: $T_p\mathcal{M} = T_pM_t \oplus \text{Span}\{n^a\}$ and the action of the projector P^a_b.

With this at hand we can define the *Projector*, $P : T_p\mathcal{M} \to T_pM$, as the linear map defined by

$$\begin{array}{rcccl} P & : & T_p\mathcal{M} & \longrightarrow & T_pM \\ & & v^a & \longmapsto & P^a_b v^b & = & v^{\bar{a}}e_{\bar{a}} \\ & & & & & = & v^a + v^b m_b m^a \\ & & & & & = & (\delta^a_b + m^a m_b)v^b \end{array} \tag{2.17}$$

[1] For the $3+1$ formalims we chose it as arbitrary (independent from n^a) and it will be associated with temporal evolution.

Therefore the projector can be expressed as,

$$P^a_b = \delta^a_b + m^a m_b \tag{2.18}$$

Note that in the case we have defined a metric on $\mathcal{M}$, g_{ab}, this is just $q^a_b = g^{ac}q_{cb} = g^{ac}(g_{cb} + n_c n_b)$, where q_{cb} is the induced metric on M, and n^a the unit normal to M. The vector $\tilde{v}^a := q^a_b v^b$ and convector $\tilde{w}_b := q^a_b w_a$ are always tangent to M, i.e. $\tilde{w}_b v^b = w_a q^a_b v^b = w_a \tilde{v}^a$ 'selects' only the tangent part of v^a.

In the case we have no metric, we ask the spacetime $\mathcal{M}$ to be topologically $M \times R$ and that there exists a function t (with nowhere vanishing gradient $(dt)_a$) such that each $t = const$ surface M_t is diffeomorphic to M. Also there exists a flow vector field t^a satisfying $t^a(dt)_a = 1$, which allow us to define "evolution", although t does not necessarily have the interpretation of time, we can defined the projector as,

$$t^a_b := P^a_b = \delta^a_b + t^a(dt)_b. \tag{2.19}$$

Note that $m^a = t^a$, we ask t^a not to be tangent to M, and also we are using the definition of the gradient 1-form dt that is normal to M in the sense that $< \mathrm{d}t, v >= 0$. This projector will be used explicitly in the next chapter (section 3.4.1).

In general for a tensor $T^{a_1 \ldots a_n}_{b_1 \ldots b_n}$ its tangent component to M can be found by

$$\tilde{T}^{a_1 \ldots a_n}_{b_1 \ldots b_n} = P^{a_1}_{c_1} \cdots P^{a_n}_{c_n} P^{d_1}_{b_1} \cdots P^{d_n}_{b_n} \; T^{a_1 \ldots a_n}_{b_1 \ldots b_n}. \tag{2.20}$$

2.1.2.4 Time evolution and adapted coordinates

To define *time derivatives* we need to define a direction for time evolution. We shall introduce a congruence of curves transversal to the foliation or equivalently a 'time evolution' vector field t^a (not necessarily orthogonal to M_t nor time-like, we just ask t^a not to be tangent to M_t). Such that t is the affine parameter $t^a \nabla_a t = 1$.

With respect to this 'time evolution' vector field t^a, we define 'time derivative' of the tensor $T^{a_1 a_2 \cdot a_n}{}_{b_1 b_2 \cdot b_n}$ as

$$\dot{T}^{a_1 a_2 \cdot a_n}{}_{b_1 b_2 \cdot b_n} = \pounds_{t^a} T^{a_1 a_2 \cdot a_n}{}_{b_1 b_2 \cdot b_n}, \tag{2.21}$$

with the Lie derivative of a tensor given by,

$$\pounds_{t^a} T^{a_1 a_2 \cdot a_n}{}_{b_1 b_2 \cdot b_n} = t^c \nabla_c T^{a_1 a_2 \cdot a_n}{}_{b_1 b_2 \cdot b_n} - \sum_{i=1}^{n} T^{a_1 \cdot c \cdot a_n}{}_{b_1 b_2 \cdot b_n} \nabla_c t^{a_i} + \sum_{i=1}^{n} T^{a_1 a_2 \cdot a_n}{}_{b_1 \cdot c \cdot b_n} \nabla_{b_i} t^c, \tag{2.22}$$

where ∇_c is any covariant derivative, in particular we can take coordinate derivative ∂_c.

With this additional structure we can define *adapted coordinates* (t, x^a) for the manifold that

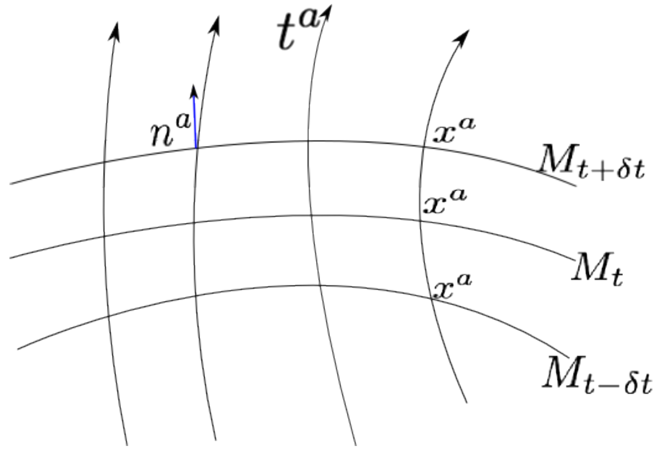

Figure 2.4: Time evolution vector field t^a.

satisfies: $t^a \nabla_a x^b = 0$ and $t^a \nabla_a t = 1$ that is x^a is constant along integral curves of t^a. In this coordinates adapted to the foliation the time derivative takes the form, $\partial/\partial t$.

Conversely, coordinates (t, x^a) adapted to the foliation define evolution vector field,

$$t^a = \frac{\partial \tilde{x}^b}{\partial t}\Big|_{x^a=const.} \tag{2.23}$$

We may decompose t^a into tangential and normal parts [], for that we use the projector (2.18),

$$P^a_b t^b = (\delta^a_b + n^a n_b) t^b = t^a + n^a (n_b t^b), \tag{2.24}$$

if we define the *laps function*,

$$N := -n_b t^b, \tag{2.25}$$

and the *shift function*,

$$N^a := P^a_b t^b. \tag{2.26}$$

With this definitions we can write,

$$\boxed{t^a = N n^a + N^a}. \tag{2.27}$$

2.2 Tetrads and connections

As we already mention in the introduction, it is a very important issue the choice of our fundamental variables, specially when gauge symmetries are present. Also, since in this thesis we shall restrict ourselves to the first order formulation, that is we choose as our fundamental variables tetrads e_{Ia} and a connection ω^I_a valued on a Lie Algebra of $SO(3,1)$ in the four dimensional case or triads e_{Ia} and a connection ω_a^I valued on a Lie Algebra of $SO(2,1)$ in the three dimensional one. We shall give a brief review of the geometrical meaning of the four dimensional variables, the three dimensional case follows directly.

2.2.1 Tetrads

Consider a 4-dimensional manifold $\mathcal{M}$, with a metric g_{ab}, with $T_p\mathcal{M}$ and $T_p^\star\mathcal{M}$ its tangent and contangent spaces at a point p, respectively. The metric can be seen as a map,

$$\begin{aligned} g_{ab} \quad : \quad & T_p\mathcal{M} \to T_p^\star\mathcal{M} \\ & v^a \mapsto g_{ab}(v^a) = v_b, \end{aligned} \tag{2.28}$$

which is the *well known fact* that the metric (and its inverse) is used to “lower” (and “raise”) indices. Moreover it allow us to define an interior product,

$$v \cdot w = g_{ab} v^a w^b, \tag{2.29}$$

so we can express the norm of vector v as $|v|^2 = v \cdot v = g_{ab} v^a v^b$.

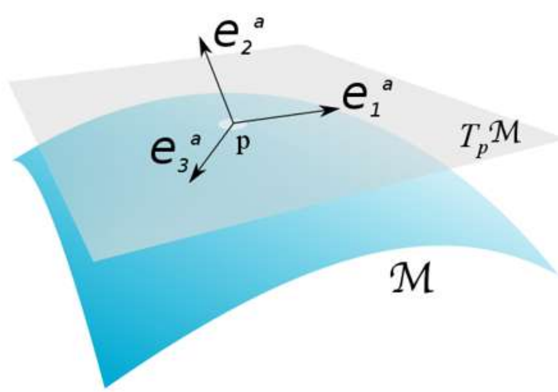

Figure 2.5: Manifold $\mathcal{M}$ with its tangent space $T_p\mathcal{M}$ at a point p.

With this at hand we can define a space-time tetrad as a set of four vector fields e_I^a labeled by an additional index $I = 0,1,2,3$ such that they provide an orthonormal basis at $T_p\mathcal{M}\ \forall p \in \mathcal{M}$,[1]

$$e_I \cdot e_J = g_{ab} e_I^a e_J^b = \eta_{IJ}. \tag{2.31}$$

We can define $e_{aI} := g_{ab} e_I^b$ and $e_a^I := \eta^{IJ} e_{aJ}$, then,

$$e_I^a e_a^J = e_I^a \eta^{JK} e_{aK} = e_I^a \eta^{JK} g_{ab} e_K^b = (g_{ab} e_I^a e_J^b)\eta^{JK} = \eta_{IK}\eta^{JK} = \delta_I^J \tag{2.32}$$

This implies that e_I^a is invertible, seen as a 4×4 matrix, and its inverse is e_a^I, this we call it the *co-tetrad*. Since $(M^\dagger)^{-1} = (M^{-1})^\dagger$ for any invertible matrix

$$e_I^a e_b^I = \delta_b^a \tag{2.33}$$

and from $g_{ab} e_I^a e_J^b = \eta_{IJ}$,

$$g_{ab} e_I^a e_J^b e_c^I e_d^J = \eta_{IJ} e_c^I e_d^J \tag{2.34}$$

but

$$g_{ab} e_I^a e_c^I e_J^b e_d^J = g_{ab} \delta_c^a \delta_d^b = \eta_{IJ} e_c^I e_d^J \tag{2.35}$$

which implies

$$g_{cd} = \eta_{IJ} e_c^I e_d^J. \tag{2.36}$$

This is consistent with the interpretation that the co-tetrad defines an invertible isomorphism between $T_p\mathcal{M}$ and a Minkowski space M_p glued at each point p:

$$\begin{aligned} e_a^J \quad &: \quad T_p\mathcal{M} \to M_p \\ & \quad\ v^a \longmapsto v^I := e_a^I v^a \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \eta_{IJ} v^I w^J &= \eta_{IJ} e_a^I v^a e_b^J v^b && (2.38)\\ &= (\eta_{IJ} e_a^I e_b^J) v^a v^b && (2.39)\\ &= g_{ab} v^a v^b && (2.40) \end{aligned}$$

[1] You can think of them at $T_p\mathcal{M}$ as

$$\underbrace{e_0^a}_{timelike}\ \underbrace{e_1^a, e_2^a, e_3^a}_{spacelike}. \tag{2.30}$$

and its inverse,

$$\begin{aligned} e^a_J \quad : \quad & M_p \to T_p\mathcal{M} \\ & v^I \longmapsto v^a := e^a_I v^I \end{aligned} \tag{2.41}$$

Note that by reversing the steps,

$$g_{ab}v^a w^b = g_{ab}e^a_I e^b_J v^J = (g_{ab}e^a_I e^b_J)v^I v^J = \eta_{IJ}v^I v^J, \tag{2.42}$$

So we can associate a physical interpretation to this description: The cotetrad represents the intertial reference frame attached to a free falling observer at p. That is, by contracting with e^I_a observers measure the tangent vector v^a_p as $v^I = e^I_a v^a$. The free falling observer uses a Minkowski metric to measure magnitudes and angles, according to the equivalence principle,

$$v \cdot w = \eta_{IJ}(e^I_a v^a)(e^J_b w^b) = (\eta_{IJ}e^I_a e^J_b)v^a w^b := g_{ab}v^a w^b \tag{2.43}$$

thus defining a metric on each point of spacetime.

With this interpretation the internal indices are vector indices on Minkowski spacetime.

Indeed, any Lorentz transformation $\Lambda^I{}_J$ on M_p has no effect on the spacetime metric:

$$\begin{aligned} \eta_{IJ}\Lambda^I{}_K e^K_a \Lambda^J{}_L e^L_b &= \Lambda^I{}_K \eta_{IJ}\Lambda^J{}_L e^K_a e^L_b \\ &= \eta_{IJ}e^K_a e^L_b \\ &= g_{ab} \end{aligned} \tag{2.44}$$

We just have new mappings $\tilde{e} = \Lambda \circ e : T_p\mathcal{M} \to M_p$, $\tilde{e}^I_a = \Lambda^I{}_J e^J_a$.

The inverse $\tilde{e}^{-1} = e^{-1} \circ \Lambda^{-1} : M_p \to T_p\mathcal{M}$ is

$$\tilde{e}^a_I = (\Lambda^I_J e^J_a)^{-1} = (e^J_a)^{-1}(\Lambda^I{}_J)^{-1} = e^a_J(\Lambda^I{}_J)^{-1} =: e^a_J \Lambda_I{}^J \tag{2.45}$$

Indeed the new vectors e^a_I also form an orthonormal set on $T_p\mathcal{M}$:

$$g_{ab}\tilde{e}^a_I\tilde{e}^b_J = g_{ab}e^a_K\Lambda_I{}^K e^b_L\Lambda^L_J \tag{2.46}$$
$$= g_{ab}e^a_K e^b_L \Lambda_I{}^K \Lambda_J{}^L \tag{2.47}$$
$$= \eta_{KL}\Lambda_I{}^K\Lambda_J{}^L = \eta_{IJ} \tag{2.48}$$

The tetrad e^a_I, also defines a mapping

$$\begin{aligned} e^a_I \quad &: \quad T^*_p M \to M*_p \\ & \quad w_a \longmapsto w_I := e^a_I w_a. \end{aligned} \tag{2.49}$$

With inverse,

$$\begin{aligned} e^I_a \quad &: \quad M^*_p \to T^*_p M \\ & \quad w_I \longmapsto w_a := e^I_a w_I. \end{aligned} \tag{2.50}$$

2.2.1.1 Time gauge

If we have a foliation of space-time, the mapping

$$n_a \longmapsto n_I := e^a_I n_a, \tag{2.51}$$

or equivalently,

$$n^a \longmapsto n^I := e^I_a n^a, \tag{2.52}$$

define a "normal" or time-like vector on Minkowski space at each point. Changing the tetrad: $\tilde{e}^I_a = \Lambda^I{}_J e^J_a$, $\tilde{e}^a_I = e^a_J \Lambda_I{}^J$, it changes or rotates the normal,

$$\tilde{n}_I = \tilde{e}^a_I n_a = \Lambda_I{}^J e^a_J n_a = \Lambda_I{}^J n_J \tag{2.53}$$

and

$$\tilde{n}^I = \tilde{e}^I_a n^a = \Lambda^I{}_J e^J_a n^a = \Lambda^I{}_J n^J. \tag{2.54}$$

We can choose e^a_0 as the 'timelik'e while $e^a_i, i = 1,2,3$ as 'spacelike'. But of course this does not necessarily mean e^a_i are tangent to the leaf M and e^a_0 in the direction of n^a. See fig []

The *time gauge* precisely consists on "aligning" e^a_0 with n^a,

$$e^a_0 = n^a \tag{2.55}$$

And consequently "putting e^a_i inside M".

Fixing this normal in Minkowski, defined by $n^I := e^I_a n^a$, so that $e^b_I n^I = e^b_I e^I_a n^a = n^b$. This implies that in the time gauge,

$$e^a_0 = n^a = e^a_I n^I \Rightarrow n^I = \delta^I_0 = (1,0,0,0). \tag{2.56}$$

Also,

$$n_a e_i^a = g_{ab} n^b e_i^a = g_{ab} e_i^a e_0^b = \eta_{i0} = 0 \tag{2.57}$$

this implies that e_i^a are tangent to M, i.e. $e_i^a \in T_pM$. From the previous equation we can also see that $n_i = n_a e_i^a = 0$ is consisten with $n_I = \eta_{IJ} n^J = (-1,0,0,0)$.

The time gauge condition, $e_0^a = n^a$, partially fixes the gauge as follows. We shall only allow transformations $\Lambda^I{}_J$ that preserves the condition $e_0^a = n^a = e_I^a n^I$, this are transformations that fix $n^I = (1,0,0,0)$:

$$n^I = \Lambda^I{}_J n^J \quad or \quad \Lambda_I{}^J n^I = n^J \tag{2.58}$$

where

$$\Lambda^I{}_J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^i{}_j & \\ 0 & & & \end{pmatrix} \quad and \quad \Lambda_I{}^J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_i{}^j & \\ 0 & & & \end{pmatrix}. \tag{2.59}$$

Indeed, $\tilde{n}^a = \tilde{e}_I^a n^I = e_J^a \Lambda_I{}^J n^I = e_J^a n^J = e_0^a = n^a$ and also $\tilde{e}_0^a = e_I^a \Lambda_0{}^J = e_0^a \Lambda_0^0 = e_0^a$. That is, transformations that fix e_0^a and rotate e_i^a, so *the gauge group is* $SO(3)$.

2.3 Connections on vector bundles

Intuitively speaking we can think of a *vector bundle E over M* in the following way. Given a manifold M (usually space-time or a hypersurface) we can construct a new manifold by 'gluing' together copies of some vector space V such that locally (for a neighbourhood $U \subset E$),

$$E = \bigcup_{p \in M} V_p \quad \text{and} \quad U \approx \mathcal{M} \times V \tag{2.60}$$

Examples are the tangent space TM, cotangent space T^*M and tensor bundles.

Another important concept is that of a *section*. Consider the bundle $E = TM$, a vector field v^a is a function $v^a : M \to TM$ such that $v^a(p) \in T_pM$ at each $p \in M$ (v^a is an abstract vector with components in a certain basis).

Generalizing, *sections* of E are functions $W^I : M \to E$ such that $W^I(p) \in V_p$. In an internal basis $e_{(I)}$, $W = W^I e_{(I)}$.

A connection D on E is a way to take 'derivatives' of sections W^I in the direction of the vector field v^a. D is a mapping $(v^a, W^I) \longmapsto D(v^a, W^I)$ such that:

i) $D(v, \alpha W) = \alpha D(v, W)$ with α constant.

ii) $D(v, W + \tilde{W}) = D(v, W) + D(v, \tilde{W})$.

iii) $D(v, fW) = (v^a \partial_a f)W + fD(v,W)$, f is a function on M, $f : M \to \mathbb{R}$.

iv) $D(v + \tilde{v}, W) = D(v,W) + D(\tilde{v},W)$.

v) $D(fv,W) = fD(v,W)$.

This mapping defines a "covariant derivative" $D_a W^I$ of a section W^I:

$$D_a W^I := D(\bullet, W). \tag{2.61}$$

Properties iv) and v) implies that this is a *one form* on M taking values on E such that:

I) $D_a(\alpha W^I) = \alpha D_a W^I$.

II) $D_a(W^I + \tilde{W}^I) = D_a W^I + D_a \tilde{W}^I$.

III) $D_a(fW^I) = (\partial_a f)W^I + fD_a W^I$.

Note that this determines a covariant derivative on all tensor bundles constructed from tensor products of V (covariant derivatives of $T^{I_1 \cdots I_n}{}_{J_1 \cdots J_n}$) if we require:

a) General Leibniz rule: $D_a(T\tilde{T}) = (D_a T)\tilde{T} + TD_a\tilde{T}$

b) Commutativity with contractions

$$D_a \left(T^{I_1 \cdots K \cdots I_n}{}_{J_1 \cdots K \cdots J_n} \right) = D_a T^{I_1 \cdots K \cdots I_n}{}_{J_1 \cdots K \cdots J_n}$$

Now a natural question arises, how does covariant derivative $D_a W^I$ look in local coordinates? (this will be useful for actual calculations).

1. Choose *local* coordinates x^a on M, this implies we have a local coordinate bases ∂_a on T_pM, then $v = v^a \partial_a$.

2. Choose local basis $e_{(I)}$ on E, then $W = W^I e_{(I)}$.

In particular, we can consider $D_a e_{(I)}$, so we can define $\omega_a^I{}_J$ by:

$$D_a e_{(I)} = \underbrace{\omega_a^I{}_J e_{(I)}}_{\textit{Linear combination}} . \tag{2.62}$$

That is, if we know these coefficients, $\omega_{a\ J}^{I}$, then we know (locally) $D_a W^I$ for all W^I,

$$\begin{aligned}
D_a W &= D_a(W^I e_{(I)}) \\
&= (\partial_a W^I) e_{(I)} + W^I D_a e_{(I)} \quad \textit{by III}) \\
&= (\partial_a W^I) e_{(I)} + W^I \omega_{a\ I}^{J} e_{(J)} \\
&= \left(\partial_a W^I + \omega_{a\ J}^{I} W^J\right) e_{(I)}
\end{aligned} \tag{2.63}$$

So the component (I) of $D_a W$ is,

$$(D_a W)^I = \partial_a W^I + \omega_{a\ J}^{I} W^J. \tag{2.64}$$

Usually physicist use the "sloppy" notation to define the covariant derivative,

$$\boxed{D_a W^I = \partial_a W^I + \omega_{a\ J}^{I} W^J.} \tag{2.65}$$

More generally for associated tensor bundles,

$$D_a T^{I_1\cdots I_n}{}_{J_1\cdots J_n} = \partial_a T^{I_1\cdots I_n}{}_{J_1\cdots J_n} + \sum_i \omega_{a\ K}^{I_i} T^{I_1\cdots K\cdots I_n}{}_{J_1\cdots J_n} - \sum_i \omega_{a\ J_i}^{K} T^{I_1\cdots I_n}{}_{J_1\cdots K\cdots J_n}. \tag{2.66}$$

Note that we have defined $\omega_{a\ J}^{I}$ by $D_a e_{(I)} = \omega_{a\ J}^{I} e_{(I)}$, so they depend on coordinates x^a on M and basis $e_{(i)}$ on fibers of E but they also define more geometrical (coordinate independent) objects. From the formula,

$$v^a D_a W^I = v^a \partial_a W^I + v^a \omega_{a\ J}^{I} W^J, \tag{2.67}$$

we can see $\omega_{a\ J}^{I}$ is a matrix-valued[1] *one-form*,

$$v^a \to \boxed{\omega_{a\ J}^{I}} \to v^a \omega_{a\ J}^{I}, \tag{2.68}$$

that is, at each point $p \in M$ "eats" a space-time vector and "spits out" a matrix in an $f-$linear way, in this way is a one-form. But these matrices also define a linear transformation on each of the fibers of E, the section W^I,

$$W^I \to \boxed{v^a \omega_{a\ J}^{I}} \to \tilde{W}^I := v^a \omega_{a\ J}^{I} W^J, \tag{2.69}$$

that is, at each space-time point $p \in M$ "eats" a vector $W^I(p) \in V_p$ on the fiber over p and "spits out" a new vector $\tilde{W}^I$. Although $\omega_{a\ J}^{I}$ depend on coordinates but these linear transformations are

[1] Note that matrices are vectors so we can construct "matrix bundles" over M and think of $\omega_{a\ J}^{I}$ as a (local) section of these bundles.

coordinate independent.

Locally $\omega_{a\ J}^{I}$ determines the covariant derivative D_a and therefore the connections D, usually (we do it also in this thesis!) $\omega_{a\ J}^{I}$ is ‘loosely’ referred as the *connection*.[1]

Example: In Yang-Mills gauge theories $\omega_{a\ J}^{I}$ is called “vector potential” and is denoted by $A_{a\ J}^{I}$. You may have seen A_a^i instead of A_a or $A_{a\ J}^{I}$ that are matrix valued one-forms and since matrices are vector spaces, so we can choose a “basis” of matrices τ_i and expand,

$$A_a = A_a^i \tau_i.$$

In general, there are many different connections D_a but if there is additional structure in the bundle E there are also *preferred connections*. For instance:

- For tangent bundle TM there is the metric g_{ab} so we can choose as preferred connection the Levi-Civita one, ∇_a, that is a unique “torsion free” connection such that,

$$\nabla_a g_{bc} = 0 \tag{2.70}$$

  is determined by the metric. In this case $\omega_{a\ J}^{I}$ when $I,J = b,c$ are indices in TM they are the Christoffel symbols $\Gamma_{a\ c}^{\ b}$.

- In the Minkowski bundle we constructed (suggested by the tetrad and co-tetrad e_a^I)

$$E = \bigcup_{p\in M} \mathcal{M}, \tag{2.71}$$

  where each finer is a copy of Minkowski space $\mathcal{M}$). There is the metric $\eta_{IJ} = \mathrm{diag}(-1,1,1,1)$. So we can choose the preferred connections

$$D_a \eta_{IJ} = 0, \tag{2.72}$$

  which implies,

$$D_a \eta_{IJ} = \partial_a \eta_{IJ} - \omega_{a\ I}^{K} \eta_{KJ} - \omega_{a\ J}^{K} \eta_{IK} = -\omega_{a\ I}^{K} \eta_{KJ} - \omega_{a\ J}^{K} \eta_{IK} = 0. \tag{2.73}$$

  If we define $\omega_{aIJ} := \omega_{a\ J}^{K} \eta_{IK}$, the previous condition give us,

$$\boxed{\omega_{aIJ} = -\omega_{aJI}}. \tag{2.74}$$

[1] $\omega_{a\ J}^{I}$ is not a tensor in the internal indices, i.e. I,J does not transform as tensor indices. Also $\omega_{a\ J}^{I}$ lives in a section of linear transformations.

The matrix ω_{aIJ} is anti-symmetric, which implies that ω_{aIJ} is valued on the Lie algebra of the Lorentz group $\mathrm{SO}(3,1)$. That is why ω_{aIJ} is called a Lorentz connection[1]. But our bundle has more structure, there is the co-tetrad e^I_a that has mixed indices a is space-time, while I is internal. This suggest that we can extend the covariant derivative[2] D_a by using ∇_a (the Levi-Civita connection) instead of ∂_a,

$$D_a v^I = \nabla_a v^I + \omega^I_{a\,J} v^J. \tag{2.84}$$

If we additionally require that,

$$D_a e^I_b = 0 \tag{2.85}$$

or equivalently,

$$D_a e^I_b = \nabla_a e^I_b + \omega^I_{a\,J} e^J_b = 0. \tag{2.86}$$

Solving for $\omega^I_{a\,J}$ in terms of the tetrad and co-tetrad,

$$\omega^I_{a\,J} = e^{bI}\nabla_a e_{bJ}. \tag{2.87}$$

[1]Note that is not unique, there are many of these.

[2]In general we can consider ∂_a as *any connection* that we can choose appropriately depending on the system at hand, as we shall see in chapter 3, we can express $\omega^I{}_J$ completely in term of the tetrads and co-tetrads as,

$$\mathrm{d}e^I + \omega^I{}_J \wedge e^J = 0 \tag{2.75}$$

we have

$$\partial_a e^I_b - \partial_a e^I_b + \omega^I_{a\,J} e^J_b - \omega^I_{b\,J} e^J_b = 0 \tag{2.76}$$

or equivalently

$$\omega^I_{a\,J} e^J_b - \omega^I_{b\,J} e^J_b = -2\partial_{[a} e^I_{b]}. \tag{2.77}$$

Multiplying by e_{cI} and permitting spaciotemporal indices

$$\begin{aligned}
\omega^I_{a\,J} e^J_b e_{cI} - \omega^I_{b\,J} e^{Je_{cI}}_b &= -2e_{cI}\partial_{[a} e^I_{b]} && (2.78)\\
\omega^I_{c\,J} e^J_a e_{bI} - \omega^I_{a\,J} e^{Je_{bI}}_c &= -2e_{bI}\partial_{[c} e^I_{a]} && (2.79)\\
\omega^I_{b\,J} e^J_c e_{aI} - \omega^I_{c\,J} e^{Je_{aI}}_b &= -2e_{aI}\partial_{[b} e^I_{c]} && (2.80)
\end{aligned}$$

Then by summing (B.5) and (B.6), and subtracting (B.4),

$$\omega^I_{c\,J} e^J_a e_{bI} = e_{cI}\partial_{[a} e^I_{b]} - e_{bI}\partial_{[c} e^I_{a]} - e_{aI}\partial_{[b} e^I_{c]} \tag{2.81}$$

multiplying by $e^a_K e^{bL}$,

$$\omega^L_{c\,K} = e^a_K e^{bL} e_{cI}\partial_{[a} e^I_b - e^a_K \partial_{[c} e^L_{a]} - e^{bL}\partial_{[b} e_{c]K} \tag{2.82}$$

In the 3−dimensional case, by using $\omega^M_c = -\frac{1}{2}\varepsilon_L{}^{KM}\omega^L_{c\,K}$ and $\tilde{\eta}^{abc}\varepsilon_{IJK}e^K_c = 2ee^{[a}_I e^{b]}_J$ where $\tilde{\eta}^{abc}$ is the 3−dimensional tensor density with weight one.

$$\omega^M_c = -\frac{1}{2}\left(\varepsilon_L{}^{KM} e^a_K e^{bL} e_{cI}\partial_{[a} e^I_{b]} - \varepsilon_L{}^{KM} e^a_K \partial_{[c} e^L_{a]} - \varepsilon_L{}^{KM} e^{bL}\partial_{[b} e_{c]K}\right) \tag{2.83}$$

This is what we called the *spin connection*. The spin Lorentz connection a connection that satisfies both,

$$\boxed{D_a\eta_{IJ} = 0 \quad \text{and} \quad D_a e^I_b = 0}. \tag{2.88}$$

The requirement of having a Lorentz connection is also related to the fact that there is additional structure. The Lorentz group "acts" on each finer,

$$v^I \to \Lambda^I{}_J v^J \qquad \Lambda^I{}_J \in SO(3,1). \tag{2.89}$$

Also, *local gauge transformations* are,

$$v^I(p) \to \Lambda^I{}_J(p)\, v^J(p). \tag{2.90}$$

Our Minkowski bundle is an example of a $\mathrm{SO}(3,1)-$bundle. We also want to relate vectors obtained from Lorentz (gauge) transformations, and connections must transform accordingly. If $v^I \to \tilde{v}^I := \Lambda^I{}_J v^J$ then $\omega^I_{a\,J}$ must transform $\omega^I_{a\,J} \to \tilde{\omega}^I_{a\,J}$ such that,

$$\tilde{D}_a\tilde{v}^I = \tilde{D}_a(\Lambda^I{}_J v^J) = \Lambda^I{}_J \tilde{D}_a v^J. \tag{2.91}$$

By demanding the previous relation, we shall see how $\omega^I_{a\,J}$ transforms under Lorentz transformations,

$$\begin{aligned}
\tilde{D}_a\tilde{v}^I &= \partial_a\tilde{v}^I + \tilde{\omega}^I_{a\,J}\tilde{v}^J \\
&= \partial_a(\Lambda^I{}_J v^J) + \tilde{\omega}^I_{a\,J}(\Lambda^I{}_J v^J) \\
&= \Lambda^I{}_J\partial_a v^J + \left(\partial_a\Lambda^I{}_K + \tilde{\omega}^I_{a\,J}\Lambda^J{}_K\right) v^K \\
&= \Lambda^I{}_J \left[\partial_a v^J + \underbrace{\Lambda_L{}^J\left(\partial_a\Lambda^L{}_K + \tilde{\omega}^L_{a\,J}\Lambda^J{}_K\right)}_{\omega^J_{a\,K}} v^K\right]
\end{aligned} \tag{2.92}$$

where $\Lambda_L{}^J = \Lambda^{-1}$, which implies that $\omega^J_{a\,K}$ transform as,

$$\omega^J_{a\,K} = \Lambda_L{}^J\partial_a\Lambda^L{}_K + \Lambda_L{}^J\tilde{\omega}^L_{a\,J}\Lambda^J{}_K \tag{2.93}$$

which can be written as,

$$\omega_a = \Lambda^{-1}\tilde{\omega}_a\Lambda + \Lambda^{-1}\partial_a\Lambda. \tag{2.94}$$

A Lorentz transformation on ω can be written as,

$$\boxed{\omega_a \to \tilde{\omega}_a = \Lambda\omega_a\Lambda^{-1} + \Lambda\partial_a\Lambda^{-1}}. \tag{2.95}$$

- In Yang-Mills theories we have a gauge Lie group G (some matrix group $\mathrm{U}(1)$, $\mathrm{SU}(2)$, $\mathrm{SU}(2)$,... etc) acting on the fiber (gauge transformations),

$$\phi(x) \to g(x)\phi(x), \tag{2.96}$$

so we have $G-$bundles. In this case we can choose as preferred connections the $A_a{}^I{}_J$ that take values on the Lie algebra of G and transforms as:

$$A_a \to \tilde{A}_a = gA_ag^{-1} + g\partial_a g^{-1} \tag{2.97}$$

so that,

$$D_a(g\phi) = gD_a\phi. \tag{2.98}$$

2.4 Action principle

In this section we review the action principle that plays a fundamental role in the formulation of physical theories. In order to do that we need to be precise about what it means to have a well posed variational principle. In particular, there are two aspects to it. The first one is to define the action by itself. This is done in the first part of this section. In the second part, we introduce the variational principle that states that physical configurations will be those that make the action stationary. In particular, we entertain the possibility that the spacetime region under consideration has non-trivial boundaries and that the allowed field configurations are allowed to vary on these boundaries. These new features require an extension of the standard, textbook, treatment. This section is based on [27; 28]

2.4.1 The Action

In particle mechanics the dynamics is specified by some action, which is a *function* of the trajectories of the particle. In turn, the action S is the time integral of the Lagriangian function L that generically depends on the coordinates and velocities of the particles. In field theory the dynamical variables, the fields, are geometrical objects defined on spacetime; now the Lagrangian has as domain this function space. In both cases, this type of objects are known as functionals. In order to properly define the action we will review what is a functional and some of its relevant properties.

A *functional* is a map from a *normed space* (a vector space with a non-negative real-valued norm [1]) into its underlying field, which in physical applications is the field of the real numbers.

[1] We need the concept of the norm of a functional to have a notion of closedness and therefore continuity and differentiability, for more details see e.g. chapter 23 of [41].

This vector space is normally a functional space, which is why sometimes a functional is considered as a function of a function.

A special class of functionals are the definite integrals that define an action by an expression of the form,

$$S[\phi] = \int_{\mathcal{M}} \mathcal{L}(\phi^\alpha, \nabla\phi^\alpha, ..., \nabla^n\phi^\alpha)\, \mathrm{d}^4V, \tag{2.99}$$

where $\phi^\alpha(x)$ are fields on spacetime, $\widetilde{\mathcal{M}}$, $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$ is a spacetime region, α is an abstract label for spacetime and internal indices, $\nabla\phi^\alpha$ their first derivatives, and $\nabla^n\phi^\alpha$ their n^{th} derivatives, and d^4V a volume element on spacetime. This integral $S[\phi]$ maps a field history $\phi^\alpha(x)$ into a real number if the Lagrangian density $\mathcal{L}$ is real-valued.

Prior to checking the well posedness of this action, we will review what it means for an action to be finite and differentiable. We say that an action is *finite* iff the integral that defines it is convergent or has a finite value when evaluated in *histories* compatible with the boundary conditions.

2.4.2 Differentiability and the variational principle

As the minimum action principle states, the classical trajectories followed by the system are those for which the action is a stationary point. This means that, to first order, the variations of the action vanish. As is well known, the origin of this emphasis on extremal histories comes from the path integral formalism where one can show that trajectories that extremise the action contribute the most to the path integral. First, let us consider some definitions:

Let $\mathcal{F}$ be a normed space of function. A functional $F : \mathcal{F} \to \mathbb{R}$ is called differentiable if we can write the finite change of the action, under the variation $\phi \to \phi + \delta\phi$, as

$$F[\phi + \delta\phi] - F[\phi] = \delta F + R, \tag{2.100}$$

where $\delta\phi \in \mathcal{F}$ (we are assuming here that vectors $\delta\phi$ belong to the space $\mathcal{F}$, so it is a linear space). The quantity $\delta F[\phi, \delta\phi]$ depends linearly on $\delta\phi$, and $R[\phi, \delta\phi] = \mathcal{O}((\delta\phi)^2)$. The linear part of the increment, δF, is called the *variation of the funcional F* (along $\delta\phi$). A stationary point $\bar{\phi}$ of a differentiable funcional $F[\phi]$ is a function $\bar{\phi}$ such that $\delta F[\bar{\phi}, \delta\phi] = 0$ for all $\delta\phi$.

As is standard in theoretical physics, we begin with a basic assumption: The dynamics is specified by an action. In most field theories the action depends only on the fundamental fields and their first derivatives. Interestingly, this is not the case for the Einstein Hilbert action of general relativity, but it *is* true for first order formulations of general relativity, which is the case that we shall analyze in the present work.

In general, we can define an action on a spacetime region $\mathcal{M}$ depending on the fields, ϕ^α and their first derivatives, $\nabla_\mu \phi^\alpha$. Thus, we have

$$S[\phi^\alpha] = \int_{\mathcal{M}} \mathcal{L}\left(\phi^\alpha, \nabla_\mu \phi^\alpha\right) \mathrm{d}^4 V \,. \tag{2.101}$$

Its variation δS is the linear part of

$$\int_{\mathcal{M}} \left[\mathcal{L}\left(\phi'^\alpha, \nabla_\mu \phi'^\alpha\right) - \mathcal{L}\left(\phi^\alpha, \nabla_\mu \phi^\alpha\right)\right] \mathrm{d}^4 V \,, \tag{2.102}$$

where $\phi'^\alpha = \phi^\alpha + \delta\phi^\alpha$. It follows that

$$\delta S[\phi^\alpha] = \int_{\mathcal{M}} \left[\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi^\alpha)}\right] \delta\phi^\alpha \, \mathrm{d}^4 V + \int_{\mathcal{M}} \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi^\alpha)} \delta\phi^\alpha\right) \mathrm{d}^4 V \,, \tag{2.103}$$

where we have integrated by parts to obtain the second term. Let us denote the integrand of the first term as: $E_\alpha := \frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi^\alpha)}\right)$. Note that the second term on the right hand side is a divergence so we can write it as a boundary term using Stokes' theorem,

$$\int_{\partial\mathcal{M}} \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi^\alpha)} \delta\phi^\alpha \, \mathrm{d}S_\mu =: \int_{\partial\mathcal{M}} \theta(\phi^\alpha, \nabla_\mu \phi^\alpha, \delta\phi^\alpha) \, \mathrm{d}^3 v \,, \tag{2.104}$$

where we have introduced the quantity θ that will be relevant in sections to follow. Note that the quantity $\delta S[\phi^\alpha]$ can be interpeted as the directional derivative of the funtion(al) S along the vector $\delta\phi$. Let us introduce the simbol $\mathbf{d}$ to denote the exterior derivative on the functional space $\mathcal{F}$. Then, we can write $\delta S[\phi] = \mathbf{d}\, S(\delta\phi) = \delta\phi(S)$, where the last equality employs the standard convention of representing the vector field, $\delta\phi$, acting on the function S.

As we mentioned before, if we want to derive in a consistent way the equations of motion for the system, the action must be differentiable. In particular, this means that we need the boundary term (2.104) to be zero. To simply demand that $\delta\phi^\alpha|_{\partial\mathcal{M}} = 0$, as is usually done in introductory textbooks, becomes too restrictive if we want to allow *all* the variations $\delta\phi^\alpha$ which preserve appropiate boundary conditions and not just variations of compact support. Thus, requiring the action to be stationary with respect to all compatible variations should yield precisely the classical equations of motion, with the respective boundary term vanishing on any allowed variation.

Let us now consider the case in which the spacetime region $\mathcal{M}$, where the action is defined, has a boundary $\partial\mathcal{M}$. We are interesting in globally hyperbolic asymptotically flat spacetimes (so that $\widetilde{\mathcal{M}} \approx \mathbb{R} \times M$, where M is a space-like non-compact hypersurface) possibly with an internal boundary, as would be the case when there is a black hole present. We can foliate the asymptotic region by time-like hyperboloids $\mathcal{H}_\rho$, corresponding to $\rho = $ const., and introduce a family of spacetime regions $\{\mathcal{M}_\rho\}_{\rho \in I \subset \mathbb{R}}$, with a boundary $\partial\mathcal{M}_\rho = M_1 \cup M_2 \cup \mathcal{H}_\rho \cup \Delta$, where Δ is an inner boundary (see Fig.2.7). This family satisfy $\mathcal{M}_\rho \subset \mathcal{M}'_\rho$ for $\rho' > \rho$ and $\mathcal{M} = \cup_\rho \mathcal{M}_\rho$. Then, the integral over $\mathcal{M}$ in (2.101) is defined as

$$S[\phi^\alpha] = \lim_{\rho\to\infty} \int_{\mathcal{M}_\rho} \mathcal{L}\left(\phi^\alpha, \nabla_\mu \phi^\alpha\right) \mathrm{d}^4V\,. \tag{2.105}$$

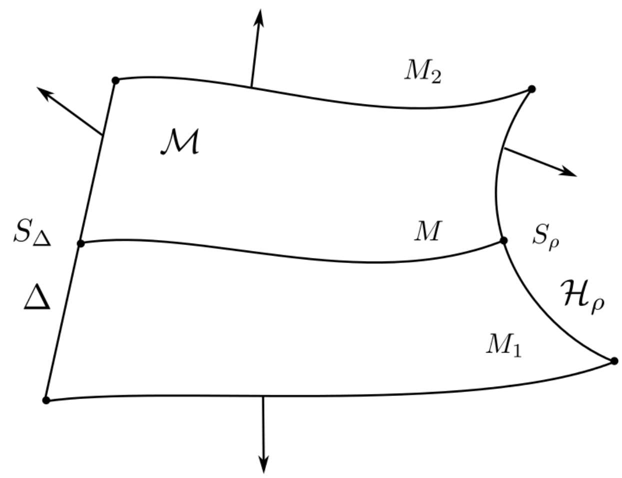

Figure 2.6: The region $\mathcal{M}_\rho$.

Now, given an action principle and boundary conditions on the fields, a natural question may arise, on whether the action principle will be well posed. So far there is no general answer, but there are examples where the introduction of a boundary term is needed to make the action principle well defined, as we shall show in the examples below. Let us then keep the discussion open and consider a generic action principle that we assume to be well defined in a region with boundaries, and with possible contributions to the action by boundary terms. Therefore, the action of such a well posed variational principle will look like,

$$S[\phi^\alpha] = \int_{\mathcal{M}} \mathcal{L}\left(\phi^\alpha, \nabla_\mu \phi^\alpha\right) \mathrm{d}^4V + \int_{\partial\mathcal{M}} \varphi(\phi^\alpha, \nabla_\mu \phi^\alpha)\, \mathrm{d}^3v\,, \tag{2.106}$$

where we have considered the possibility that there is contribution to the action coming from the boundary $\partial\mathcal{M}$. Thus, the variation of this *extended* action becomes,

$$\delta S[\phi^\alpha] = \int_{\mathcal{M}} E_\alpha\, \delta\phi^\alpha\, \mathrm{d}^4V + \int_{\partial\mathcal{M}} \theta(\phi^\alpha, \nabla_\mu\phi^\alpha, \delta\phi^\alpha)\, \mathrm{d}^3v + \int_{\partial\mathcal{M}} \delta\varphi(\phi^\alpha, \nabla_\mu\phi^\alpha)\, \mathrm{d}^3v\,. \tag{2.107}$$

The action principle will be well posed if the first term is finite and $\varphi(\phi^\alpha)$ is a boundary term that makes the action well defined under appropriate boundary conditions. That is, when the action is evaluated along *histories* that are compatible with the boundary conditions, the numerical value of the integral should be finite, and in the variation (2.107), the contribution from the boundary terms must vanish. Now, asking $\delta S[\phi^\alpha] = 0$, for arbitrary variations $\delta\phi$ of the fields, implies that the fields must satisfy

$$E_\alpha = 0\,,$$

the *Euler-Lagrange* equations of motion.

Note that in the "standard approach", i.e. when one simply considers variations, say, of compact support such that $\delta\phi^\alpha|_{\partial\mathcal{M}} = 0$, we can always add a term of the form $\nabla_\mu\chi^\mu$ to the Lagrangian density,

$$\mathcal{L} \to \mathcal{L} + \nabla_\mu\chi^\mu\,, \tag{2.108}$$

with χ arbitrary. The relevant fact here is that this term will *not* modify the equations of motion since the variation of the action becomes,

$$\delta S = \delta\int_{\mathcal{M}} \mathcal{L}\ \mathrm{d}^4V + \delta\int_{\mathcal{M}} \nabla_\mu\chi^\mu\ \mathrm{d}^4V = \delta\int_{\mathcal{M}} \mathcal{L}\ \mathrm{d}^4V + \int_{\partial\mathcal{M}} \delta\chi^\mu\ \mathrm{d}S_\mu\,, \tag{2.109}$$

thus, by the boundary conditions, $\delta\phi^\alpha|_{\partial\mathcal{M}} = 0$, the second term of the right-hand side vanishes, that is, $\delta\chi^\mu|_{\partial\mathcal{M}} = 0$. Therefore, it does not matter which boundary term we add to the action; it will not modify the equations of motion.

On the contrary, when one considers variational principles of the form (2.106), consistent with arbitrary (compatible) variations in spacetime regions with boundaries, we cannot just add arbitrary total divergences/boundary term to the action, but only those that preserve the action principle well-posedness, in the sense mentioned before. Adding to the action any other term that does not satisfy this condition will spoil the differentiability properties of the action and, therefore, one would not obtain the equations of motion in a consistent manner.

This concludes our review of the action principle. Let us now recall how one can get a consistent covariant Hamiltonian formulation, once the action principle at hand is well posed.

2.5 Covariant Hamiltonian Formalism and conserved charges

In this section we give a self-contained review of the *covariant Hamiltonian formalism* (CHF) taking special care of the cases where boundaries are present. It contains three parts. In the first one, we introduce the relevant structure in the definition of the covariant phase space, starting from the action principle. In particular, we see that boundary terms that appear in the 'variation' of the action are of particular relevance to the construction of the symplectic structure. We shall pay special attention to the presence of boundary terms in the original action and how that gets reflected in the Hamiltonian formulation. We prove the first result of this thesis. In the second part, we recall the issue of symmetries of the theory. That is, when there are certain symmetries of the underlying spacetime, these get reflected in the Hamiltonian formalism. Of particular relevance is the construction of the corresponding conserved quantities, that are both conserved and play an important role of being the generators of such symmetries. In particular we focus our attention on the symmetries generated by certain vector fields, closely related to the issue of diffeomorphism invariance. In the third part we compare and contrast these Hamiltonian conserved quantities with the so-called Noether symmetries and charges. We show how they are related and comment on the fact that, contrary to the Hamiltonian charges, the corresponding 'Noetherian' quantities *do* depend on the existence of boundary terms in the original action.

2.5.1 Covariant Phase Space

In this part we shall introduce the relevant objects that define the covariant phase space. If the theory under study has a *well posed initial value formulation*, then, given the initial data we have a unique solution to the equations of motion. In this way we have an isomorphism I between the space of solutions to the equations of motion, Γ, and the space of all valid initial data, the 'canonical phase space' $\tilde{\Gamma}$. In this even dimensional space we can construct a nondegenerate, closed $2-$form $\tilde{\Omega}$, the symplectic form. Together, the phase space and the symplectic form constitute a symplectic manifold $(\tilde{\Gamma}, \tilde{\Omega})$.

We can bring the symplectic structure to the space of solutions, via the pullback I^* of $\tilde{\Omega}$ and define a corresponding 2-form on Γ. In this way the space of solutions is equipped with a *natural* symplectic form, Ω, since the mapping is independent of the reference Cauchy surface one is using to define I. Together, the space of solutions and its symplectic structure (Γ, Ω) are known as the *covariant phase space* (CPS).

However, most of the field theories of interest present *gauge* symmetries. This fact is reflected on the symplectic form Ω, making it *degenerate*. When this is the case, Ω is only a pre-symplectic form, to emphasize the degeneracy. It is only after one gets rid of this degeneracy, by means of an appropriate quotient, that one recovers a physical non-degenerate symplectic structure. Let us now see how one can arrive to such description from the action principle.

Before proceeding we shall make some remarks regarding notation. It has proved to be useful to use differential forms to deal with certain diffeomorphism invariant theories, and we shall do that here. However, when working with differential forms in field theories one has to distinguish between the exterior derivative $\mathbf{d}$ in the infinite dimensional covariant phase space, and the 'standard' exterior derivative on the spacetime manifold, denoted by d. In this context, differential forms in the CPS act on vectors tangent to the space of solutions Γ. We use δ or $\delta\phi$ to denote these tangent vectors to , to be consistent with the standard notation used in the literature. We hope that no confusion should arise by such a choice. Let us now recall some basic constructions on the covariant phase space.

Taking as starting point an action principle,

$$S[\phi^A] = \int_{\mathcal{M}} \mathbf{L}\,, \tag{2.110}$$

where the Lagrangian density, $\mathbf{L}$, is a $4-$form, that depends on fields ϕ^A and their derivatives. The fields ϕ^A are certain $n-$forms (with $n \leq 4$) in the $4-$dimensional spacetime manifold, $\mathcal{M}$, with boundary, $\partial\mathcal{M}$, and $A, B, \ldots$ are internal indices. Then, the variation of the action can be written as[1],

$$\mathbf{d}S(\delta) = \delta S = \int_{\mathcal{M}} E_A \wedge \delta\phi^A + \int_{\mathcal{M}} \mathrm{d}\theta(\delta\phi^A)\,, \tag{2.111}$$

where E_A are the Euler-Lagrange equations of motion forms and $\delta\phi^A$ is an arbitrary vector on the CPS, that can be thought to point 'in the direction that ϕ^A changes'. The 1-form (in CPS) θ depends on ϕ^A, $\delta\phi^A$ and their derivatives, for simplicity we do not write it explicitly. Note that we are using $\delta\phi^A$ and δ, to denote *the same* object[2]. For simplicity in the notation, sometimes the ϕ^A part is dropped out. Here we wrote both for clarity. The second term of the RHS is obtained after integration by parts, and using Stokes' theorem it can be written as,

$$\Theta(\delta\phi^A) := \int_{\mathcal{M}} \mathrm{d}\theta(\delta\phi^A) = \int_{\partial\mathcal{M}} \theta(\delta\phi^A)\,. \tag{2.112}$$

This term can be seen as a $1-$form in the covariant phase space, acting on vectors $\delta\phi^A$ and returning a real number. Also it can be seen as a potential for the symplectic structure, that we already mentioned in the preamble of this section and shall define below. For such a reason, we will call this term, $\Theta(\delta\phi^A)$ a *symplectic potential* associated to a boundary $\partial\mathcal{M}$, and the integrand, $\theta(\delta\phi^A)$, is the *symplectic potential current*[3].

Note that from Eqs. (2.111) and (2.112), in the space of solutions $E_A = 0$, $\mathbf{d}S = \Theta(\delta\phi^A)$.

If we want to derive in a consistent way the equations of motion for the system, the action must be differentiable. In particular, this means that we need the boundary term (2.112) to be zero. To simply demand that $\delta\phi^A|_{\partial\mathcal{M}} = 0$, becomes too restrictive if we want to allow *all* the variations which preserve appropriate boundary conditions and not just variations of compact support. Thus,

[1]This notation is a little bit unusual, so we shall clarify a little bit more. Here we are considering two spaces: the space-time $\mathcal{M}$ and the infinite dimensional covariant phase space CPS, we can define exterior derivatives in both, d and $\mathbf{d}$, and wedge products $\wedge$ and $\bigwedge$ respectively. In equation (2.111), S is a functional, when we applied $\mathbf{d}$, $\mathbf{d}S$ it becomes a 1-form in CPS, then when evaluated on a vector field in CPS, δ, $\mathbf{d}S(\delta)$ is again a functional, also this is equal to δS, the vector field acting on the functional, as it is standard: $\mathrm{d}f(X) = Xf$, where f is a function, X a vector field and d the exterior derivative. In the right-hand side of eq. (2.111), $\delta\phi^A$ is a vector on CPS, but since ϕ^a are certain $n-$forms (with $n \leq 4$) in the $4-$dimensional space-time manifold, $\delta\phi^A$ is also a $n-$form in the space-time, and the wedge product is that of the space-time.

[2]

[3]Usually, a symplectic potential is defined as an integral of θ over a spatial slice M, see, for example, [**?**]. Here, we are extending this definition since, as we shall show, in order to construct a symplectic structure it is important to consider the integral over the whole boundary $\partial\mathcal{M}$.

requiring the action to be stationary with respect to all compatible variations should yield precisely the classical equations of motion, with the respective boundary term vanishing on any allowed variation.

If the original action is not well defined, the introduction of a boundary term could be needed. In that case the action becomes,

$$S[\phi^A] = \int_{\mathcal{M}} \mathbf{L} + \int_{\mathcal{M}} \mathrm{d}\varphi\,, \tag{2.113}$$

where the boundary term in general depends on fields, as well as of their derivatives, and is chosen in such a way that the new action is differentiable and finite, for allowed field configurations, and we have a well posed variational principle,

$$\delta S = \int_{\mathcal{M}} E_A \wedge \delta\phi^A + \int_{\mathcal{M}} \mathrm{d}\left[\theta(\delta\phi^A) + \delta\varphi\right]. \tag{2.114}$$

When we have added a boundary term, the symplectic potential associated to this well posed action changes as $\Theta \to \Theta + \int_{\mathcal{M}} \mathrm{d}\delta\varphi$, equivalently we can consider,

$$\tilde{\Theta}(\delta) := \int_{\partial\mathcal{M}} [\theta(\delta) + \delta\varphi]\,. \tag{2.115}$$

From this equation we can see that besides the boundary term added to the action, to make it well defined, we can always add a term, $\mathrm{d}Y$, to the symplectic potential current that will not change $\tilde{\Theta}$. Thus, the most general symplectic potential can be written as,

$$\tilde{\Theta}(\delta) = \int_{\partial\mathcal{M}} [\theta(\delta) + \delta\varphi + \mathrm{d}Y(\delta)] =: \int_{\partial\mathcal{M}} \tilde{\theta}(\delta)\,. \tag{2.116}$$

Now, we take the exterior derivative of the symplectic potential, $\tilde{\Theta}$, acting on tangent vectors δ_1 and δ_2 at a point γ of the phase space,

$$\mathbf{d}\,\tilde{\Theta}(\delta_1,\delta_2) = \delta_1\tilde{\Theta}(\delta_2) - \delta_2\tilde{\Theta}(\delta_1) = 2\int_{\partial\mathcal{M}} \delta_{[1}\tilde{\theta}(\delta_{2]})\,. \tag{2.117}$$

From this expression we can define a spacetime $3-$form, *the symplectic current* $\tilde{J}(\delta_1,\delta_2)$, to be the integrand of the RHS of (2.117),

$$\tilde{J}(\delta_1,\delta_2) := \delta_1\tilde{\theta}(\delta_2) - \delta_2\tilde{\theta}(\delta_1)\,. \tag{2.118}$$

In particular, when we have added a boundary term to the action, and taking into account the ambiguities, the symplectic current becomes,

$$\tilde{J}(\delta_1,\delta_2) = J(\delta_1,\delta_2) + 2\big(\delta_{[1}\delta_{2]}\varphi + \delta_{[1}\mathrm{d}Y(\delta_{2]})\big)\,. \tag{2.119}$$

where

$$J(\delta_1,\delta_2) := \delta_1\theta(\delta_2) - \delta_2\theta(\delta_1)\,, \tag{2.120}$$

is the symplectic current associated to the action (2.110).

Now, the term $\delta_{[1}\delta_{2]}\varphi$ vanishes by antisymmetry, because δ_1 and δ_2 commute when acting on functions. Note that the last term of the RHS of (2.119) can be written as $\mathrm{d}\chi(\delta_1,\delta_2) = 2\delta_{[1}\mathrm{d}Y(\delta_{2]})$ due to d and δ_i commuting. Since d and $\mathbf{d}$ act on different spaces, the spacetime and the space of fields, respectively, they are independent. In this way $\tilde{J}(\delta_1,\delta_2)$ is determined as

$$\tilde{J}(\delta_1,\delta_2) = J(\delta_1,\delta_2) + \mathrm{d}\chi(\delta_1,\delta_2)\,. \tag{2.121}$$

This ambiguity will appear explicitly in the examples that we shall consider below.

Therefore we conclude that, *when we add a boundary term to the original action it will not change the symplectic current*, and this result holds independently of the specific boundary conditions. This is the first result of this thesis.

Recall that in the space of solutions, $\mathbf{d}S(\delta) = \tilde{\Theta}(\delta)$, therefore from eqs. (2.117) and (2.118),

$$0 = \mathbf{d}^2 S(\delta_1,\delta_2) = \mathbf{d}\tilde{\Theta}(\delta_1,\delta_2) = 2\int_{\mathcal{M}} \delta_{[1}\mathrm{d}\tilde{\theta}(\delta_{2]}) = \int_{\mathcal{M}} \mathrm{d}\tilde{J}(\delta_1,\delta_2). \tag{2.122}$$

Since we are integrating over *any* region $\mathcal{M}$, we can conclude that $\tilde{J}$ is closed, i.e. $\mathrm{d}\tilde{J} = 0$. Note that $\mathrm{d}\tilde{J} = \mathrm{d}(J + \mathrm{d}\chi) = \mathrm{d}J$ depends only on θ. Using Stokes' theorem, and taking into account the orientation of $\partial\mathcal{M}$ (see Fig. 2.7), we have

$$0 = \int_{\mathcal{M}} \mathrm{d}\tilde{J}(\delta_1,\delta_2) = \int_{\mathcal{M}} \mathrm{d}J(\delta_1,\delta_2) = \oint_{\partial\mathcal{M}} J(\delta_1,\delta_2) = \left(-\int_{M_1} + \int_{M_2} - \int_{\Delta} + \int_{\mathcal{I}}\right) J, \tag{2.123}$$

where $\mathcal{M}$ is bounded by $\partial\mathcal{M} = M_1 \cup M_2 \cup \Delta \cup \mathcal{I}$, M_1 and M_2 are space-like slices, Δ is an inner boundary and $\mathcal{I}$ an outer boundary.

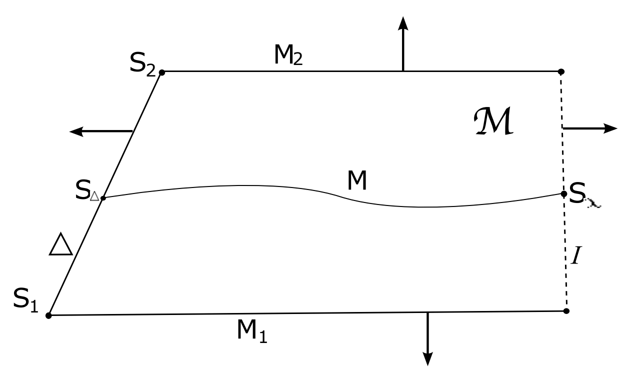

Figure 2.7: The region $\mathcal{M}$.

Now consider the following two possible scenarios: First, consider the case when there is no internal boundary, only a boundary $\mathcal{I}$ at infinity. In some instances the asymptotic conditions ensure that the integral $\int_{\mathcal{I}} J$ vanishes, in which case from (2.123), one gets

$$\oint_{\partial \mathcal{M}} J(\delta_1, \delta_2) = \left(- \int_{M_1} + \int_{M_2} \right) J = 0\,, \tag{2.124}$$

which implies that $\int_M J$ is independent of the Cauchy surface. This allows us to define a *conserved* pre-symplectic form over an arbitrary space-like surface M,

$$\bar{\Omega}(\delta_1, \delta_2) = \int_M J(\delta_1, \delta_2)\,. \tag{2.125}$$

Note that in (2.123) at the end we only have a contribution from J, not from the complete $\tilde{J}$, and for that reason the pre-symplectic form does not depend on φ (the contribution of the topological, total derivative, terms in the action) nor χ (the contribution of total derivative terms in $\tilde{J}$).

One should have special care in the case when the symplectic current is of the form $J = J_0 + \mathrm{d}\alpha$, as we shall now demonstrate. Our previous arguments, see (2.121) and (2.123), show that the $\mathrm{d}\alpha$ term does not appear in the symplectic structure. It follows then that, when $J_0 = 0$, the symplectic structure is trivial $\bar{\Omega} = 0$, by construction, so that in the definition (2.125), it is only the J_0 part of J that contributes to $\bar{\Omega}$. It should be obvious that this conclusion is valid also in the case when there is an internal boundary Δ. We shall further comment on this case below.

Let us now consider with more details the case when we have an internal boundary. Now, the integral $\int_\Delta J$ may no longer vanish under the boundary conditions, as is the case with the isolated horizon boundary conditions (more about this below). The "next best thing" is that this integral is "controllable". Let us be more specific. If, after imposing boundary conditions, the integral takes the form,

$$\int_\Delta J = \int_\Delta \mathrm{d}j = \int_{\partial\Delta} j\,, \tag{2.126}$$

we can still define a *conserved* pre-symplectic structure. From (2.123), and assuming the integral over the outer boundary vanishes, we now have

$$\left(- \int_{M_1} + \int_{M_2} - \int_\Delta \right) J = \left(- \int_{M_1} + \int_{M_2} \right) J - \left(\int_{S_1} - \int_{S_2} \right) j = 0\,, \tag{2.127}$$

where S_1 and S_2 are the intersections of space-like surfaces M_1 and M_2 with the inner boundary Δ, respectively. Therefore we can define the *conserved* pre-symplectic structure as,

$$\bar{\Omega} = \int_M J + \int_S j\,. \tag{2.128}$$

Note that by construction, the two form $\bar{\Omega}$ is closed, so it is justified to call it a (pre-)symplectic structure.

Let us end this section by further commenting on the case when the symplectic current contains a total derivative. In the literature, the symplectic structure is sometimes defined, from the beginning, as an integral of $\tilde{J}$ over a spatial hypersurface M, but we have shown that this is correct only if $\tilde{J}$ does not have a total derivative term, and the action does not have a boundary term. Let us now describe the argument that one sometimes encounters in this context, in the simple case where $J = \mathrm{d}\alpha$. In this case one could *postulate* the existence of a pre-symplectic structure $\tilde{\Omega}_M$ as follows. Let us define

$$\tilde{\Omega}_M(\delta_1, \delta_2) := \int_M \mathrm{d}\alpha(\delta_1, \delta_2)\,, \tag{2.129}$$

we have, from (2.123) that $\tilde{\Omega}_M$ is independent on M only if $\int_{\mathfrak{I}} \mathrm{d}\alpha$ *and* $\int_{\Delta} \mathrm{d}\alpha$ vanish. In this case the object $\tilde{\Omega}_M$ does define a conserved two-form that satisfies the definition of a pre-symplectic structure. It should be stressed though that such an object does not follow from the systematic derivation we followed, starting from an action principle. It can instead be viewed and a possible freedom that exists in the covariant Hamiltonian formalism. It is indeed interesting to explore the possible physical consequences of introducing the object $\tilde{\Omega}_M$. As we shall show in forthcoming sections, there is one instance in which one can postulate such a two-form, that satisfies the conditions for being conserved, but as we shall show in detail, one does run into inconsistencies when postulating such object for topological terms.

To summarize, in this part we have developed in detail the covariant Hamiltonian formalism in the presence of boundaries. As we have seen, there might be a contribution to the (pre-)symplectic structure coming from the boundaries. Finally, we have shown that the addition of boundary terms to the action does not modify the conserved (pre-)symplectic structure of the theory, independently of the boundary conditions imposed. This is the first result of this note. As a remark, we have also noted that under certain circumstances, one could introduce a conserved symplectic structure that results from the existence of a total derivative term in the symplectic current.

2.5.2 Symmetries and conserved charges

Let us now explore how the covariant Hamiltonian formulation can deal with the existence of symmetries, and their associated conserved quantities. Before that, let us recall the standard notion of a Hamiltonian vector field (HVF) in Hamiltonian dynamics. A Hamiltonian vector field Z is defined as a symmetry of the symplectic structure, namely

$$\pounds_Z \Omega = 0. \tag{2.130}$$

From this condition and the fact that $\mathbf{d}\Omega = 0$ we have,

$$\pounds_Z \Omega = Z \cdot \mathbf{d}\Omega + \mathbf{d}(Z \cdot \Omega) = \mathbf{d}(Z \cdot \Omega) = 0. \tag{2.131}$$

where $Z \cdot \Omega \equiv i_Z \Omega$ means the *contraction* of the 2-form Ω with the vector field Z. Note that $(Z \cdot \Omega)(\delta) = \Omega(Z, \delta)$ is a one-form on Γ acting on an arbitrary vector δ. We can denote it as $X(\delta) := \Omega(Z, \delta)$. From the previous equation we can see that $X = Z \cdot \Omega$ is closed, $\mathbf{d}X = 0$. It follows from (2.131) and from the Poincaré lemma that locally (on the CPS), there exists a function H such that $X = \mathbf{d}H$. We call this function, H, the *Hamiltonian*, that generates the infinitesimal canonical transformation defined by Z. Furthermore, and by its own definition, H is a *conserved quantity* along the flow generated by Z.

Note that the directional derivative of the Hamiltonian H, along an arbitrary vector δ can be written in several ways,

$$X(\delta) = \mathbf{d}H(\delta) = \delta H, \tag{2.132}$$

some of which will be used in-distinctively in what follows.

So far this vector field Z is an arbitrary Hamiltonian vector field on Γ. Later on we will relate it to certain space-time symmetries. For instance, for field theories that possess a symmetry group, such as the Poincaré group for field theories on Minkowski spacetime, there will be corresponding Hamiltonian vector fields associated to the generators of the symmetry group. In this thesis we are interested in exploring gravity theories that are diffeomorphism invariant. That is, such that the diffeomorphism group acts as a (kinematical) symmetry of the action. Of particular relevance is then to understand the role that these symmetries have in the Hamiltonian formulation. In particular, one expects that diffeomorphisms play the role of *gauge* symmetries of the theory. However, the precise form in which diffeomorphisms can be regarded as gauge or not, depends on the details of the theory, and is dictated by the properties of the corresponding Hamiltonian vector fields. Another important issue is to separate those diffeomorphisms that are gauge from those that represent truly physical canonical transformations that change the system. Those true motions could then be associated to symmetries of the theory. For instance, in the case of asymptotically flat spacetimes, *some* diffeomorphism are regarded as gauge, while others represent nontrivial transformations at infinity and can be associated to the generators of the Poincaré group. In the case when the vector field Z generates time evolution, one expects H to be related to the energy, the ADM energy at infinity. Other conserved Hamiltonian charges can thus be found, and correspond to the generators of the asymptotic symmetries of the theory. In what follows we shall explore the aspects of the theory that allow us to separate the notion of gauge from standard symmetries of the theory.

2.5.2.1 Gauge and degeneracy of the symplectic structure

In the standard treatment of constrained systems, one starts out with the kinematical phase space Γ_{kin}, and there exists a constrained surface $\bar{\Gamma}$ consisting of points that satisfy the constraints present in the theory. One then notices that the pullback of Ω, the symplectic structure to $\bar{\Gamma}$ is degenerate (for first class constraints). These degenerate directions represent the gauge directions where two points are physically indistinguishable. In the covariant Hamiltonian formulation we are considering here, the starting point is the space Γ of solutions to *all* the equations of motion, where a (pre-)symplectic structure is naturally defined, as we saw before. We call this a pre-symplectic structure since it might be degenerate. We say that $\bar{\Omega}$ is degenerate if there exist vectors Z_i such that $\bar{\Omega}(Z_i, X) = 0$ for all X. We call Z_i a degenerate direction (or an element of the kernel of $\bar{\Omega}$). If $\bar{\Omega}$ is degenerate we have a gauge system, with a gauge submanifold generated by the degenerate directions Z_i (it is immediate to see that they satisfy the local integrability conditions to generate a submanifold).

Note that since we are on the space of solutions to the field equations, tangent vectors X to Γ must be solutions to the *linearized* equations of motion. Since the degenerate directions Z_i generate infinitesimal gauge transformations, configurations ϕ' and ϕ on Γ, related by such transformations, are physically indistinguishable. That is, $\phi' \sim \phi$ and, therefore, the quotient $\hat{\Gamma} = \Gamma/\sim$ constitutes the physical phase space of the system. It is only in the reduced phase space $\hat{\Gamma}$ that one can define a non-degenerate symplectic structure Ω.

In the next subsection we explain how vector fields are the infinitesimal generators of transformations on the space-time in general. Then we will point out when these transformations are diffeomorphisms and moreover, when these are also gauge symmetries of the system.

2.5.2.2 Diffeomorphisms and Gauge

Let us start by recalling the standard notion of a diffeomorphism on the manifold $\mathcal{M}$. Later on, we shall see how, for diffeomorphism invariant theories, the induced action on phase space of certain diffeomorphisms becomes gauge transformations.

There is a one-to-one relation between vector fields on a manifold and families of transformations of the manifold onto itself. Let φ be a one-parameter group of transformations on $\mathcal{M}$, the map $\varphi_\tau : \mathcal{M} \to \mathcal{M}$, defined by $\varphi_\tau(x) = \varphi(x, \tau)$, is a differentiable mapping. If ξ is the infinitesimal generator of φ and $f \in C^\infty(\mathcal{M})$, $\varphi_\tau^* f = f \circ \varphi_\tau$ also belongs to $C^\infty(\mathcal{M})$; then the Lie derivative of f along ξ, $\pounds_\xi f = \xi(f)$, represents the rate of change of the function f under the family of transformations φ_τ. That is, the vector field ξ is the generator of infinitesimal diffeomorphisms. Now, given such a vector field, a natural question is whether there exists a vector field Z_ξ on the CPS that represents the induced action of the infinitesimal diffeos? As one can easily see, the answer is in the affirmative.

In order to see that, let us go back a bit to Section 2.4. The action is defined on the space of histories (the space of all possible configurations) and, after taking the variation, the vectors $\delta\phi^\alpha$ lie on the tangent space to the space of histories. It is only after we restrict ourselves to the space of solutions Γ, that $\mathbf{d}S(\delta) = \delta S = \Theta(\delta\phi^A)$. Now these $\delta\phi^A$ represent *any* vector on $T_{\phi^A}\Gamma$ (tangent space to Γ at the point ϕ^A). As we already mentioned, these $\delta\phi^A$ can be seen as "small changes" in the fields. What happens if we want the infinitesimal change of fields to be generated by a particular group of transformations (e.g. spatial translations, boosts, rotations, etc)? There is indeed a preferred tangent vector for the kind of theories we are considering. Given ξ, consider

$$\delta_\xi \phi^A := \pounds_\xi \phi^A \,. \tag{2.133}$$

From the geometric perspective, this is the natural candidate vector field to represent the induced action of infinitesimal diffeomorphisms on Γ. The first question is whether such objects are indeed tangent vectors to Γ. It is easy to see that, *for kinematical diffeomorphism invariant theories, Lie derivatives satisfy the linearized equations of motion*.[1] Of course, in the presence of boundaries such vectors must preserve the boundary conditions of the theory in order to be admissible (more about this below). For instance, in the case of asymptotically flat boundary conditions, the allowed vector fields should preserve the asymptotic conditions.

Let us suppose that we have prescribed the phase space and pre-symplectic structure $\bar{\Omega}$, and a vector field $\delta_\xi := \pounds_\xi \phi^A$. The question we would like to pose is: when is such vector a degenerate direction of $\bar{\Omega}$? The equation that such vector δ_ξ must satisfy is then:

$$\bar{\Omega}(\delta_\xi, \delta) = 0\,, \qquad \forall\, \delta\,. \tag{2.134}$$

This equation will, as we shall see in detail below once we consider specific boundary conditions, impose some conditions on the behaviour of ξ on the boundaries. An important signature of diffeomorphism invariant theories is that Eq.(2.134) *only has contributions from the boundaries*. Thus, the vanishing of such terms will depend on the behaviour of ξ there. In particular, if $\xi = 0$ on the boundary, the corresponding vector field is guaranteed to be a degenerate direction and therefore to generate *gauge* transformations. In some instances, non vanishing vectors at the boundary also satisfy Eq. (2.134) and therefore define gauge directions.

Let us now consider the case when ξ is non vanishing on $\partial\mathcal{M}$ and Eq. (2.134) is not zero. In that case, we should have

$$\bar{\Omega}(\delta, \delta_\xi) = \mathbf{d}H_\xi(\delta) = \delta H_\xi \,, \tag{2.135}$$

for some function H_ξ. This function will be the generator of the symplectic transformation gener-

[1] See, for instance [65]. When the theory is not diffeomorphism invariant, such Lie derivatives are admissible vectors only when the defining vector field ξ is a symmetry of the background spacetime.

ated by δ_ξ. In other words, H_ξ is the *Hamiltonian conserved charge* associated to the symmetry generated by ξ.
Remark: One should make sure that Eq. (2.135) is indeed well defined, given the degeneracy of $\bar{\Omega}$. In order to see that, note that one can add to δ_ξ an arbitrary 'gauge vector' Z and the result in the same: $\bar{\Omega}(\delta_\xi + Z, \delta) = \bar{\Omega}(\delta_\xi, \delta)$. Therefore, if such function H_ξ exists (and we know that, locally, it does), it is insensitive to the existence of the gauge directions so it must be constant along those directions and, therefore, projectable to $\hat{\Gamma}$. Thus, one can conclude that even when H_ξ is defined through a degenerate pre-symplectic structure, it is indeed a physical observable defined on the reduced phase space.

This concludes our review of the covariant phase space methods and the definition of gauge and Hamiltonian conserved charges for diffeomorphism invariant theories. In the next part we shall revisit another aspect of symmetries on covariant theories, namely the existence of Noether conserved quantities, which are also associated to symmetries of field theories.

2.5.3 Diffeomorphism invariance: Noether charge

Let us briefly review some results about the Noether current 3-form J_N and its relation to the symplectic current J. For that, we shall rely on [36]. We know that to any Lagrangian theory invariant under diffeomorphisms we can associate a corresponding Noether current 3-form. Consider infinitesimal diffeomorphism generated by a vector field ξ. These diffeomorphisms induce the infinitesimal change of fields, given by $\delta_\xi \phi^A := \pounds_\xi \phi^A$. From (2.111) it follows that the corresponding change in the lagrangian four-form is given by

$$\pounds_\xi \mathbf{L} = \mathrm{E}_A \wedge \pounds_\xi \phi^A + \mathrm{d}\theta(\phi^A, \pounds_\xi \phi^A) \,. \tag{2.136}$$

On the other hand, using Cartan's formula, we obtain

$$\pounds_\xi \mathbf{L} = \xi \cdot \mathrm{d}\mathbf{L} + \mathrm{d}(\xi \cdot \mathbf{L}) = \mathrm{d}(\xi \cdot \mathbf{L}) \,, \tag{2.137}$$

since $\mathrm{d}\mathbf{L} = 0$, in a four-dimensional spacetime. Now, we can define a *Noether current 3-form* as

$$J_N(\delta_\xi) = \theta(\delta_\xi) - \xi \cdot \mathbf{L} \,, \tag{2.138}$$

where we are using the simplified notation $\theta(\delta_\xi) := \theta(\phi^A, \pounds_\xi \phi^A)$. From the equations (2.136) and (2.137) it follows that on the space of solutions, $\mathrm{d}J_N(\delta_\xi) = 0$, so at least locally one can define a corresponding Noether charge density 2-form Q_ξ relative to ξ as

$$J_N(\delta_\xi) = \mathrm{d}Q_\xi \,. \tag{2.139}$$

Following [**?**], the integral of Q_ξ over some compact surface S is the Noether charge of S relative to ξ. As we saw in the previous chapter there are ambiguities in the definition of θ (2.116) , that produce ambiguities in Q_ξ. As we saw in the section 2.4.1, θ is defined up to an exact form: $\theta \to \theta + \mathrm{d}Y(\delta)$. Also, the change in Lagrangian $\mathbf{L} \to \mathbf{L} + \mathrm{d}\varphi$ produces the change $\theta \to \theta + \delta\varphi$. These transformations affect the symplectic current in the following way

$$J(\delta_1, \delta_2) \to J(\delta_1, \delta_2) + \mathrm{d}\big(\delta_2 Y(\delta_1) - \delta_1 Y(\delta_2)\big)\,. \tag{2.140}$$

The contribution of φ vanishes, as before, and as we have shown in section 2.4.1. The above transformation leaves invariant the symplectic structure. It is easy to see that the two changes, generated by Y and φ contribute to the following change of Noether current 3-form

$$J_N(\delta_\xi) \to J_N(\delta_\xi) + \mathrm{d}Y(\delta_\xi) + \delta_\xi \varphi - \xi \cdot \mathrm{d}\varphi\,, \tag{2.141}$$

and the corresponding Noether charge 2-form changes as

$$Q_\xi \to Q_\xi + Y(\delta_\xi) + \xi \cdot \varphi + \mathrm{d}Z\,. \tag{2.142}$$

The last term in the previous expression is due to the ambiguity present in (2.139). This arbitrariness in Q_ξ was used in [**?**] to show that the Noether charge form of a general theory of gravity arising from a diffeomorphism invariant Lagrangian, in the second order formalism, can be decomposed in a particular way.

Since $\mathrm{d}J_N(\delta_\xi) = 0$ it follows, as in (2.123), that

$$0 = \int_{\mathcal{M}} \mathrm{d}J_N(\delta_\xi) = \oint_{\partial\mathcal{M}} J_N(\delta_\xi) = \left(-\int_{M_1} + \int_{M_2} - \int_{\Delta} + \int_{\mathcal{I}}\right) J_N(\delta_\xi), \tag{2.143}$$

and we see that if $\int_\Delta J_N(\delta_\xi) = \int_{\mathcal{I}} J_N(\delta_\xi) = 0$ then the previous expression implies the existence of the conserved quantity (independent on the choice of M),

$$\int_M J_N(\delta_\xi) = \int_{\partial M} Q_\xi\,. \tag{2.144}$$

Note that the above results are valid only on shell.

In the covariant phase space, and for ξ arbitrary and fixed, we have [**?**]

$$\delta J_N(\delta_\xi) = \delta\theta(\delta_\xi) - \xi \cdot \delta\mathbf{L} = \delta\theta(\delta_\xi) - \xi \cdot \mathrm{d}\theta(\delta)\,. \tag{2.145}$$

Since, $\xi \cdot \mathrm{d}\theta = £_\xi \theta - \mathrm{d}(\xi \cdot \theta)$ and $\delta\theta(\delta_\xi) - £_\xi \theta(\delta) = J(\delta, \delta_\xi)$ by the definition of the symplectic current J (2.118), it follows that the relation between the symplectic current J and the Noether current 3-form J_N is given by

$$J(\delta, \delta_\xi) = \delta J_N(\delta_\xi) - \mathrm{d}(\xi \cdot \theta(\delta)) \,. \tag{2.146}$$

We shall use this relation in the following sections, for the various actions that describe first order general relativity, to clarify the relation between the Hamiltonian and Noether charges. We shall see that, in general, a Noether charge does not correspond to a Hamiltonian charge generating symmetries of the phase space.

Chapter 3

Palatini action in a 3D asymptotically flat space time

"The most beautiful experience we can have is the mysterious. It is the fundamental emotion which stands at the cradle of true art and true science."
—Albert Einstein.

Idealized and reduced models have been useful in analyzing and studying, in a simplified arena, some aspects of the (3+1) classical and quantum geneal relativity. To be more precise, we can consider solutions to the Einstein equations that are invariant under certain symmetries. An outstanding example is the (2+1)-dimensional case which, apart from being a lot more simpler than the (3+1) one, has been solved in many different contexts and by different approaches, so we can compare our results throughout these different paths and gain some insight.

In the context of Loop Quantum Gravity, the first order formulation of General Relativity allow us to write the theory as a SU(2) connection theory (both the real or complex formulations, depending if the Barbero-Immirzi parameter is a real or complex parameter), becoming the starting point of the canonical quantization. Also it is mandatory the use of the first order formulation of General Relativity if we want to couple fermionic matter. Thus it becomes relevant to fully understand the classical aspects of this formulation. Moreover, if we want stability conditions to be satisfied in the quantum theory, we must check whether the classical hamiltonian is bounded from below. In the second order metric formulation this boundeness has been proved [11; 47], we want to check and compare it with the first order general relativity results obtained in the present work. This subject also shed some light on the discussion about the energy of Minkowski space-time.

We point out the main results of this chapter, that will be reported on [26]:

- We derive the asymptotically flat conditions for the first order variables.

- We prove that the 3-dimensional Palatini action with boundary term[1], which give us the same equations of motion that the 3-dimensional Einstein-Hilbert action, has a well posed action principle, is finite and differentiable under the asymptotically flat boundary conditions. Moreover if we introduce an additional boundary term to the action to make it explicitly Lorentz invariant we find that the resulting action is equivalent to the Einstein-Hilbert action with Gibbons-Hawking term.

- We prove that the energy is bounded from below and above, through the covariant hamiltonian formalism (CHF) of first order gravity in an asymptotically 3-dimensional flat spacetime. Agreeing with previous results in the metric variables via Regge-Teitelboim methods [11]. Although CHF provides an elegant and short derivation for the energy (and other relevant symmetries as discussed in [27]), this quantity is determined up to a constant, that shifts the region in which the energy is bounded.

- We also prove that the energy is bounded bounded from below and above with the Canonical formalism (following two different $2+1$ decompositions), but in contrast with the CHF, here there is no ambiguity in the election of the constant, the energy is given directly from the hamiltonian. Our results agree with those of [47].

- We propose a Chern-Simons action with boundary term valued on the Lie algebra of $ISO(2,1)$ that lead us to the well posed manifestly Lorentz invariant Palatini action previously introduced. And at each stage we prove that we obtain all the same relevant quantities, in particular the energy. This action may served to further study some topological aspects of the theory. But we shall leave it to forthcoming works.

3.1 Asymptotic structure: subtleties in 3 dimensions

Intuitively speaking, in $(3+1)$ dimensions we can think of an asymptotically flat spacetime as an spacetime with matter content in a bounded region outside of which the metric approaches the Minkowski metric. In the formal definition we say a smooth space-time metric g on $\mathcal{R}$ is *weakly asymptotically flat at spatial infinity* if there exist a Minkowski metric η such that, outside a spatially compact world tube, $(g-\eta)$ admits an asymptotic expansion[2] to order 1 and $\lim_{r^m\to\infty}(g-\eta)=0$.

[1]In analogy to the four dimensional case [7] and as previously introduced in the literature ....

[2]The explicit form of the expansion depends on the coordinates. In 3-dimensions and cylindrical coordinates, as we use through the present work, an asymptotic expansion to order m of a function f has the form,

$$f(r,\theta)=\sum_{n=0}^{m}\frac{{}^{n}f(\theta)}{r^n}+o(r^{-m}), \tag{3.1}$$

In a (2+1) spacetime the situation is a bit different, if we consider a mass distribution, say a point particle at the origin, $r = 0$, outside this region, $r > 0$, the metric does not approach a flat metric, *it is flat*. So, how can we define an asymptotically flat space-time?

In order to define an (2+1) asymptotically flat spacetime consider the 'closest thing' to the $(3+1)$ picture. Consider the solution of a point particle of mass M at the origin,

$$ds^2 = -dt^2 + r^{-8GM}(dr^2 + r^2 d\theta^2) \quad \text{for } r > 0 \tag{3.3}$$

where t, r, θ are the cylindrical coordinates, $t \in (-\infty, +\infty)$, $r \in [0, \infty)$, and $\theta \in [0, 2\pi)$. This metric is flat everywhere except at the origin. To see that, we can define $\rho \equiv \frac{r^\alpha}{\alpha}$, $\bar{\theta} \equiv \alpha\theta$ with $\alpha \equiv 1 - 4GM$. So the metric takes the form,

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\bar{\theta}^2, \tag{3.4}$$

from which the flatness of the metric is apparent. This is due to the fact that in a three-dimensional manifold satisfying Einstein's equations, whenever $T_{ab} = 0$ the Riemann tensor is zero, i.e. the spacetime is flat on those points[1].

One can see that $\bar{\theta} \in [0, 2\pi\alpha)$ with $(0 < \alpha \leq 1)$ so there is a deficit angle which, despite the local flatness for $r > 0$, makes this spacetime not globally equivalent to Minkowski space (due to the conic singulalrity).

We are looking for a metric that at spatial infinity approaches that of a point particle at the origin (3.3). So we can define a 2+1 space-time to be asymptotically flat if the line element admits an expansion of the form[2] [47],

$$\begin{aligned} ds^2 \;=\; & -\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right) dt^2 + r^{-\beta}\left[\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right) dr^2 + r^2\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right) d\theta^2\right] \\ & + \mathcal{O}(r^{-1-\beta/2}) dt d\theta, \end{aligned} \tag{3.5}$$

Note that in the asymptotic region (when $r \to \infty$) the previous line element approaches to the background metric (in cartesian coordinates),

where r and θ are the coordinates on cylinders with $r = const$ and the remainder $o(r^{-m})$ has the property that

$$\lim_{r\to\infty} r \; o(r^{-m}) = 0. \tag{3.2}$$

[1] We know that the Riemann tensor can be split into its trace and trace-free part, the Ricci tensor and scalar, and the Weyl tensor respectively. In 3-dimensions the Weyl tensor is identically zero, and by Einstein's equations $T_{ab} = 0$ implies that the Ricci tensor and scalar are also zero. Therefore the Riemann tensor is zero, so locally the space-time is flat. Note also that here we are dealing with asymptotically flat space-time, in contrast to the conformally flat picture where the vanishing of the Cotton tensor is equivalent the metric being conformally flat.

[2] A word on notation, $\mathcal{O}(r^{-m})$ means that those terms *include* a term proportional to r^{-m} and terms that decay faster, in contrast with $o(r^{-m})$ that only includes terms that decay faster than r^{-m}.

$$\bar{\eta}_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & r^{-\beta} & 0 \\ 0 & 0 & r^{-\beta} \end{pmatrix}. \tag{3.6}$$

We are approaching spatial infinity by some one-parameter family of boundaries of regions $\mathcal{M}_\rho \subset \mathcal{M}$ (cylinders throughout the present work, since they are more suited for hamiltonian methods, as we plan to use in the following sections, and also the use of hyperboloids in the $2+1$ context is less natural than in the 4D case [7; 25; 27], due to the lack of asympototic Lorentz invariance, since, unless $M = 0$, the asymptotically flat spacetime previously defined is no globally isometric to the three dimensional Minkowski space). $\{\mathcal{M}_\rho | \rho > 0\}$ are an increasing family, i.e. $\mathcal{M}_\rho \subset \mathcal{M}_{\rho'}$ whenever $\rho < \rho'$ and such that they cover $\mathcal{M}$ ($\bigcup_\rho \mathcal{M}_\rho = \mathcal{M}$). This procedure of taking a finite region $\mathcal{M}_\rho$ represent a cut-off for space-time and then we remove it by the limiting process $\rho \to \infty$. We take $\rho = r + \mathcal{O}(r^0)$. This is called a 'cylindrical cut-off' in [46].

3.2 The action and the boundary conditions of the first order variables

We can consider the Palatini action in three dimensions, whose equations of motion are equivalent to those given by the three dimensional Einstein-Hilbert action. Now the dynamical variables instead of the metric are a triad e and a Lorentz connection ω, both valued on the Lie algebra of $SO(2,1)$. Additionally we add to the Palatini action a boundary term in order to have a well posed action principle, that is, we want the action to be finite when evaluated on histories compatible with the boundary conditions and also differentiable.[1]

As we have emphasized, we want to begin with a well posed action principle, we can begin with the three dimensional analog of the four dimensional Palatini well posed action [7], that is, *the Standard Palatini action with boundary term* (SPB),

$$S_{SPB}[e,\omega] = -\frac{1}{\kappa}\int_{\mathcal{M}} e^I \wedge F_I \; - \frac{1}{\kappa}\int_{\partial\mathcal{M}} e^I \wedge \omega_I. \tag{3.7}$$

Now the natural question arises, is the boundary term Lorentz invariant? We can answer this in two ways. The first is by noting that we can perform a Lorentz transformation on the internal indices in (3.14), (3.15) and we still have an asymptotically flat configuration. So, in a sense, the internal directions are 'arbitrary', therefore without a loss of generality we can fix on the boundary one of the internal directions $\partial_a n^I = 0$ as in the 4-dimensional case [7; 18], and the boundary term will be invariant under the residual gauge transformations. On the other hand we can add the following

[1]For further explanation about what does it mean an action to be differentiable check out [27].

term to the action,

$$\alpha \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathrm{d} n_L \tag{3.8}$$

with this addition, when $\alpha = 1$, the boundary term in (3.7) becomes[1],

$$\int_{\partial \mathcal{M}} e^I \wedge \omega_I + \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathrm{d} n_L = \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L. \tag{3.10}$$

So instead of the action (3.7) we can begin with the *manifestly Lorentz invariant well posed action* (LIP)[2],

$$S_{LIP}[e,\omega] = -\frac{1}{\kappa} \int_{\mathcal{M}} e^I \wedge F_I \;\; - \frac{1}{\kappa} \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L. \tag{3.11}$$

Note that the *general Palatini action* contains both the SPB and LIP cases, when $\alpha = 0$ and $\alpha = 1$ respectively, we shall use it to compare both actions,

$$S_{GP}[e,\omega] = -\frac{1}{\kappa} \int_{\mathcal{M}} e^I \wedge F_I \;\; - \int_{\partial \mathcal{M}} e^I \wedge \omega_I - \alpha \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathrm{d} n_L \tag{3.12}$$

Moreover we can show that (3.8) is a constant when evaluated on asymptotically flat boundary conditions (see Appendix 3.7 for the details on the derivation), so it does not spoil the finiteness nor differentiability of the action. Therefore (3.11) is still a well posed action. Further, the term (3.10) is related the Gibbons-Hawking term needed for the Einstein-Hilbert action to be well posed and the action (3.11) is *the same* as the Einstein-Hilbert action with Gibbons-Hawking term [47].

As in the four dimensional case this is a first order action, we only have first derivatives on our configuration variables, that is why we also refer to these variables as first order variables.

Some comments are in order. We are writing the action in a way that is independent of the Lie group G on which is defined [60], which does not need the existence of a metric to be defined. In the case of an arbitrary G, e_{aI} can no longer be think of as the cotriad. The action (3.7) is then a functional of an $\pounds_G-$valued connection one-form ω_a^I and an $\pounds_G^*-$valued covector field e_{aI}. Where $\pounds_G-$ stands out for the Lie algebra of G and $\pounds_G^\star-$ its dual. When we chose $G = SO(2,1)$ we recover three-dimensional general relativity and we can think of e_{aI} as a cotriad. This coincidence is exclusive of the three-dimensional case.

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[1]

$$\begin{aligned} \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L &= \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IK}{}_L e_I \wedge n_K \left(\mathrm{d} n^K + \varepsilon^L{}_{MN} \omega^M n^N \right) \\ &= \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathrm{d} n_L + \int_{\partial \mathcal{M}} \frac{1}{n \cdot n} \varepsilon^{IKL} e_I \wedge n_K \varepsilon_{LMN} \omega^M n^N \end{aligned} \tag{3.9}$$

[2]Note the global minus sign, this is introduced since the Einstein Hilbert action with Gibbons Hawking term is equivalent to this action with minus sign (see appendix 3.8 for more details), so we can compare our results here with those obtained in the second order formulation [11; 47].

3.2.1 Fall-off conditions

To check that, in fact, the previous action is well posed we need to specify the boundary conditions on the first order variables e and ω, in this case asymptotically flat boundary conditions.

From the line element (3.5),

$$\begin{aligned} ds^2 &= -\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right)dt^2+r^{-\beta}\left[\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right)dr^2+r^2\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right)d\theta^2\right] \\ &\quad +\mathcal{O}(r^{-1-\beta/2})dtd\theta, \end{aligned} \tag{3.13}$$

we can find the fall-off conditions of g_{ab} as in [11; 47], with $a,b,c=0,1,2$ spacetime indices, and therefore remembering that $g_{ab}=\eta_{IJ}e^I_a e^J_b$ where $\eta_{IJ}=\mathrm{diag}(-1,1,1)$ is the Minkowski metric, the fall-off conditions of the first order variables.

We can assume the co-triads and the triads admit an asymptotic expansion of the form[1]

$$e^I_a=\delta^0_a\left({}^o\bar{e}^I_0+\frac{{}^1\bar{e}^I_0(\theta)}{r}+o(r^{-1})\right)+r^{-\beta/2}\left({}^o\bar{e}^I_{\bar{a}}+\frac{{}^1\bar{e}^I_{\bar{a}}(\theta)}{r}+o(r^{-1})\right)\delta^{\bar{a}}_a, \tag{3.14}$$

and

$$e^a_I=\delta^a_0\left({}^o\bar{e}^0_I+\frac{{}^1\bar{e}^0_I(\theta)}{r}+o(r^{-1})\right)+r^{\beta/2}\left({}^o\bar{e}^{\bar{a}}_I+\frac{{}^1\bar{e}^{\bar{a}}_I(\theta)}{r}+o(r^{-1})\right)\delta_{\bar{a}}a. \tag{3.15}$$

We define,

$${}^0e^I_a:={}^0\bar{e}^I_0\delta^0_a+r^{-\beta/2}\,{}^0\bar{e}^I_{\bar{a}}\delta^{\bar{a}}_a \quad \text{and} \quad {}^1e^I_a:=\frac{{}^1\bar{e}^I_0}{r}\delta^0_a+r^{-\beta/2}\frac{{}^1\bar{e}^I_{\bar{a}}}{r}\delta^{\bar{a}}_a \tag{3.16}$$

such that $\bar{\eta}_{ab}=\eta_{IJ}\,{}^0e^I_a\,{}^0e^J_b$ given by (3.6), where $\eta_{IJ}=\mathrm{diag}(-1,1,1)$ is the Minkowski metric.

As for the triads, we assume that the connection ω^I_a admit an expansion of the form,

$$\omega^I_a={}^o\bar{\omega}^I_a+\frac{{}^1\bar{\omega}^I_a(\theta)}{r}+\frac{{}^2\bar{\omega}^I_a(\theta)}{r^2}+o(r^{-2}), \tag{3.17}$$

Even though this expansion seems different from that of the triad, we can check that this expansion is derived from that of the triad and co-triad by means of the condition, De = 0, to first order.

Now we have to remember that any connection D can be written as $D=\mathring{\mathcal{D}}+\omega$, where $\mathring{\mathcal{D}}$ is any other connection. When there is a 'preferred' connection available, we can write all the other connections as that one plus a vector potential ω. Since there is no canonical choice of this *standard flat connection*, $\mathring{\mathcal{D}}$, within this particular problem it will be convenient to choose that

[1] A tensor field $T^{a...b}{}_{c...d}$ will be said to admit an asymptotic expansion to order m if all its component in the *Cartesian* chart x^a do so. Note that appart from the $r^{-\beta}$ factor in the spatial part of (3.5) the components in cartesian coordinates admit an expansion of order 1 in analogy with the standard definition of an asymptotically flat spacetime for 4 dimensional spacetimes [7; 25; 27], and also we assume that the first order variables, appart from a factor of $r^{-\beta/2}$, do so.

$\mathring{\mathcal{D}}_a {}^0\bar{e}^I_b = 0$. Using local coordinates and a local trivialization of $E = U_{\mathcal{M}} \times SO(2,1)$, where $U_{\mathcal{M}}$ is an open set on $\mathcal{M}$, the components of the connection for the condition of the compatibility of the triad with the connection, $De = 0$ will look like,

$$D_a e^I_b = \mathring{\mathcal{D}}_a e^I_b + \varepsilon^{IJK} \omega_{aJ} e_{bK} = 0. \tag{3.18}$$

From (3.18) is a standar calculation to see that the spin connection can be written in terms of the triad as[1],

$$\omega^M_c = -\frac{1}{2}\left(\varepsilon_L{}^{KM} e^a_K e^{bL} e_{cI} \mathring{\mathcal{D}}_{[a} e^I_{b]} - \varepsilon_L{}^{KM} e^a_K \mathring{\mathcal{D}}_{[c} e^L_{a]} - \varepsilon_L{}^{KM} e^{bL} \mathring{\mathcal{D}}_{[b} e_{c]K}\right). \tag{3.19}$$

The leading term of the spin connection can be found from the previous equation considering the leading terms of the triad and cotriad,

$$^{Leading}\omega^M_c = -\frac{1}{2}\left(\varepsilon_L{}^{KM\,0} e^a_K {}^0e^{bL\,0} e_{cI} \mathring{\mathcal{D}}_{[a} {}^0e^I_{b]} - \varepsilon_L{}^{KM\,0} e^a_K \mathring{\mathcal{D}}_{[c} {}^0e^L_{a]} - \varepsilon_L{}^{KM\,0} e^{bL} \mathring{\mathcal{D}}_{[b} {}^0e_{c]K}\right). \tag{3.20}$$

where $\mathring{\mathcal{D}}_b {}^0\bar{e}^I_a = 0$. Note that from (3.14),

$$\mathring{\mathcal{D}}_b {}^0e^I_a = \mathring{\mathcal{D}}_b({}^0\bar{e}^I_0 \delta^0_a) + \mathring{\mathcal{D}}_b(r^{-\beta/2\,0} \bar{e}^I_{\bar{a}} \delta^{\bar{a}}_a) = \mathring{\mathcal{D}}_b(r^{-\beta/2})\,{}^0\bar{e}^I_{\bar{a}} \delta^{\bar{a}}_a = (\partial_b r^{-\beta/2})\,{}^0\bar{e}^I_{\bar{a}} \delta^{\bar{a}}_a \tag{3.21}$$

but $\partial_b r^{-\beta/2} = -\frac{1}{2}\beta r^{-1-\beta/2}\partial_b r$. Therefore,

$$\mathring{\mathcal{D}}_b {}^0e^I_a = (-\frac{1}{2}\beta r^{-1-\beta/2}\partial_b r)\,{}^0\bar{e}^I_{\bar{a}} \delta^{\bar{a}}_a = (-\frac{1}{2}\beta r^{-1}\partial_b r)\,{}^0e^I_{\bar{a}} \delta^{\bar{a}}_a \tag{3.22}$$

Taking into account the previous equation and the fall-off conditions (3.14) and (3.15), equation (3.20) becomes (using that $\partial_0 r = 0$),

$$^{Leading}\omega^M_c = \frac{\beta}{2r} \varepsilon_L{}^{KM\,0} \bar{e}^{\bar{a}}_K {}^0\bar{e}^L_{\bar{c}} \delta^{\bar{c}}_c, \tag{3.23}$$

then by considering the expansion (4.37) we can see that,

$$^1\bar{\omega}^M_c = \frac{\beta}{2} \partial_{\bar{a}} r \varepsilon_L{}^{KM\,0} \bar{e}^{\bar{a}}_K {}^0\bar{e}^L_{\bar{c}} \delta^{\bar{c}}_c. \tag{3.24}$$

Which implies that $\frac{^1\bar{\omega}^M_c}{r}$ is the leading term of ω^M_c and also ${}^0\omega^M_c = 0$ as well as ${}^1\omega^M_0 = 0$.

[1]See Appendix B for the derivation.

3.2.2 Well posedness of the action

As we already mentioned, beginning with a well posed action principle under asymptotically flat boundary conditions, we want to find an expression for the energy under various approaches. We want to analyse whether this results coincide with those in the second order formalism [11; 47] and also the relation and differences among the different paths we take: the covariant hamiltonian formalism (CHF), and the canonical one, but taking two different $2+1-$decompositions.

But first we have to check that the action principle we are working with is well posed, i.e. finite and differentiable under asymptotically flat boundary conditions. With the fall-off conditions of the first order variables found in section 3.2.1 we are ready to undertake this task.

3.2.2.1 Finiteness

Since the term (3.8) is a finite constant when evaluated on the boundary[1], it does not spoil finiteness, then is only necessary to cheek that the action (3.7) is finite, so the manifestly gauge invariant action (3.11) is finite. The action (3.7) can be rewritten as,

$$\begin{aligned} S[e,\omega] &= -\frac{1}{\kappa}\int_{\mathcal{M}} e^I\wedge F_I \; -\frac{1}{\kappa}\int_{\partial\mathcal{M}} e^I\wedge\omega_I \\ &= -\frac{1}{\kappa}\int_{\mathcal{M}}\left(e^I\wedge \mathrm{d}\omega_I+\frac{1}{2}\varepsilon_I{}^{JK}e^I\wedge\omega_J\wedge\omega_K\right)\; -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} e^I\wedge\omega_I \end{aligned} \qquad (3.25)$$

since $F_I=\mathrm{d}\omega_I+\frac{1}{2}\varepsilon_I{}^{JK}\omega_J\wedge\omega_K$ and,

$$\mathrm{d}(e^I\wedge\omega_I)=\mathrm{d}e^I\wedge\omega_I-e^I\wedge\mathrm{d}\omega_I \;\Rightarrow\; e^I\wedge\mathrm{d}\omega_I=\mathrm{d}e^I\wedge\omega_I-\mathrm{d}(e^I\wedge\omega_I). \qquad (3.26)$$

Therefore,

$$\begin{aligned} S[e,\omega] &= -\frac{1}{\kappa}\int_{\mathcal{M}}\left(\mathrm{d}e^I\wedge\omega_I+\frac{1}{2}\varepsilon_I{}^{JK}e^I\wedge\omega_J\wedge\omega_K-\mathrm{d}(e^I\wedge\omega_I)\right)\; -\frac{1}{\kappa}\int_{\partial\mathcal{M}} e^I\wedge\omega_I \\ &= -\frac{1}{\kappa}\int_{\mathcal{M}}\left(\mathrm{d}e^I\wedge\omega_I+\frac{1}{2}\varepsilon_I{}^{JK}e^I\wedge\omega_J\wedge\omega_K\right). \end{aligned} \qquad (3.27)$$

The leading term of the previous equation is,

$${}^0S[e,\omega]=-\frac{1}{\kappa}\int_{\mathcal{M}}\left(\mathrm{d}\,{}^0e^I\wedge{}^1\omega_I+\frac{1}{2}\varepsilon_I{}^{JK}\,{}^0e^I\wedge{}^1\omega_J\wedge{}^1\omega_K\right), \qquad (3.28)$$

but we already used the compatibility condition with the triad to first order to obtain the fall-off

[1] See appendix 3.7 for the details.

conditions on ω, (3.18), which also can be written as,

$$\mathrm{d}\,{}^0e^I - \varepsilon^I{}_{JK}\,{}^1\omega^K \wedge {}^0e^J = 0 \tag{3.29}$$

therefore, we can rewrite (3.28) as,

$$\begin{aligned} {}^0S[e,\omega] &= -\frac{1}{\kappa}\int_{\mathcal{M}} \left(\mathrm{d}\,{}^0e^I \wedge {}^1\omega_I - \frac{1}{2}\varepsilon_I{}^{JK}\,{}^0e^I \wedge {}^1\omega_J \wedge {}^1\omega_K\right) \\ &= -\frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}\,{}^0e^I \wedge {}^1\omega_I - \frac{1}{2}\mathrm{d}\,{}^0e^I \wedge {}^1\omega_I = -\frac{1}{\kappa}\int_{\mathcal{M}} \frac{1}{2}\mathrm{d}\,{}^0e^I \wedge {}^1\omega_I. \end{aligned} \tag{3.30}$$

Now, using (3.22) and (3.24) the leading term is,

$$\frac{1}{4\kappa}\int_{\mathcal{M}} \mathring{\mathcal{D}}_a\,{}^0e^I_b\,{}^1\omega^K_c\,\tilde{\varepsilon}^{abc}d^3x = 0, \tag{3.31}$$

since[1] ${}^1\omega^K_0 = 0$, $\mathring{\mathcal{D}}_0\,{}^0e^I_{\bar{a}} = 0$ and $\mathring{\mathcal{D}}_a\,{}^0e^I_0 = 0$. On the other hand note that we could have chosen to write (3.27), also using $De = 0$ to first order, as,

$${}^0S[e,\omega] = -\frac{1}{4\kappa}\int_{\mathcal{M}} \varepsilon^I{}_{JK}\,{}^0e^I \wedge {}^1\omega^J \wedge {}^1\omega^K = -\frac{1}{4\kappa}\int_{\mathcal{M}} \varepsilon^I{}_{JK}\,{}^0e^I_a\,{}^1\omega^J_b\,{}^1\omega^K_c\,\tilde{\varepsilon}^{abc}d^3x. \tag{3.32}$$

In the previous equation, using (3.16) and (3.24), the only nonvanishing term is

$${}^0S[e,\omega] = -\frac{1}{4\kappa}\int_{\mathcal{M}} \varepsilon^I{}_{JK}\,{}^0\bar{e}^I_0 \frac{{}^1\bar{\omega}^J_{\bar{b}}}{r}\frac{{}^1\bar{\omega}^K_{\bar{c}}}{r}\varepsilon^{0\bar{b}\bar{c}}\,r\,dr\,d\theta\,dt = \int_{\mathcal{M}} \mathcal{O}(r^{-1})dr. \tag{3.33}$$

Our region of integration $\mathcal{M}$ is bounded by $\partial\mathcal{M} = M_1 \cup M_2 \cup \mathcal{I}$ with its corresponding orientation. In order to check finiteness it is enough to check that the integral over a spatial hypersurface is finite. This is true since we are integrating over a finite time interval where the Cauchy surfaces M_1 and M_2 are asymptotically time-translated with respect to each other. Such spacetimes $\mathcal{M}$ are referred to as *cylindrical slabs* [7] or as *cylindrical temporal cut-off* [**?**].

Note that on a Cauchy slice the only dependency on r of the previous equation is due to ${}^1\omega^K_c = \mathcal{O}(r^{-1})$, so the integral over r goes as $\int \mathcal{O}(r^{-1})dr$ that *may* logarithmically diverge in the limit $r \to \infty$, but we already proved on (3.31) that this term is zero. Then the next to leading terms decay faster on r so in the limit $r \to \infty$ they go to zero. Therefore the integral is finite *even off shell*.

[1] ${}^1\omega^K_0 = 0$ is zero from the fall off conditions on ω, $\mathring{\mathcal{D}}_a\,{}^0e^I_0 = 0$ because ${}^0e^I_0 = {}^0\bar{e}^I_0$ and $\mathring{\mathcal{D}}_0\,{}^0e^I_{\bar{a}} = 0$ because we ask the condition of the compatibility of the triad with the connection to be satisfied to first order to find the fall-off conditions on ω,

$$D_0\,{}^0e^I_b = \mathring{\mathcal{D}}_0\,{}^0e^I_b + \varepsilon^{IJK}\,{}^1\omega_{0J}\,{}^0e_{bK} = 0.$$

since ${}^1\omega^K_0 = 0$ then $\mathring{\mathcal{D}}_0\,{}^0e^I_b = 0$.

3.2.2.2 Differentiability

In order an action to be differentiable the variation of the action needs to take the form,

$$\delta S[e,\omega] = \int_{\mathcal{M}} [\mathbf{E}_e \wedge \delta e + \mathbf{E}_\omega \wedge \delta\omega] + \int_{\partial M} \tilde{\theta}(e^I, \omega^I, \delta e^I, \delta\omega^I), \tag{3.34}$$

and in order $\mathbf{E}_e$ and $\mathbf{E}_\omega$ to be the Euler-Lagrange equations of motion, the boundary term needs to be zero when evaluated on solutions compatible with the boundary conditions . Since the term (3.8) is constant when evaluated on those solutions, its variation is zero so it does not spoil differentiability. Therefore we only need to check if the action (3.7) is differentiable.

The variation of the 3-dimensional Palatini action with boundary term (3.7) is,

$$\delta S[e,\omega] = -\frac{1}{\kappa}\int_{\mathcal{M}} \left[\delta e^I \wedge F_I + e^I \wedge \delta F_I\right] - \frac{1}{\kappa}\int_{\partial\mathcal{M}} \left[\delta e^I \wedge \omega_I + e^I \wedge \delta\omega_I\right], \tag{3.35}$$

but

$$\delta F_I = \mathrm{d}\delta\omega_I + \frac{1}{2}\varepsilon_I{}^{JK}\delta\omega_J \wedge \omega_K + \frac{1}{2}\varepsilon_I{}^{JK}\omega_J \wedge \delta\omega_K = \mathrm{d}\delta\omega_I + \varepsilon_I{}^{JK}\delta\omega_J \wedge \omega_K \tag{3.36}$$

then, the variation becomes,

$$\delta S[e,\omega] = -\frac{1}{\kappa}\int_{\mathcal{M}} \delta e^I \wedge F_I - \frac{1}{\kappa}\int_{\mathcal{M}} \left(\mathrm{d}e^J + \varepsilon^{JIK} e_I \wedge \omega_K\right) \wedge \delta\omega_J - \frac{1}{\kappa}\int_{\partial\mathcal{M}} \delta e^I \wedge \omega_I. \tag{3.37}$$

If the boundary term is zero under the boundary conditions, the action is said to be differentiable and the equations of motion are,

$$F_I = 0 \;\text{ and }\; De^J = \mathrm{d}e^J + \varepsilon^{JIK} e_I \wedge \omega_K = 0. \tag{3.38}$$

Then the boundary term is,

$$-\frac{1}{\kappa}\int_{\partial\mathcal{M}} \delta e^I \wedge \omega_I = -\frac{1}{\kappa}\left(-\int_{M_1} + \int_{M_2} + \int_{\mathcal{I}}\right) \delta e^I \wedge \omega_I \tag{3.39}$$

where we are considering that our integration region $\mathcal{M}$ is bounded by $\partial\mathcal{M} = M_1 \cup M_2 \cup \mathcal{I}$ with its corresponding orientation. We are taking, as usual, $\delta e^I = \delta\omega_I = 0$ on the space-like surfaces M_1 and M_2. We are left only with the integral on the time-like boundary $\mathcal{I}$. Remember that we are approaching spatial infinity by a family of cylinders, C_r with $r = const$, in the limit when $r \to \infty$. To check differentiability we have to prove that

$$\lim_{r\to\infty} \int_{C_r} \delta e^I \wedge \omega_I = 0, \tag{3.40}$$

when evaluated on histories compatible with the asymptotically flat boundary conditions. It is enough to check the behaviour of the leading term (the next to leading terms decay 'faster' as r goes to infinity). The leading term of (3.40) in components is,

$$\lim_{r\to\infty}\int_{C_r}\delta\,{}^0e^I_a\frac{{}^1\omega_{bI}}{r}\varepsilon^{ab}rd\theta dt=\lim_{r\to\infty}\int_{C_r}\left(\delta\,{}^0e^I_0\,{}^1\omega_{\bar{b}I}-\delta\,{}^0e^I_{\bar{b}}\,{}^1\omega_{0I}\right)\varepsilon^{0\bar{b}}d\theta dt=0. \tag{3.41}$$

We have used that ${}^1\bar{\omega}^I_0=0$ and $\delta\,{}^0e^I_0=0$ (since ${}^0e^I_0={}^0\bar{e}^I_0$ and ${}^0\bar{e}^I_a$ is our fixed flat frame at the asymptotic region). The next to leading terms decay faster, since ${}^1e^I_a=\mathcal{O}(r^{-1})$ and ${}^1\omega^K_c=\mathcal{O}(r^{-1})$, they are proportional to

$$\lim_{r\to\infty}\int_{C_r}\left(\frac{1}{r}+\mathcal{O}(r^{-2})\right)d\theta dt=0. \tag{3.42}$$

Therefore the action is also differentiable under asymptotically flat boundary conditions.

3.3 Covariant analysis

In this section we shall follow covariant hamiltonian formalism developed in section 2.5 [27]. From the variation of the action (3.37), we can identify the symplectic potential,

$$\tilde{\Theta}(e^I,\omega^I,\delta e^I,\delta\omega^I):=\int_{\partial M}\tilde{\theta}(e^I,\omega^I,\delta e^I,\delta\omega^I)=-\frac{1}{\kappa}\int_{\partial M}\delta e^I\wedge\omega_I, \tag{3.43}$$

and its associated symplectic current,

$$J(\delta_1,\delta_2):=2\delta_{[1}\tilde{\theta}(\delta_{2]})=-\frac{1}{\kappa}\left(\delta_2e^I\wedge\delta_1\omega_I-\delta_1e^I\wedge\delta_2\omega_I\right). \tag{3.44}$$

Since J is closed over any region $\mathcal{M}$,

$$0=\int_{\mathcal{M}}\mathrm{d}J(\delta_1,\delta_2)=\oint_{\partial\mathcal{M}}J(\delta_1,\delta_2)=\left[-\int_{M_1}+\int_{M_2}+\int_{\mathcal{I}}\right]J(\delta_1,\delta_2) \tag{3.45}$$

here we are considering the region $\mathcal{M}$ is bounded by $\partial_{\mathcal{M}}=M_1\cup M_2\cup\mathcal{I}$, M_1 and M_2 are space-like slices and $\mathcal{I}$ an outer boundary, in particular we shall consider configurations that are asymptotically flat. We are assuming no internal boundary.

In order to have a *conserved symplectic current* and therefore a *conserved pre-symplectic form*, independent of the Cauchy surface, we have to check that $\int_{\mathcal{I}}J=0$, that is there is no current leakage' at infinity.

Taking into account the asymptotically flat boundary conditions previously derived, we can see that the leading terms of $\int_{\mathcal{I}}J$ are,

$$\int_{\mathcal{I}}{}^0J(\delta_1,\delta_2)=-\frac{1}{\kappa}\lim_{r\to\infty}\int_{C_r}\left(\delta_2\,{}^0e^I\wedge\delta_1\,{}^1\omega_I-\delta_1\,{}^0e^I\wedge\delta_2\,{}^1\omega_I\right). \tag{3.46}$$

Following the same arguments as in (3.41), that is using ${}^1\bar{\omega}_0^I = 0$ and $\delta\, {}^0e_0^I = 0$, and noticing that the previous equation becomes,

$$\int_{\mathcal{I}} {}^0J(\delta_1,\delta_2) = -\frac{1}{\kappa}\lim_{r\to\infty}\int_{C_r}\left(\delta_2\, {}^0e_0^I\delta_1\, {}^1\omega_{\bar{a}I} - \delta_2\, {}^0e_{\bar{a}}^I\delta_1\, {}^1\omega_{0I} - \delta_1\, {}^0e_0^I\delta_2\, {}^1\omega_{\bar{a}I} + \delta_1\, {}^0e_{\bar{a}}^I\delta_2\, {}^1\omega_{0I}\right)\tilde{\varepsilon}^{0\bar{a}}, \tag{3.47}$$

we can see that

$$\int_{\mathcal{I}} {}^0J(\delta_1,\delta_2) = 0. \tag{3.48}$$

But on the other hand note that

$$\int_{\mathcal{I}} {}^0J(\delta_1,\delta_2) = -\frac{1}{\kappa}\lim_{r\to\infty}\int_{C_r}\left(\delta_2\, {}^0e_a^I\delta_1\frac{{}^1\bar{\omega}_{bI}}{r} - \delta_1\, {}^0e_a^I\wedge\delta_2\frac{{}^1\bar{\omega}_{bI}}{r}\right)\varepsilon^{ab}rd\theta dt \tag{3.49}$$

is independent of r. Therefore the next to leading terms goes as,

$$\int_{\mathcal{I}} J(\delta_1,\delta_2) = -\frac{1}{\kappa}\lim_{r\to\infty}\int_{C_r}\mathcal{O}(r^{-1})\varepsilon^{ab}d\theta dt = 0. \tag{3.50}$$

Then the symplectic current is conserved.

Now we can define a *conserved pre-symplectic form* over an arbitrary space-like surface M,

$$\tilde{\Omega}(\delta_1,\delta_2) = \int_M J(\delta_1,\delta_2) = -\frac{1}{\kappa}\int_M \delta_2 e^I\wedge\delta_1\omega_I - \delta_1 e^I\wedge\delta_2\omega_I \tag{3.51}$$

Once we have $\tilde{\Omega}(\delta_1,\delta_2)$, we can analyse the symmetries of the theory and their associated conserved charges. In particular we are interested in the conserved charge associated with the asymptotic time translations, i.e. the ADM energy.

Since one of our goals is to compare the resulting expression for the energy through the covariant and canonical formalism, we need to be sure that the conventions in both schemes are in agreement. We discuss this point in the next section.

3.3.0.3 Link between covariant and canonical approaches

The symplectic structure is essential in order to have a hamiltonian description. In a coordinate basis asscociated with the configuration variables, the fields ϕ^A, the symplectic form can also may be difined by

$$\bar{\Omega} = \mathrm{d}\Pi_A\wedge\mathrm{d}\phi^A, \tag{3.52}$$

where Π_A is the momenta canonically conjugated to ϕ^A. This $\bar{\Omega}$ is consistent with all our derivations in the covariant phase space. But up to now we haven't said 'who are' our ϕ^A and Π_A.

It is well known that in the first order formulation of General Relativity one of our configuration variables is the canonically conjugated to the other. For instance, in the connection approach, ω is chosen to be the configuration variable and it turns out that e will be its canonical momenta. Or the other way around if we chose the geometrodynamics picture.

To compare with the results obtained by the canonical formalism, first we have to chose if we want to work in the connection or geometrodynamics approach. We chose the first one, that is $\phi^A = \omega^I$ and $\Pi_A = e_I$. From (3.51),

$$\tilde{\Omega}(\delta_1, \delta_2) = -\frac{1}{2\kappa}\int_M \delta_2 \underbrace{e^I}_{\Pi^A} \wedge \delta_1 \underbrace{\omega_I}_{\phi^A} - \delta_1 e^I \wedge \delta_2 \omega_I = -\bar{\Omega} \tag{3.53}$$

In order to compare our expressions for the energy, through all the approaches we have to use $\bar{\Omega} = -\tilde{\Omega}$.

3.3.1 The hamiltonian and the energy

Consider infinitesimal diffeomorphisms generated by a vector field ξ, these diffeomorphisms induce an infinitesimal change in the fields given by $\delta_\xi := (\pounds_\xi e, \pounds_\xi \omega)$.

We said ξ is a hamiltonian vector field iff $\bar{\Omega}(\delta, \delta_\xi)$ is closed, $\mathbf{d}\,\Omega = 0$, and the Hamiltonian H_ξ is defined by,

$$\bar{\Omega}(\delta, \delta_\xi) = \delta H_\xi = \mathbf{d} H. \tag{3.54}$$

Where $\mathbf{d}$ is the exterior derivative on the covariant phase space[1], which is different from the exterior derivative on spacetime d.

So H_ξ is a conserved quantity along the flow generated by ξ. And when we consider the case when ξ generates asymptotic time translations of the space-time, which induces time evolution on the covariant phase space generated by the vector field $\delta_\xi := (\pounds_\xi e, \pounds_\xi \omega)$. H_ξ is the energy.

3.3.1.1 The energy

From eq. (3.51),

$$\begin{aligned} \bar{\Omega}(\delta, \delta_\xi) &= -\tilde{\Omega}(\delta, \delta_\xi) = \frac{1}{\kappa}\int_M \delta_\xi e^I \wedge \delta\omega_I - \delta e^I \wedge \delta_\xi \omega_I && (3.55)\\ &= \frac{1}{\kappa}\int_M \pounds_\xi e^I \wedge \delta\omega_I - \delta e^I \wedge \pounds_\xi \omega_I && (3.56)\end{aligned}$$

by using $\pounds_\xi \phi^A = \xi \cdot \mathrm{d}\phi^A + \mathrm{d}(\xi \cdot \phi^A)$

$$\bar{\Omega}(\delta, \delta_\xi) = \frac{1}{\kappa}\int_M \left[(\xi \cdot \mathrm{d}e^I) \wedge \delta\omega_I + \mathrm{d}(\xi \cdot e^I) \wedge \delta\omega_I - \delta e^I \wedge (\xi \cdot \mathrm{d}\omega_I) - \delta e^I \wedge \mathrm{d}(\xi \cdot \omega_I)\right]. \tag{3.57}$$

[1] see [27] for further details and definitions.

Now we have to use that at infinity ξ should approach a time-translation Killing vector field of the asymptotic flat spacetime. In particular this means that in the asymptotic region ξ^a is orthogonal to the spacelike surface. Therefore $\xi \cdot e^I = e^I_0$, $\xi \cdot \omega_I = \omega_{0I}$ but for the leading term we have seen ${}^1\bar{\omega}_{0I} = 0$, also $\mathring{\bar{\mathcal{D}}}_a {}^0e^I_b$ only has spatial components so $\xi \cdot \mathrm{d}\,{}^0e^I = 0$. With this at hand we can see that[1],

$$\begin{aligned} \bar{\Omega}(\delta,\delta_\xi) &= \frac{1}{\kappa}\int_M \mathrm{d}(\xi \cdot e^I) \wedge \delta\omega_I && (3.58)\\ &= \frac{1}{\kappa}\int_M \mathrm{d}\left[(\xi \cdot e^I)\delta\omega_I\right] - (\xi \cdot e^I)\mathrm{d}\delta\omega_I && (3.59)\end{aligned}$$

Note that the second term of the previous equation in components becomes,

$$-\int_M {}^0\bar{e}^I_0 \mathring{\bar{\mathcal{D}}}_{[\bar{b}|}\delta\omega_{|\bar{c}]I}\varepsilon^{\bar{b}\bar{c}} r dr d\theta \tag{3.60}$$

but

$$\mathring{\bar{\mathcal{D}}}_{[\bar{b}|}\delta\omega^M_{|\bar{c}]} = \delta\beta\left[r^{-2}\partial_{[\bar{b}|} r \partial_{\bar{a}} r + r^{-1}\partial_{[\bar{b}|}\partial_{\bar{a}} r\right]\varepsilon_{LK}{}^{M}\,{}^0\bar{e}^K_{\bar{d}}\,{}^0\bar{e}^L_{|\bar{c}]}\eta^{\bar{a}\bar{d}}, \tag{3.61}$$

so

$${}^0\bar{e}^I_0 \mathring{\bar{\mathcal{D}}}_{[\bar{b}|}\delta\omega_{|\bar{c}]I}\varepsilon^{\bar{b}\bar{c}} = \delta\beta\left[r^{-2}\partial_{[\bar{b}|} r \partial_{\bar{a}} r + r^{-1}\partial_{[\bar{b}|}\partial_{\bar{a}} r\right]\underbrace{\varepsilon_{LKI}\,{}^0\bar{e}^K_{\bar{d}}\,{}^0\bar{e}^L_{|\bar{c}]}\,{}^0\bar{e}^I_0}_{\bar{e}\varepsilon_{|\bar{c}]\bar{d}0}}\eta^{\bar{a}\bar{d}}\underbrace{\varepsilon^{\bar{b}\bar{c}}}_{\varepsilon^{0\bar{b}\bar{c}}}, \tag{3.62}$$

here $\bar{e} = \sqrt{-\eta} = 1$ where η_{ab} is the Minkowski metric associated with the fixed frame $\bar{e}^a_I$ at the asymptotic region, also $\varepsilon_{|\bar{c}]\bar{d}0}\varepsilon^{0\bar{b}\bar{c}} = -2\delta^{\bar{b}}_{|\bar{d}]}$. Thus by antisymmetry in the space-time indices this term vanishes.

From (3.58) and the previous argument the presymplectic form is,

$$\begin{aligned} \bar{\Omega}(\delta,\delta_\xi) &= \frac{1}{\kappa}\int_M \mathrm{d}\left[(\xi \cdot e^I)\delta\omega_I\right] = \frac{1}{\kappa}\int_{\partial M}(\xi \cdot e^I)\delta\omega_I \\ &= \lim_{r\to\infty}\left[\frac{1}{\kappa}\int_{\partial M} {}^0e^I_0\,\delta\frac{{}^1\omega_{\bar{c}I}}{r}\tilde{\varepsilon}^{0\bar{c}} + \int_{\partial M}\mathcal{O}(r^{-1})d\theta\right], && (3.63)\end{aligned}$$

with

$$\delta\left(\frac{{}^1\omega^M_{\bar{c}}}{r}\right) = \frac{1}{2r}\delta\beta\,\partial_{\bar{a}} r\,\varepsilon_{LK}{}^{M}\,{}^0\bar{e}^K_{\bar{d}}\,{}^0\bar{e}^L_{\bar{c}}\eta^{\bar{a}\bar{d}}. \tag{3.64}$$

Then, by (3.54), the variation of the hamiltonian, and therefore of its corresponding associated

[1] This is the only non-vanishing term to first order.

conserved quantity, the energy, is

$$\delta H_\xi = \bar{\Omega}(\delta,\delta_\xi) = \frac{1}{2\kappa}\lim_{r\to\infty}\int_{\partial M} {}^0\bar{e}^I_0\left(\delta\beta r^{-1}\partial_{\bar{a}} r\varepsilon_{LKI}{}^0\bar{e}^K_{\bar{d}}{}^0\bar{e}^L_{\bar{c}}\eta^{\bar{a}\bar{d}}\right)\tilde{\varepsilon}^{\bar{c}}$$
$$= \frac{1}{2\kappa}\int_{\partial M}\frac{1}{r}\delta\beta\underbrace{\left(\varepsilon_{LKI}{}^0\bar{e}^I_0{}^0\bar{e}^K_{\bar{d}}{}^0\bar{e}^L_{\bar{c}}\right)}_{\bar{e}\varepsilon_{\bar{c}\bar{d}0}}\eta^{\bar{a}\bar{d}}\partial_{\bar{a}} r\underbrace{\tilde{\varepsilon}^{\bar{c}}}_{\tilde{\varepsilon}^{0\bar{c}}} \tag{3.65}$$

here we are using the identity $\varepsilon_{LKI}{}^0\bar{e}^I_0{}^0\bar{e}^K_{\bar{d}}{}^0\bar{e}^L_{\bar{c}} = \bar{e}\tilde{\varepsilon}_{0\bar{c}\bar{d}}$ where $\bar{e} = \sqrt{-\eta} = 1$ with η the determinant of η_{ab}, the Minkowski metric associated with the fixed frame $\bar{e}^a_I$ at the asymptotic region. Also $\partial^{\bar{a}} r =: r^{\bar{a}}$ can be seen as the normal to the cylinders $r = const$. On the other hand $\eta^{\bar{a}\bar{d}}\partial_{\bar{a}} r\tilde{\varepsilon}^{0\bar{c}} = r^{\bar{d}}\tilde{\varepsilon}^{0\bar{c}}$, so we can use $\tilde{\varepsilon}_{abc}r^c = \tilde{\varepsilon}_{ab}$. With all this we can see that the previous equation (3.65) is,

$$\delta H_\xi = \frac{1}{2\kappa}\int_{\partial M}\frac{1}{r}\delta\beta\bar{e}\tilde{\varepsilon}_{\bar{c}\bar{d}0}r^{\bar{d}}\tilde{\varepsilon}^{0\bar{c}} = \frac{1}{2\kappa}\int_{\partial M}\frac{1}{r}\delta\beta\bar{e}\underbrace{\varepsilon_{0\bar{c}}\varepsilon^{0\bar{c}}}_{1} r d\theta = \frac{1}{2\kappa}\delta\beta\int_{\partial M} d\theta \tag{3.66}$$

Also note that $\partial M = C_t$, M a space like slice at "time" t, and C_t a circle with radius r at time t. We can write the energy,

$$\delta H_\xi = \frac{\delta\beta}{2\kappa}\int_{C_t} d\theta \tag{3.67}$$

taking $\kappa = 8\pi G$,

$$\delta H_\xi = \frac{\delta\beta}{2(8\pi G)}2\pi = \frac{\delta\beta}{8G}. \tag{3.68}$$

Since the previous expression only gives the variation, the energy will always be determined up to a constant,

$$E = \frac{\beta}{16G} + const \tag{3.69}$$

Following [11; 47], $\beta \in [0,2)$, we can choose this constant to be zero for the energy of Minkowski space-time to be zero,

$$E \in \left[0, \frac{1}{4G}\right]. \tag{3.70}$$

which coincides with the result obtained in [11] throughout Regge-Teitelboim method. Although CHF is elegant only provide us with the variation of the energy, so we have and indeterminacy in the election of the constant that may shift the region in which the energy is bounded. Thats why we shall also analyse this action through the canonical analysis, where the hamiltonian is completely determined by the Legendre transformation.

3.4 Canonical analysis

In the case of theories that can be formulated without the need of a metric, we have two choices for a $2+1$ decompoisition. The first one, we shall refere to it as the *Witten approach*[1], it does not need the existence of a metric, we only ask the spacetime $\mathcal{M}$ to be topologically $\Sigma \times R$ and that there exists a function t (with nowhere vanishing gradient $(dt)_a$) such that each $t = const$ surface M_t is diffeomorphic to Σ. Also there exists a flow vector field t^a satisfying $t^a(dt)_a = 1$, which allow us to define "evolution", although t does not necessarily have the interpretation of time[2].

The second one, we shall refere to it as *Ashtekar-Barbero-Varadarajan approach*[3]. In this approach, following closely the $3+1$ decomposition of the first order variables, besides the ingredients of the Witten approach we are assuming the existence of a metric g_{ab} and therefore a unit normal to the Cauchy surfaces. This introduce additional information to that in Witten's decomposition. In particular, we can decompose any tensor into its normal and tangential part, in particular t^a can be decomposed as $t^a = Nn^a + N^a$, where N and N^a are the laps and shift functions. Now we have additional information, the freedom of choosing any foliation and any vector field t^a, that is coded in the laps and shift functions.

A comment in notation, in what follows we use $\tilde{\varepsilon}^{abc}$ as the Levi-Civita tensor density of weight $+1$ instead of $\tilde{\eta}^{abc}$, more commonly used in the $3-$ dimensional case, this to avoid confusion with the flat metric $\bar{\eta}_{ab}$ (3.6), or with the Minkowki metric (either with internal or spacetime indices). Also we refer to a Cauchy slice as M following the notation in [27].

3.4.1 Witten's approach

In order to make the canonical analysis (a la Witten) of the 3-dimensional Palatini action, we write action (3.11) it in components

$$
\begin{aligned}
S_{PB}[e,\omega] &= -\frac{1}{2\kappa}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc} e_{aI} F^I_{bc} - \frac{1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\omega^I_b \tilde{\varepsilon}^{ab} - \frac{\alpha}{\kappa}\int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_{aI} n_K \mathring{\mathcal{D}}_b n_L \tilde{\varepsilon}^{ab} && (3.71)\\
&= -\frac{1}{2\kappa}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc} e_{aI} F^I_{bc} - \frac{1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\omega^I_b \tilde{\varepsilon}^{ab} + \frac{\alpha}{\kappa}\int_{\partial\mathcal{M}} \frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL} e_{aI} \mathring{\mathcal{D}}_b n_L \tilde{\varepsilon}^{ab} && (3.72)
\end{aligned}
$$

For this decomposition we shall follow the analysis in [60], taking enough care of the boundary term, the ones coming from the Palatini action and the boundary terms in (3.71). Using that

[1] Following [15] refering to Witten's paper [67], for more details on the analysis in the case where there is no boundary see [60].

[2] Since the $2+1$ Palatini action based on an arbitrary Lie group G (3.7) is a theory independent of a spacetime metric, we can still define evolution from one $t = const$ surface to the next using the Lie derivative along t^a.

[3] In [15] the authors discuss the differences in the canonical analysis, particularly in the constraints, following Witten's vs Ashtekar's approaches. Thats whay we call it Ashtekar-Barbero-Varadarajan approach.

$\tilde{\varepsilon}^{abc} = 3t^{[a}\tilde{\varepsilon}^{bc]}dt$ and $\tilde{\varepsilon}^{ab} = 2t^{[a}\tilde{\varepsilon}^{b]}dt$

$$\begin{aligned}
S_{PB}[e,\omega] &= -\frac{1}{2\kappa}\int dt \int_M (t^a\tilde{\varepsilon}^{bc} + t^b\tilde{\varepsilon}^{ca} + t^c\tilde{\varepsilon}^{ab})e_{aI}F^I_{bc} - \frac{1}{\kappa}\int dt \int_{C_t}(t^a\tilde{\varepsilon}^b - t^b\tilde{\varepsilon}^a)e_{aI}\omega^I_b \\
&\quad + \frac{\alpha}{\kappa}\int dt \int_{C_t}(t^a\tilde{\varepsilon}^b - t^b\tilde{\varepsilon}^a)\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}e_{aI}\mathring{\bar{\mathcal{D}}}_b n_L \qquad (3.73) \\
&= -\frac{1}{\kappa}\int dt \int_M \left[\frac{1}{2}\underbrace{(t^a e_{aI})}_{(t\cdot e)^I}F^I_{bc}\tilde{\varepsilon}^{bc} + t^b\tilde{\varepsilon}^{ca}e_{aI}F^I_{bc}\right] \\
&\quad -\frac{1}{\kappa}\int dt \int_{C_t}\left[\underbrace{(t^a e_{aI})}_{(t\cdot e)^I}\omega^I_b\tilde{\varepsilon}^b - \underbrace{(t^b\omega^I_b)}_{(t\cdot\omega)^I}e_{aI}\tilde{\varepsilon}^a\right] \qquad (3.74) \\
&\quad + \frac{\alpha}{\kappa}\int dt \int_{C_t}\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}\left[(t^a e_{aI})\mathring{\bar{\mathcal{D}}}_b n_L\tilde{\varepsilon}^b - (t^b\mathring{\bar{\mathcal{D}}}_b n_L)e_{aI}\tilde{\varepsilon}^a\right] \qquad (3.75)
\end{aligned}$$

Taking into account the following standard relations,

$$\begin{aligned}
F^I_{bc} &= 2\partial_{[b}\omega^I_{c]} + [\omega_b,\omega_c]^I = \partial_b\omega_c - \partial_c\omega_b + [\omega_b,\omega_c]^I \qquad (3.76) \\
\mathcal{D}_b\omega^I_c &= \partial_b\omega^I_c + [\omega_b,\omega_c]^I \qquad (3.77) \\
t^b F^I_{bc} &= \pounds_{\vec{t}}\omega^I_c - \mathcal{D}_c(t\cdot\omega)^I \qquad (3.78)
\end{aligned}$$

the second term of the bulk part can be written as,

$$\begin{aligned}
\tilde{\varepsilon}^{ca}e_{aI}t^b F^I_{bc} &= (\pounds_{\vec{t}}\omega^I_c)\tilde{\varepsilon}^{ca}e_{aI} - \mathcal{D}_c(\omega\cdot t)^I\tilde{\varepsilon}^{ca}e_{aI} \qquad (3.79) \\
&= (\pounds_{\vec{t}}\omega^I_c)\tilde{\varepsilon}^{ca}e_{aI} - \mathcal{D}_c[(\omega\cdot t)^I\tilde{\varepsilon}^{ca}e^I_{aI}] + (\omega\cdot t)^I\mathcal{D}_c(\tilde{\varepsilon}^{ca}e_{aI}). \qquad (3.80)
\end{aligned}$$

Then the action takes the form,

$$
\begin{aligned}
S_{PB}[e,\omega] &= -\frac{1}{\kappa}\int dt\int_M \left[\frac{1}{2}\underbrace{(t^a e_{aI})}_{(t\cdot e)_I} F^I_{bc}\tilde{\varepsilon}^{bc} + (\pounds_{\vec{t}}\omega^I_c)\tilde{\varepsilon}^{ca}e_{aI} - \mathcal{D}_c[(\omega\cdot t)^I\tilde{\varepsilon}^{ca}e_{aI}] + (\omega\cdot t)^I\mathcal{D}_c(\tilde{\varepsilon}^{ca}e_{aI})\right] \\
&\quad -\frac{1}{\kappa}\int dt\int_{C_t}\left[\underbrace{(t^a e_{aI})}_{(t\cdot e)_I}\omega^I_b\tilde{\varepsilon}^b - \underbrace{(t^b\omega^I_b)}_{(t\cdot\omega)^I}e_{aI}\tilde{\varepsilon}^a\right] \qquad (3.81) \\
&\quad +\frac{\alpha}{\kappa}\int dt\int_{C_t}\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}\left[(t^a e_{aI})\mathring{\tilde{\mathcal{D}}}_b n_L\tilde{\varepsilon}^b - (t^b\mathring{\tilde{\mathcal{D}}}_b n_L)e_{aI}\tilde{\varepsilon}^a\right] \qquad (3.82) \\
&= -\frac{1}{\kappa}\int dt\int_M\left[\frac{1}{2}\underbrace{(t^a e_{aI})}_{(t\cdot e)_I}F^I_{bc}\tilde{\varepsilon}^{bc} + (\pounds_{\vec{t}}\omega^I_c)\tilde{\varepsilon}^{ca}e_{aI} + (\omega\cdot t)^I\mathcal{D}_c(\tilde{\varepsilon}^{ca}e_{aI})\right] \\
&\quad +\frac{1}{\kappa}\int dt\int_M\mathcal{D}_c[(\omega\cdot t)^I\tilde{\varepsilon}^{ca}e_{aI}] - \frac{1}{\kappa}\int dt\int_{C_t}\left[\underbrace{(t^a e_{aI})}_{(t\cdot e)_I}\omega^I_b\tilde{\varepsilon}^b - \underbrace{(t^b\omega^I_b)}_{(t\cdot\omega)^I}e_{aI}\tilde{\varepsilon}^a\right] \qquad (3.83) \\
&\quad +\frac{\alpha}{\kappa}\int dt\int_{C_t}\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}\left[(t^a e_{aI})\mathring{\tilde{\mathcal{D}}}_b n_L\tilde{\varepsilon}^b - (t^b\mathring{\tilde{\mathcal{D}}}_b n_L)e_{aI}\tilde{\varepsilon}^a\right]. \qquad (3.84)
\end{aligned}
$$

Strictly speaking we begin with an action valid for *any* Lie group (e is not related to the metric unless we identify the group with $SO(2,1)$ so this action can be defined without the need of a metric), more over in Witten's decomposition we are not assuming the existence of a metric. Therefore we can not use the Stokes theorem that needs the normal to the surface. So we left the term $\int_M \mathcal{D}_c[(\omega\cdot t)^I\tilde{\varepsilon}^{ca}e_{aI}]$ indicated.

In order to proceed to the Legendre transformation we need to calculate the momenta,

$$\Pi^c_I = \frac{\delta\mathcal{L}}{\delta(\pounds_{\vec{t}}\omega^I_c)} = \frac{1}{\kappa}\tilde{\varepsilon}^{ca}e_{aI}, \qquad (3.85)$$

then the canonical Hamiltonian is[1],

$$\begin{aligned}
H[e,\omega] &= \int_M \left[(\pounds_{\vec{t}}\omega_c^I)\Pi_I^c - \mathcal{L}\right] \\
&= +\frac{1}{\kappa}\int_M \left[\frac{1}{2}\underbrace{(t^a e_{aI})}_{(t\cdot e)_I} F_{bc}^I \tilde{\varepsilon}^{bc} + (\omega\cdot t)^I \mathcal{D}_c(\tilde{\varepsilon}^{ca} e_{aI})\right] \\
&\quad -\frac{1}{\kappa}\int_M \mathcal{D}_c[(\omega\cdot t)^I \tilde{\varepsilon}^{ca} e_{aI}] + \frac{1}{\kappa}\int_{C_t}\left[\underbrace{(t^a e_{aI})}_{(t\cdot e)_I}\omega_b^I\tilde{\varepsilon}^b - \underbrace{(t^b\omega_b^I)}_{(t\cdot\omega)^I} e_{aI}\tilde{\varepsilon}^a\right] \\
&\quad -\frac{\alpha}{\kappa}\int_{C_t}\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}\left[(t^a e_{aI})\mathring{\bar{\mathcal{D}}}_b n_L \tilde{\varepsilon}^b - (t^b \mathring{\bar{\mathcal{D}}}_b n_L) e_{aI}\tilde{\varepsilon}^a\right].
\end{aligned} \tag{3.86}$$

We can see that the following contraints

$$F_{bc}^I \tilde{\varepsilon}^{bc} \approx 0 \;\text{ and }\; \mathcal{D}_c(\tilde{\varepsilon}^{ca} e_{aI}) \approx 0, \tag{3.87}$$

are first class, and also they are the pull-back to M with $\tilde{\varepsilon}^{ab}$ of the equations of motion (3.38).

On the constraint surface,

$$\begin{aligned}
H[e,\omega] &= -\frac{1}{\kappa}\int_M \mathcal{D}_c[(\omega\cdot t)^I \tilde{\varepsilon}^{ca} e_{aI}] + \frac{1}{\kappa}\int_{C_t}\left[\underbrace{(t^a e_{aI})}_{(t\cdot e)_I}\omega_b^I\tilde{\varepsilon}^b - \underbrace{(t^b\omega_b^I)}_{(t\cdot\omega)^I} e_{aI}\tilde{\varepsilon}^a\right] \\
&\quad -\frac{\alpha}{\kappa}\int_{C_t}\varepsilon^{IL}\left[(t^a e_{aI})\mathring{\bar{\mathcal{D}}}_b \frac{n_L}{\sqrt{n\cdot n}}\tilde{\varepsilon}^b - \left(t^b \mathring{\bar{\mathcal{D}}}_b \frac{n_L}{\sqrt{n\cdot n}}\right) e_{aI}\tilde{\varepsilon}^a\right].
\end{aligned} \tag{3.88}$$

that is, the boundary terms are the only non-vanishing terms.

Now if we take into account the asymptotically flat boundary conditions, the leading term of $(\omega\cdot t)^I$ is zero and also $t^b\mathring{\bar{\mathcal{D}}}_b(r^c\,{}^0e_{cL}) = 0$. In the timelike boundary as well as in the boundary of M (circles for each time t, C_t) the normal to the surface is r^a, then $n_L/\sqrt{n\cdot n} = r^c e_{cL}$. So the only non-vanishing leading term comes from,

$$H[e,\omega] = \frac{1}{\kappa}\int_{C_t}(t\cdot e)_I\omega_b^I\tilde{\varepsilon}^b - \frac{\alpha}{\kappa}\int_{C_t}\varepsilon^{IL}(t^a e_{aI})\underbrace{\mathring{\bar{\mathcal{D}}}_b(r^c e_{cL})}_{r^c\mathring{\bar{\mathcal{D}}}_b e_{cL} + e_{cL}\mathring{\bar{\mathcal{D}}}_b r^c}\tilde{\varepsilon}^b. \tag{3.89}$$

As in the covariant case, if we want this hamiltonian to generate asympotic time translations and therefore its conserved quantity to be the *energy*, t^a has to approach a time-translation Killing vector field of the asymptotic flat spacetime, which also translates in t being orthogonal to M.

[1]Note that the bulk part of this hamiltonian coincides with that given in [60].

Using this and the fall-off conditions (3.14) and (3.24), the hamiltonian is given by[1],

$$H[e,\omega] = \lim_{r\to\infty}\int_{C_t}\left[\underbrace{\frac{1}{\kappa}{}^0e_{0I}\frac{{}^1\omega^I_{\bar{b}}}{r}\tilde{\varepsilon}^{\bar{b}}}_{H_1}\underbrace{-\frac{\alpha}{\kappa}\int_{C_t}\varepsilon^{IL\,0}e_{0I}\left({}^0e_{cL}\mathring{\mathcal{D}}_b r^c\right)\tilde{\varepsilon}^b}_{H_2}+\mathcal{O}(r^{-\beta/2})\right] \tag{3.91}$$

For the first term of the right hand side of previous equation, since the volume element associated to C_t goes as $rd\theta$, the leading term of the previous equation does not depend on r, and the next to leading terms go as $\mathcal{O}(r^{-1})$ so in the limit they vanish leaving us just with the leading term,

$$H_1 = \frac{1}{\kappa}\lim_{r\to\infty}\int_{C_t}{}^0e_{0I}\frac{{}^1\bar{\omega}^I_{\bar{b}}}{r}\tilde{\varepsilon}^{\bar{b}} = \frac{1}{2\kappa}\lim_{r\to\infty}\int_{C_t}{}^0e_{0I}\frac{1}{r}\beta\partial_{\bar{a}}r\varepsilon_L{}^{KI\,0}\bar{e}^{\bar{a}}_K{}^0\bar{e}^L_{\bar{b}}\tilde{\varepsilon}^{\bar{b}} \tag{3.92}$$

Note that appart from $\delta\beta\leftrightarrow\beta$ this expression *is the same* as (3.65). Using the same steps we can see that (taking $\kappa = 8\pi G$),

$$H_1 = \frac{\beta}{2\kappa}\int_{C_t}d\theta = \frac{\beta}{2(8\pi G)}2\pi = \frac{\beta}{8G}. \tag{3.93}$$

[1]The term (that comes from eq. (3.89)),

$$\begin{aligned}
{}^{Leading}\lim_{r\to\infty}-\frac{\alpha}{\kappa}\int_{C_t}\varepsilon^{IL}(t^a e_{aI})r^c\mathring{\mathcal{D}}_b e_{cL}\tilde{\varepsilon}^b &= \lim_{r\to\infty}\left[-\frac{\alpha}{\kappa}\int_{C_t}\varepsilon^{IL\,0}e_{0I}r^c\Big(-\frac{\beta}{2r}r^{-\beta/2}\partial_b r\,{}^0\bar{e}_{\bar{c}L}\delta^{\bar{c}}_c\Big)\tilde{\varepsilon}^{0b}+\mathcal{O}(r^{-1})\right]\\
&= \lim_{r\to\infty}\left[\frac{\alpha\beta}{2\kappa}\int_{C_t}\underbrace{\varepsilon^{IL\,0}e_{0I}{}^0\bar{e}_{\bar{c}L}}_{\bar{e}\tilde{\varepsilon}_{0\bar{c}}}\frac{1}{r}r^{-\beta/2}r^{\bar{c}}\partial_b r\tilde{\varepsilon}^{0b}+\mathcal{O}(r^{-1})\right]\\
&= \lim_{r\to\infty}\left[\frac{\alpha\beta}{2\kappa}\int_{C_t}\frac{1}{r}r^{-\beta/2}r^{\bar{c}}\partial_b r\underbrace{\tilde{\varepsilon}_{0\bar{c}}\tilde{\varepsilon}^{0b}}_{\delta^b_{\bar{c}}}rd\theta+\mathcal{O}(r^{-1})\right]\\
&= \lim_{r\to\infty}\left[\frac{\alpha\beta}{2\kappa}\int_{C_t}r^{-\beta/2}(+1)d\theta+\mathcal{O}(r^{-1})\right]\\
&= \lim_{r\to\infty}\left[\frac{\alpha\beta}{2\kappa}r^{-\beta/2}2\pi+\mathcal{O}(r^{-1})\right]\\
&= \lim_{r\to\infty}\left[\mathcal{O}(r^{-\beta/2})+\mathcal{O}(r^{-1})\right] = 0 \;\; iff \;\; \beta>0
\end{aligned} \tag{3.90}$$

For the second term of the right hand side,

$$
\begin{aligned}
H_2 &= \lim_{r\to\infty}\left[-\frac{\alpha}{\kappa}\int_{C_t}\varepsilon^{IL\,0}e_{0I}\left({}^0e_{cL}\mathring{\tilde{\mathfrak{D}}}_b r^c\right)\tilde{\varepsilon}^b\right] \\
&= -\frac{\alpha}{\kappa}\lim_{r\to\infty}\int_{C_t}\underbrace{\varepsilon^{IL\,0}e_{0I}\,{}^0e_{cL}}_{\bar{e}\tilde{\varepsilon}^{0b}}\underbrace{\mathring{\tilde{\mathfrak{D}}}_b r^c}_{\partial_b r^c}\tilde{\varepsilon}^{0c} \\
&= -\frac{\alpha}{\kappa}\lim_{r\to\infty}\int_{C_t}\underbrace{\tilde{\varepsilon}^{0b}\tilde{\varepsilon}^{0c}}_{\delta_c^b}(\partial_b r^c)r\mathrm{d}\theta \\
&= -\frac{\alpha}{\kappa}\lim_{r\to\infty}\int_{C_t}\underbrace{(\partial_c r^c)}_{1/r}r\mathrm{d}\theta = -\frac{\alpha}{2\kappa}\int_{C_t}2d\theta
\end{aligned}
\tag{3.94}
$$

Using (3.92) and (3.94), we can see that the hamiltonian (3.91) is given by,

$$
H = H_1 + H_2 = \frac{\beta}{2\kappa}\int_{C_t}d\theta - \frac{\alpha}{2\kappa}\int_{C_t}2d\theta = -\frac{1}{2\kappa}\int_{C_t}(2\alpha-\beta)d\theta. \tag{3.95}
$$

When $\alpha = 1$ we recover the results of [47],

$$
H = -\frac{1}{2\kappa}\int_{C_t}(2-\beta)d\theta \tag{3.96}
$$

Following [11; 47], $\beta\in[0,2)$, we can see that energy is bounded from below and above by,

$$
E \in \left[-\frac{1}{4G}, 0\right]. \tag{3.97}
$$

Otherwise when $\alpha = 0$ we recover that of [11], that is the same as the covariant case considered in this work when considering the indetermination constant as zero.

In both cases, our analysis here and that given in [47] the starting point is a well posed action, the Palatini action with boundary term and the Einstein-Hilbert action with Gibbons-Hawking term. Also note that the addition of the boundary term (3.8) is essential in order to first order action, in this case LIP, to be equivalent to the Einstein-Hilbert action with Gibbons-Hawking term and therefore lead to the same expression for the energy.

Even though both actions, SPB and LIP, lead to the same classical equations of motion, the Einstein's equations of motion, they do not completely agree at the hamiltonian level, they differ up to a constant.

We should emphasize the difference between this result where the hamiltonian and therefore the energy is *completely* determined by the Legendre transform, in contrast with the covariant formalism where we only get the variation of the hamiltonian, so the energy it is always determined up to a constant (3.166).

3.4.2 Barbero-Varadarajan's approach

As in the Witten's decomposition, we begin with the well posed manifestly Lorentz invariant Palatini action,

$$S_{LIP}[e,\omega] = -\frac{1}{2\kappa}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc} e_{aI} F^I_{bc} - \frac{1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\omega^I_b \tilde{\varepsilon}^{ab} - \frac{\alpha}{\kappa}\int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_{aI} n_K \mathring{\tilde{\mathcal{D}}}_b n_L \tilde{\varepsilon}^{ab} \tag{3.98}$$

Using $\tilde{\varepsilon}^{abc}\varepsilon_{IJK}e^K_c = 2ee^{[a}_I e^{b]}_J$, which implies $e\,\varepsilon^{LKM}e^a_L e^b_K = \tilde{\varepsilon}^{abc}e^M_c$. The well posed Palatini action can be written,

$$S_{LIP}[e,\omega] = -\frac{1}{2\kappa}\int_{\mathcal{M}} e\varepsilon^{LKI} e^b_L e^c_K F_{bcI} - \frac{1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\omega^I_b \tilde{\varepsilon}^{ab} - \frac{\alpha}{\kappa}\int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_{aI} n_K \mathring{\tilde{\mathcal{D}}}_b n_L \tilde{\varepsilon}^{ab} \tag{3.99}$$

As we already mention, to make a standard $2+1$ *decomposition*, we assume the existence of a metric and thus we can introduce a projector $q^b_a = \delta^b_a + n_a n^b$ which projects down all the fields in their spacelike and normal components respectively. In particular we can decompose $t^a = n^a N + N^a$.

To begin with, we have to use q^b_a to project all the dinamical variables appearing in the action. First we shall decompose the integrand of the bulk term of the previous equation,

$$e\varepsilon^{LKI} e^b_L e^c_K F_{bcI} = e\varepsilon^{LKI} e^a_L e^d_K \delta^b_a \delta^c_d F_{bcI} = e\varepsilon^{LKI} e^a_L e^d_K (q^b_a - n_a n^b)(q^c_d - n_d n^c) F_{bcI} \tag{3.100}$$

with q_{ab} the induced metric and n^a the normal to the $2-$dimensional Cauchy slices. Now using $n^a = (t^a - N^a)/N$, also $\mathcal{E}^I_a = q^b_a e^I_b$ and $\mathcal{F}^I_{ab} = q^c_a q^d_b F^I_{cd}$ are the projections of e and F to the Cauchy slice, and $n_K := n^a e_{aK}$, then the integrand of the bulk term becomes,

$$e\varepsilon^{LKI} e^b_L e^c_K F_{bcI} = e\varepsilon^{LKI}\left[\mathcal{E}^b_L \mathcal{E}^c_K \mathcal{F}_{bcI} - \frac{2}{N}\mathcal{E}^b_L n_K t^c F_{bcI} + \frac{2}{N}\mathcal{E}^b_L n_K N^c \mathcal{F}_{bcI}\right]. \tag{3.101}$$

which implies that the decomposed bulk term is,

$$-\frac{1}{2\kappa}\int_{\mathcal{M}} e\varepsilon^{LKI} e^b_L e^c_K F_{bcI} = -\frac{1}{2\kappa}\int_{\mathcal{M}} e\varepsilon^{LKI}\left[\mathcal{E}^b_L \mathcal{E}^c_K \mathcal{F}_{bcI} - \frac{2}{N}\mathcal{E}^b_L n_K t^c F_{bcI} + \frac{2}{N}\mathcal{E}^b_L n_K N^c \mathcal{F}_{bcI}\right]. \tag{3.102}$$

Now we shall decompose the boundary term,

$$-\frac{1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\omega^I_b \tilde{\varepsilon}^{ab} - \frac{\alpha}{\kappa}\int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_{aI} n_K \mathring{\tilde{\mathcal{D}}}_b n_L \tilde{\varepsilon}^{ab}. \tag{3.103}$$

We begin with the integrand of the standard boundary term, $e_{aI}\omega^I_b\tilde{\varepsilon}^{ab}$,

$$e_{aI}\omega^I_b \tilde{\varepsilon}^{ab} = \delta^c_a \delta^d_b \tilde{\varepsilon}^{ab} e_{cI}\omega^I_d = (q^c_a - n_a n^c)(q^d_b - n_b n^d) e_{cI}\omega^I_d \tilde{\varepsilon}^{ab} \tag{3.104}$$

but $\tilde{\varepsilon}^{ab} = 2Nn^{[a}\tilde{\varepsilon}^{b]}dt$, then

$$\begin{aligned} e_{aI}\omega^I_b\tilde{\varepsilon}^{ab} &= N\Big[q^c_a q^d_b e_{cI}\omega^I_d(n^a\tilde{\varepsilon}^b - n^b\tilde{\varepsilon}^a) - q^c_a n_b n^d e_{cI}\omega^I_d(n^a\tilde{\varepsilon}^b - n^b\tilde{\varepsilon}^a) \\ &\quad - q^d_b n_a n^c e_{cI}\omega^I_d(n^a\tilde{\varepsilon}^b - n^b\tilde{\varepsilon}^a) + n_a n^c n_b n^d e_{cI}\omega^I_d(n^a\tilde{\varepsilon}^b - n^b\tilde{\varepsilon}^a)\Big]\, dt. \end{aligned} \tag{3.105}$$

Note that most of the terms vanishes due to $q^c_a n^a = 0$ or by antisymmetry of the indices, the non vanishing terms are,

$$e_{aI}\omega^I_b\tilde{\varepsilon}^{ab} = -N\left[q^c_a n_b n^d n^b\tilde{\varepsilon}^a - q^d_b n_a n^c n^a\tilde{\varepsilon}^b\right] e_{cI}\omega^I_d dt. \tag{3.106}$$

Since n^a is the normal to the spacelike surfaces M (and the splitting in the boundary is compatible with the spacetime one), $n_a n^a = -1$. Also we use $n^a = (t^a - N^a)/N$, $\mathcal{E}^I_a = q^b_a e^I_b$ and $\mathcal{W}^I_a = q^b_a\omega^I_b$, the integrand of the boundary term becomes,

$$\begin{aligned} e_{aI}\omega^I_b\tilde{\varepsilon}^{ab} &= -N\left[\mathcal{E}_{aI}\frac{1}{N}(t^d - N^d)(n_b n^b)\omega^I_d\tilde{\varepsilon}^a - \frac{1}{N}(t^c - N^c)\omega^I_d e_{cI}(n_a n^a)\tilde{\varepsilon}^b\right] dt \\ &= -(n_b n^b)\left[t^d\omega^I_d\mathcal{E}_{aI}\tilde{\varepsilon}^a - N^d\omega^I_d\mathcal{E}_{aI}\tilde{\varepsilon}^a + t^c e_{cI}\mathcal{W}^I_d\tilde{\varepsilon}^d - N^c e_{cI}\mathcal{W}^I_d\tilde{\varepsilon}^d\right] dt \end{aligned} \tag{3.107}$$

which implies that the decomposed standard boundary term is,

$$-\frac{1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\omega^I_b\tilde{\varepsilon}^{ab} = -\frac{1}{\kappa}\int_{\partial\mathcal{M}}\left[t^d\omega^I_d\mathcal{E}_{aI}\tilde{\varepsilon}^a - N^d\omega^I_d\mathcal{E}_{aI}\tilde{\varepsilon}^a + t^c e_{cI}\mathcal{W}^I_d\tilde{\varepsilon}^d - N^c e_{cI}\mathcal{W}^I_d\tilde{\varepsilon}^d\right] dt \tag{3.108}$$

Now we decompose the integrand of the additional boundary term (3.8), $\frac{1}{n\cdot n}\varepsilon^{IKL}e_{aI}n_K\mathring{\tilde{\mathcal{D}}}_b n_L\tilde{\varepsilon}^{ab}$,

$$
\begin{aligned}
\frac{1}{n\cdot n}\varepsilon^{IKL}e_{aI}n_K\mathring{\tilde{\mathcal{D}}}_b n_L\tilde{\varepsilon}^{ab} &= \frac{1}{n\cdot n}\varepsilon^{IKL}e_{cI}\delta^c_a n_K\delta^d_b\mathring{\tilde{\mathcal{D}}}_d n_L\tilde{\varepsilon}^{ab} \\
&= \frac{1}{n\cdot n}\varepsilon^{IKL}e_{cI}n_K\mathring{\tilde{\mathcal{D}}}_d n_L(q^c_a - n_a n^c)(q^d_b - n_b n^d)\tilde{\varepsilon}^{ab} \\
&= -\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}e_{cI}\mathring{\tilde{\mathcal{D}}}_d n_L(q^c_a - n_a n^c)(q^d_b - n_b n^d)(n^a\tilde{\varepsilon}^b - n^b\tilde{\varepsilon}^a)dt \\
&= \frac{N}{\sqrt{n\cdot n}}\varepsilon^{IL}e_{cI}\mathring{\tilde{\mathcal{D}}}_d n_L\left(q^c_a\underbrace{(n_b n^b)}_{-1}n^d\tilde{\varepsilon}^a + q^d_b\underbrace{(n_a n^a)}_{-1}n^c\tilde{\varepsilon}^b\right)dt \\
&= -\frac{N}{\sqrt{n\cdot n}}\varepsilon^{IL}e_{cI}\mathring{\tilde{\mathcal{D}}}_d n_L\left(q^c_a n^d\tilde{\varepsilon}^a + q^d_b n^c\tilde{\varepsilon}^b\right)dt \\
&= -\frac{N}{\sqrt{n\cdot n}}\varepsilon^{IL}e_{cI}\mathring{\tilde{\mathcal{D}}}_d n_L q^c_a\left(\frac{t^d - N^d}{N}\right)\tilde{\varepsilon}^a \\
&\quad +\frac{N}{\sqrt{n\cdot n}}\varepsilon^{IL}e_{cI}\mathring{\tilde{\mathcal{D}}}_d n_L q^d_b\left(\frac{t^c - N^c}{N}\right)\tilde{\varepsilon}^b dt \\
&= -\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}\mathcal{E}_{aI}\mathring{\tilde{\mathcal{D}}}_d n_L\left(t^d - N^d\right)\tilde{\varepsilon}^a \\
&\quad +\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}e_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L(t^c - N^c)\tilde{\varepsilon}^b dt \\
&= -\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}\left[\left(\mathcal{E}_{aI}t^d\mathring{\tilde{\mathcal{D}}}_d n_L - \mathcal{E}_{aI}N^d\mathring{\tilde{\mathcal{D}}}_d n_L\right)\tilde{\varepsilon}^a\right. \\
&\quad \left.+\left(t^c e_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L - N^c\mathcal{E}_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L\right)\tilde{\varepsilon}^b\right]dt
\end{aligned}
\tag{3.109}
$$

for the previous equation we used $n^c = \frac{1}{N}(t^c - N^c)$, n^a is normal to a spacelike surface so $n_a n^a = -1$, $\mathring{\tilde{\mathcal{D}}}_d$ is spatial so $q^d_b\mathring{\tilde{\mathcal{D}}}_d = \mathring{\tilde{\mathcal{D}}}_d$, and $\mathcal{E}^I_a = q^b_a e^I_b$. Thus the decomposed boundary term ((3.8)) is,

$$
\begin{aligned}
-\frac{\alpha}{\kappa}\int_{\partial\mathcal{M}}\frac{1}{n\cdot n}\varepsilon^{IKL}e_{aI}n_K\mathring{\tilde{\mathcal{D}}}_b n_L\tilde{\varepsilon}^{ab} &= \frac{\alpha}{\kappa}\int_{\partial\mathcal{M}}\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL}\left[\left(\mathcal{E}_{aI}t^d\mathring{\tilde{\mathcal{D}}}_d n_L - \mathcal{E}_{aI}N^d\mathring{\tilde{\mathcal{D}}}_d n_L\right)\tilde{\varepsilon}^a\right. \\
&\quad \left.+\left(t^c e_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L - N^c\mathcal{E}_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L\right)\tilde{\varepsilon}^b\right]dt
\end{aligned}
\tag{3.110}
$$

Using (3.102), (3.108), (3.110) and $e = \sqrt{-g} = N\sqrt{|q|} = N\mathcal{E}$ with q the determinant of the induced metric q_{ab} on M and $\mathcal{E}$ the determinant of $\mathcal{E}^a_I$, we can rewrite the action (3.71) as,

$$
\begin{aligned}
S_{PB}[e,\omega] &= -\frac{1}{2\kappa}\int dt\int_M N\mathcal{E}\varepsilon^{LKI}\left[\mathcal{E}^b_L\mathcal{E}^c_K\mathcal{F}_{bcI}-\frac{2}{N}\mathcal{E}^b_L n_K t^c F_{bcI}+\frac{2}{N}\mathcal{E}^b_L n_K N^c\mathcal{F}_{bcI}\right]\\
&\quad -\frac{1}{\kappa}\int dt\int_{\partial M}\left[t^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a-N^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a+t^c e_{cI}\mathcal{W}^I_d\varepsilon^d-N^c e_{cI}\mathcal{W}^I_d\varepsilon^d\right]\\
&\quad +\frac{\alpha}{\kappa}\int dt\int_{\partial M}\frac{1}{n\cdot n}\varepsilon^{IL}\left[\left(\mathcal{E}_{aI}t^d\mathring{\tilde{\mathcal{D}}}_d n_L-\mathcal{E}_{aI}N^d\mathring{\tilde{\mathcal{D}}}_d n_L\right)\tilde{\varepsilon}^a\right.\\
&\quad \left.+\left(t^c e_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L-N^c\mathcal{E}_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L\right)\tilde{\varepsilon}^b\right]
\end{aligned}
\tag{3.111}
$$

As in the Witten decomposition, we use (3.78) to rewrite the second term of the bulk part of the action,

$$
\begin{aligned}
N\mathcal{E}\varepsilon^{LKI}\left(\frac{2}{N}\mathcal{E}^b_L n_K t^c F^I_{cb}\right) &= \mathcal{E}\varepsilon^{LKI}2\mathcal{E}^b_L n_K \pounds_{\vec{t}}\omega^I_b-\mathcal{E}\varepsilon^{LKI}2\mathcal{E}^b_L n_K\mathcal{D}_b(t\cdot\omega)^I\\
&= 2\mathcal{E}\varepsilon^{LKI}\left[\mathcal{E}^b_L n_K\pounds_{\vec{t}}\omega^I_b+\mathcal{D}_b\left(\mathcal{E}^b_L n_K\right)(t\cdot\omega)^I\right]\\
&\quad -\mathcal{D}_b\left[\mathcal{E}\varepsilon^{LKI}2\mathcal{E}^b_L n_K(t\cdot\omega)^I\right]
\end{aligned}
\tag{3.112}
$$

Then the action can be written,

$$
\begin{aligned}
S_{PB}[e,\omega] &= -\frac{1}{2\kappa}\int dt\int_M\left[N\mathcal{E}\varepsilon^{LKI}\mathcal{E}^b_L\mathcal{E}^c_K\mathcal{F}_{bcI}+2\mathcal{E}\varepsilon^{LKI}\left(\mathcal{E}^b_L n_K\pounds_{\vec{t}}\omega^I_b+\mathcal{D}_b\left(\mathcal{E}^b_L n_K\right)(t\cdot\omega)^I\right.\right.\\
&\quad \left.\left.+\mathcal{E}^b_L n_K N^c\mathcal{F}_{bcI}\right)\right]+\frac{1}{2\kappa}\int dt\int_M\mathcal{D}_b\left[\mathcal{E}\varepsilon^{LKI}2\mathcal{E}^b_L n_K(t\cdot\omega)^I\right]\\
&\quad -\frac{1}{\kappa}\int dt\int_{\partial M}\left[t^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a-N^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a+t^c e_{cI}\mathcal{W}^I_d\varepsilon^d-N^c e_{cI}\mathcal{W}^I_d\varepsilon^d\right]\\
&\quad +\frac{\alpha}{\kappa}\int dt\int_{\partial M}\frac{1}{n\cdot n}\varepsilon^{IL}\left[\left(\mathcal{E}_{aI}t^d\mathring{\tilde{\mathcal{D}}}_d n_L-\mathcal{E}_{aI}N^d\mathring{\tilde{\mathcal{D}}}_d n_L\right)\tilde{\varepsilon}^a\right.\\
&\quad \left.+\left(t^c e_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L-N^c\mathcal{E}_{cI}\mathring{\tilde{\mathcal{D}}}_b n_L\right)\tilde{\varepsilon}^b\right]
\end{aligned}
\tag{3.113}
$$

To find the hamiltonian we need to calculate the momenta to perform the Legendre transformation,

$$
\Pi^b_I=\frac{\delta\mathcal{L}}{\delta(\pounds_{\vec{t}}\omega^I_b)}=\frac{1}{\kappa}\mathcal{E}\varepsilon^{LKI}\mathcal{E}^b_L n_K
\tag{3.114}
$$

Then,

$$
\begin{aligned}
H[e,\omega] &= \int_M \left[(\pounds_{\vec{t}}\omega^I_c)\Pi^c_I - \mathcal{L} \right] \\
&= +\frac{1}{2\kappa}\int_M \left[N\mathcal{E}\varepsilon^{LKI}\mathcal{E}^b_L\mathcal{E}^c_K\mathcal{F}_{bcI} + 2\mathcal{E}\varepsilon^{LKI}\left[\mathcal{D}_b\left(\mathcal{E}^b_L n_K\right)(t\cdot\omega)^I + \mathcal{E}^b_L n_K N^c\mathcal{F}_{bcI}\right]\right] \\
&\quad -\frac{1}{2\kappa}\int_M \mathcal{D}_b\left[\mathcal{E}\varepsilon^{LKI}2\mathcal{E}^b_L n_K(t\cdot\omega)^I\right] \\
&\quad +\frac{1}{\kappa}\int_{\partial M}\left[t^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a - N^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a + t^c e_{cI}\mathcal{W}^I_d\varepsilon^d - N^c e_{cI}\mathcal{W}^I_d\varepsilon^d\right] \\
&\quad -\frac{\alpha}{\kappa}\int_{\partial M}\frac{1}{n\cdot n}\varepsilon^{IL}\left[\left(\mathcal{E}_{aI}t^d\mathring{\bar{\mathcal{D}}}_d n_L - \mathcal{E}_{aI}N^d\mathring{\bar{\mathcal{D}}}_d n_L\right)\tilde{\varepsilon}^a\right. \\
&\quad \left. +\left(t^c e_{cI}\mathring{\bar{\mathcal{D}}}_b n_L - N^c\mathcal{E}_{cI}\mathring{\bar{\mathcal{D}}}_b n_L\right)\tilde{\varepsilon}^b\right]
\end{aligned} \tag{3.115}
$$

Note that within this decomposition we have 'more structure', now we have three constraints

$$
\varepsilon^{LKI}\mathcal{E}^b_L\mathcal{E}^c_K\mathcal{F}_{bcI}\approx 0,\;\; \varepsilon^{LKI}\mathcal{D}_b\left(\mathcal{E}^b_L n_K\right)\approx 0 \;\text{ and }\; \mathcal{E}^b_L n_K\mathcal{F}_{bcI}\approx 0, \tag{3.116}
$$

instead of the two found by the Witten approach (3.87).

On the constraint surface we are left only with the boundary term,

$$
\begin{aligned}
H &= -\frac{1}{2\kappa}\int_M \mathcal{D}_b\left[\mathcal{E}\varepsilon^{LKI}2\mathcal{E}^b_L n_K(t\cdot\omega)^I\right] \\
&\quad +\frac{1}{\kappa}\int_{\partial M}\left[t^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a - N^d\omega^I_d\mathcal{E}_{aI}\varepsilon^a + t^c e_{cI}\mathcal{W}^I_d\varepsilon^d - N^c e_{cI}\mathcal{W}^I_d\varepsilon^d\right] \\
&\quad -\frac{\alpha}{\kappa}\int_{\partial M}\frac{1}{n\cdot n}\varepsilon^{IL}\left[\left(\mathcal{E}_{aI}t^d\mathring{\bar{\mathcal{D}}}_d n_L - \mathcal{E}_{aI}N^d\mathring{\bar{\mathcal{D}}}_d n_L\right)\tilde{\varepsilon}^a\right. \\
&\quad \left. +\left(t^c e_{cI}\mathring{\bar{\mathcal{D}}}_b n_L - N^c\mathcal{E}_{cI}\mathring{\bar{\mathcal{D}}}_b n_L\right)\tilde{\varepsilon}^b\right]
\end{aligned} \tag{3.117}
$$

Now considering the asymptotically flat boundary conditions, the leading term of $(t\cdot\omega)^I = 0$ and also since $\mathring{\bar{\mathcal{D}}}_d$ is spatial $t^d\mathring{\bar{\mathcal{D}}}_d n_L = 0$. So we are left with

$$
\begin{aligned}
H &= \lim_{r\to\infty}\left\{ -\frac{1}{\kappa}\int_{\partial M}\left[N^{\bar{d}}\,{}^1\mathcal{W}^I_{\bar{d}}\,{}^0\mathcal{E}_{\bar{a}I}\varepsilon^{\bar{a}} - t^c\,{}^0e_{cI}\,{}^1\mathcal{W}^I_{\bar{d}}\varepsilon^{\bar{d}} + N^{\bar{c}}\,{}^0\mathcal{E}_{\bar{c}I}\,{}^1\mathcal{W}^I_{\bar{d}}\varepsilon^{\bar{d}}\right]\right. \\
&\quad -\frac{\alpha}{\kappa}\int_{\partial M}\frac{1}{n\cdot n}\varepsilon^{IL}\left[-{}^0\mathcal{E}_{aI}N^d\mathring{\bar{\mathcal{D}}}_d n_L\tilde{\varepsilon}^a + \left(t^c\,{}^0e_{cI}\mathring{\bar{\mathcal{D}}}_b n_L - N^c\,{}^0\mathcal{E}_{cI}\mathring{\bar{\mathcal{D}}}_b n_L\right)\tilde{\varepsilon}^b\right] \\
&\quad \left. +\mathcal{O}(r^{-1})\right\}
\end{aligned} \tag{3.118}
$$

In addition to the fall-off conditions on e and ω, now we have to take into account the the behaviour of the laps N and shift N^a functions on the asymptotic region for time-translations (following

[11; 47]),

$$N = 1 + \mathcal{O}(r^{-1}) \tag{3.119}$$

$$N^a = \mathcal{O}(r^{-1-\beta}), \tag{3.120}$$

Note that in the asymptotic region the projections $\mathcal{E}^I_a = q^b_a e^I_b$ and $\mathcal{W}^I_a = q^b_a \omega^I_b$ coincide with $e^I_{\bar{a}}$ and $\omega^I_{\bar{a}}$. With conditions (3.119),(3.120) and considering the order of leading terms of e and ω: ${}^1\omega^I_{\bar{d}} = \mathcal{O}(r^{-1}) = {}^1\mathcal{W}^I_{\bar{d}}$, ${}^0e_{\bar{c}I} = \mathcal{O}(r^{-\beta/2}) = {}^0\mathcal{E}_{\bar{a}I}$, and that $\varepsilon^{\bar{d}} = \mathcal{O}(r)$. Note that to first order the first and third terms in (3.118) decay as,

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} N^{\bar{d}}\, {}^1\omega^I_{\bar{d}}\, {}^0\mathcal{E}_{\bar{a}I}\varepsilon^{\bar{a}} &= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-1-\beta})\mathcal{O}(r^{-1})\mathcal{O}(r^{-\beta/2})\mathcal{O}(r) && (3.121)\\
&= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-1-3\beta/2}) = 0 && (3.122)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} N^{\bar{c}}\, {}^0\mathcal{E}_{\bar{c}I}\, {}^1\mathcal{W}^I_{\bar{d}}\varepsilon^{\bar{d}} &= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-1-\beta})\mathcal{O}(r^{-\beta/2})\mathcal{O}(r^{-1})\mathcal{O}(r) && (3.123)\\
&= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-1-3\beta/2}) = 0, && (3.124)
\end{aligned}$$

respectively. And the fourth and sixth terms decay as,

$$\begin{aligned}
\lim_{r\to\infty} \frac{\alpha}{\kappa}\int_{\partial M} \frac{1}{n\cdot n}\varepsilon^{IL}\left[{}^0\mathcal{E}_{aI}N^d \mathring{\bar{\mathcal{D}}}_d n_L \tilde{\varepsilon}^a\right] &= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-\beta/2})\mathcal{O}(r^{-1-\beta})\mathcal{O}(r^{-1-\beta/2})\mathcal{O}(r) \\
&= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-1-2\beta}) = 0 && (3.125)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{r\to\infty} \frac{\alpha}{\kappa}\int_{\partial M} \frac{1}{n\cdot n}\varepsilon^{IL}\left[N^c\, {}^0\mathcal{E}_{cI} \mathring{\bar{\mathcal{D}}}_b n_L \tilde{\varepsilon}^b\right] &= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-1-\beta})\mathcal{O}(r^{-\beta/2})\mathcal{O}(r^{-1-\beta/2})\mathcal{O}(r) \\
&= \lim_{r\to\infty} \frac{1}{2\kappa}\int_{\partial M} \mathcal{O}(r^{-1-2\beta}) = 0. && (3.126)
\end{aligned}$$

Therefore, H can be written as,

$$H = \lim_{r\to\infty}\left\{-\frac{1}{\kappa}\int_{\partial M}\left[-t^c\, {}^0e_{cI}\, {}^1\mathcal{W}^I_{\bar{d}}\varepsilon^{\bar{d}}\right] - \frac{\alpha}{\kappa}\int_{\partial M}\frac{1}{n\cdot n}\varepsilon^{IL}\left[t^c\, {}^0e_{cI}\mathring{\bar{\mathcal{D}}}_b n_L\tilde{\varepsilon}^b\right] + \mathcal{O}(r^{-1})\right\}$$

As in the previous sections, if we want this hamiltonian to generate asympotic time translations and therefore its conserved quantity to be the *energy*, t^a has to approach a time-translation Killing vector field of the asymptotic flat spacetime, which also translates in t being orthogonal to M. In that case the previous expression coincides with (3.89) from

Therefore, H can be written as,

$$\begin{aligned} H &= \lim_{r\to\infty}\left\{-\frac{1}{\kappa}\int_{\partial M}\left[-t^{c\,0}e_{cI}\,{}^{1}\mathcal{W}^{I}_{\bar{d}}\varepsilon^{\bar{d}}\right]-\frac{\alpha}{\kappa}\int_{\partial M}\frac{1}{n\cdot n}\varepsilon^{IL}\left[t^{c\,0}e_{cI}\mathring{\tilde{\mathcal{D}}}_{b}n_{L}\tilde{\varepsilon}^{b}\right]+\mathcal{O}(r^{-1})\right\} \\ &= \lim_{r\to\infty}\left\{\frac{1}{\kappa}\int_{\partial M}{}^{0}e_{0I}\frac{{}^{1}\bar{\omega}^{I}_{\bar{d}}}{r}\varepsilon^{\bar{d}}-\frac{\alpha}{\kappa}\int_{C_t}\frac{1}{\sqrt{n\cdot n}}\varepsilon^{IL\,0}e_{0I}\left({}^{0}e_{cL}\mathring{\tilde{\mathcal{D}}}_{b}r^{c}\right)\tilde{\varepsilon}^{b}\right\}. \end{aligned} \tag{3.127}$$

Which is exactly the same term as (3.91), the one found by the Witten's decomposition. Therefore the hamiltonian is the same as (**??**),

$$H=-\frac{1}{2\kappa}\int_{C_t}(2\alpha-\beta)d\theta \tag{3.128}$$

Following [11; 47], $\beta\in[0,2)$, we can see, when $\alpha=1$, that energy is bounded from below and above by,

$$E\in\left[-\frac{1}{4G},0\right]. \tag{3.129}$$

Note that at the end of the day, the result for the energy is the same in both decompositions as expected, this is due to the fact that at the asymptotic region the direction of t^a coincides with n^a, and also the laps y shift functions decay in such a way. This may not be true for other conserved quantities as the angular momentum, but we shall leave the discussion to forthcoming works.

3.5 Chern-Simons

Now we shall analyse the Chern-Simons theory based on the Poincaré group $ISO(2,1)$ which, as pointed out in [60], at the action level is *equivalent* up to a boundary term to the $2+1$ Palatini theory based on $SO(2,1)$. A natural question would be whether this boundary term coincides with the one we add to the Palatini action to make it well posed, or what would be needed to recover, from Chern-Simons action, the Lorentz invariant well posed Palatini action we previously introduced.

As we show below, beginning from standard Chern-Simons action, by taking $A^i_a:=(e_{aI},\omega^I_a)$, we obtain Palatini action plus a boundary term that is proportional to that in (3.7), although it does not have the correct relative factors, so we do not have a well posed Palatini action. A posible solution is to see whether we can add some additional boundary term to the Chern-Simons action. We explore that possibility in what follows.

3.5.1 The action

On one hand we can remember that for *any* Lie Group G, the Palatini action based on G, (3.71), is defined in terms of ω_a^I and e_{aI} which are $\pounds_G-$ and $\pounds_G^{\star}-$ valued 1-forms. Therefore we can construct the inhomogeneous Lie algebra $\pounds_{IG}$ associated with G and define a $\pounds_{IG}-$valued connection 1-form A_a^i by,

$$A_a^i := (e_{aI}, \omega_a^I) = (e_a, \omega_a)^i. \tag{3.130}$$

With this connection A_a^i we can formulate the standard Chern-Simons theory plus the addition of a boundary term,

$$\begin{aligned} {}^{IG}S_{CSB}[A] &= \underbrace{-\frac{1}{2\kappa}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc} k_{ij}\left(A_a^i \partial_b A_c^j + \frac{1}{3} A_a^i \left[A_b, A_c\right]^j\right)}_{{}^{IG}S_{CS}[A]} + \underbrace{\frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}} k_{ij} A_a^i f_b^j \tilde{\varepsilon}^{ab}}_{{}^{IG}S_B[A]} && (3.131) \\ &= {}^{IG}S_{CS}[A] + {}^{IG}S_B[A] && (3.132) \end{aligned}$$

where f_b^j is a spacetime 1-form and an arbitrary internal function/vector valued in $\pounds_{IG}$, in particular we can choose $f_b^j := (e_{bJ}, \frac{\bar{\alpha}_2}{n \cdot n} n_K \mathrm{d}_b n_L \varepsilon^{JKL})$ with $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are constants to be determined.

Using that $k_{ij}(\alpha, v)^i (\beta, w)^j = \alpha_I w^I + \beta_I v^I$, where k_{ij} is the Killing Cartan metric on the Lie algebra, and the Lie bracket on $\pounds_{IG}$ is given by $[(\alpha, v), (\beta, w)]^i := \left(-\{v, \beta\} + \{w, \alpha\}, [v, w]^i\right)$ which implies,

$$[A_b, A_c]^j = [(e_b, \omega_b), (e_c, \omega_c)]^j = \left(-\{\omega_b, e_c\}_I + \{\omega_c, e_b\}_I, [\omega_b, \omega_c]^I\right). \tag{3.133}$$

Then the Chern-Simons action can be written,

$${}^{IG}S_{CS}[A] = -\frac{1}{2\kappa}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc}\left\{e_{aI}\partial_b \omega_c^I + \omega_a^I \partial_b e_{cI} + \frac{1}{3}\left(e_{aI}[\omega_b, \omega_c]^I + \omega_a^I\left(-\{\omega_b, e_c\}_I + \{\omega_c, e_b\}_I\right)\right)\right\}. \tag{3.134}$$

Now using that,

$$[v, w]^I := C^I{}_{JK} v^J w^K \quad \text{and} \quad \{v, \beta\}_I := C^K{}_{JI} v^J \beta_K, \tag{3.135}$$

are the Lie bracket and coadjoint bracket associated with $\pounds_G$ and $C^I{}_{JK}$ are the structure constants,

$$\begin{aligned} \omega_a^I\left(-\{\omega_b, e_c\}_I + \{\omega_c, e_b\}_I\right) &= -\omega_a^I C^K{}_{JI} \omega_b^J e_{cK} + \omega_a^I C^K{}_{JI} \omega_c^J e_{bK} \\ &= -[\omega_b, \omega_a]^K e_{cK} + [\omega_c, \omega_a]^K e_{bK} && (3.136) \end{aligned}$$

thus,

$$ {}^{IG}S_{CS}[A] = -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{abc}\left\{e_{aI}\partial_b\omega_c^I + \omega_a^I\partial_b e_{cI} + \frac{1}{3}\left(e_{aI}[\omega_b,\omega_c]^I - [\omega_b,\omega_a]^K e_{cK} + [\omega_c,\omega_a]^K e_{bK}\right)\right\}. \tag{3.137} $$

renaming indices and integrating by parts the second term of the RHS,

$$ \begin{aligned} {}^{IG}S_{CS}[A] &= -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{abc}\left[e_{aI}(2\partial_b\omega_c^I) + e_{aI}[\omega_b,\omega_c]^I + \partial_b(\omega_a^I e_{cI})\right] \\ &= -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{abc}\left[e_{aI}F_{bc}^I + \partial_b(\omega_a^I e_{cI})\right]. \end{aligned} \tag{3.138} $$

Note that even though we recover Palatini action with a boundary term, this action is not well posed as (3.71), so we can see what is the effect of the boundary term introduced in (3.131),

$$ {}^{IG}S_B[A] = \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}} k_{ij}A_a^i f_b^j\tilde{\varepsilon}^{ab}. \tag{3.139} $$

Considering $A_a^i := (e_{aI},\omega_a^I)$ and $f_b^j := (e_{bJ}, \frac{\bar{\alpha}_2}{n\cdot n}n_K \mathrm{d}_b n_L \varepsilon^{JKL})$, $k_{ij}(\alpha,v)^i(\beta,w)^j = \alpha_I w^I + \beta_I v^I$, we can see that,

$$ {}^{IG}S_B[A] = \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}} k_{ij}A_a^i f_b^j\tilde{\varepsilon}^{ab} \tag{3.140} $$

$$ = \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}}\left(e_{aI}\frac{\bar{\alpha}_2}{n\cdot n}n_K\mathrm{d}_b n_L\varepsilon^{IKL} + \omega_a^I e_{bI}\right)\tilde{\varepsilon}^{ab}. \tag{3.141} $$

Now we can determine the constants $\bar{\alpha}_1$ and $\bar{\alpha}_2$, from

$$ \begin{aligned} {}^{IG}S_{CSB}[A] &= {}^{IG}S_{CS}[A] + {}^{IG}S_B[A] \\ &= -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{abc}\left[e_{aI}F_{bc}^I + \partial_b(\omega_a^I e_{cI})\right] + \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}}\left(e_{aI}\frac{\bar{\alpha}_2}{n\cdot n}n_K\mathrm{d}_b n_L\varepsilon^{IKL} + \omega_a^I e_{bI}\right)\tilde{\varepsilon}^{ab} \\ &= -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{abc}e_{aI}F_{bc}^I - \frac{1}{2\kappa}\int_{M}\tilde{\varepsilon}^{abc}\partial_b(\omega_a^I e_{cI}) + \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}}\omega_a^I e_{bI}\tilde{\varepsilon}^{ab} \\ &\quad + \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\frac{\bar{\alpha}_2}{n\cdot n}n_K\mathrm{d}_b n_L\varepsilon^{IKL} \\ &= -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{abc}e_{aI}F_{bc}^I + \frac{1}{2\kappa}\int_{\partial M}\omega_a^I e_{bI}\tilde{\varepsilon}^{ab} + \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}}\omega_a^I e_{bI}\tilde{\varepsilon}^{ab} \\ &\quad + \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\frac{\bar{\alpha}_2}{n\cdot n}n_K\mathrm{d}_b n_L\varepsilon^{IKL} \\ &= -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{abc}e_{aI}F_{bc}^I + \frac{1}{\kappa}\int_{\partial M}\left[\frac{1}{2}+\bar{\alpha}_1\right]\omega_a^I e_{bI}\tilde{\varepsilon}^{ab} + \frac{\bar{\alpha}_1\bar{\alpha}_2}{\kappa}\int_{\partial\mathcal{M}} e_{aI}\frac{1}{n\cdot n}n_K\mathrm{d}_b n_L\varepsilon^{IKL} \end{aligned} $$

In order to recover the well posed Lorentz invariant Palatini action, S_{LIP} (3.71), the constants $\bar{\alpha}_1$ and $\bar{\alpha}_2$ must satisfy, $\frac{1}{2}+\bar{\alpha}_1 = 1$ which implies $\bar{\alpha}_1 = \frac{1}{2}$ on the other hand $\bar{\alpha}_1\bar{\alpha}_2 = -\alpha$ thus

$\bar{\alpha}_2 = -2\alpha$, where as in previous sections α is a switch, when $\alpha = 1$ we recover S_{LIP}, for $\alpha = 0$ we recover the usual S_{SPB}.

3.5.2 Fall-off conditions and finiteness

For the asymptotic conditions of A_a^i note that when we define the Chern-Simons theory in terms of connections valued on the inhomogeneous Lie algebra of $SO(2,1)$, and then split the connection into $\pounds_G \otimes \pounds_G^\star$, the splitting is only on the internal indices and since the asymptotic conditions are defined over the spacetime (they are defined up to a transformation in the internal space) we can choose that $A_a^i = (e_{aI}, \omega_a^I)$ decay according to (3.14) and (4.37).

Now we have to see whether with this asymptotic conditions we have a well posed action principle, that is finite and differentiable.

Since we know the Palatini $SO(2,1)$ action is finite under these asymptotic conditions, by reversing steps in the previous section on the relation between ${}^{IG}S_{CS}$ and ${}^{G}S_P$, we can see that the finiteness of the former is equivalent to the finitenes of the later. And we already proved in (3.2.2.1) that the Palatini action with boundary term, the same as (3.138), is finite under the boundary conditions.

It is left to check if ${}^{IG}S_{CS}$ is differentiable under these asymptotically flat boundary conditions.

3.5.3 Differentiability

Taking the variation of the Chern-Simons action valued on the Lie algebra of ISO(2,1),

$$\begin{aligned}
\delta\, {}^{IG}S_{CS}[A] &= \delta\left[\frac{1}{2}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc} k_{ij}\left(A_a^i \partial_b A_c^j + \frac{1}{3} A_a^i\,[A_b, A_c]^j\right)\right] \\
&= \frac{1}{2}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc} k_{ij}\left[(\delta A_a^i)\partial_b A_c^j + \underbrace{A_a^i \partial_b \delta A_c^j}_{\partial_b(A_a^i \delta A_c^j) - \delta A_c^j \partial_b A_a^i} + \frac{1}{3}\delta\left(A_a^i [A_b, A_c]^j\right)\right]
\end{aligned} \tag{3.142}$$

but, using that $[A_b, A_c]^j := C^j{}_{mn} A_b^m A_c^n$, $k_{ij} C^j{}_{mn} = C_{imn}$ and $C_{imn} = C_{[imn]}$,

$$\begin{aligned}
\frac{\delta}{3}\left(\tilde{\varepsilon}^{abc} k_{ij} A_a^i [A_b, A_c]^j\right) &= \frac{\delta}{3}\left(\tilde{\varepsilon}^{abc} k_{ij} A_a^i C^j{}_{mn} A_b^m A_c^n\right) \\
&= \frac{\tilde{\varepsilon}^{abc} k_{ij} C^j{}_{mn}}{3}\left(\delta A_a^i A_b^m A_c^n + A_a^i \delta A_b^m A_c^n + A_a^i A_b^m \delta A_c^n\right) \\
&= \tilde{\varepsilon}^{abc} C_{imn} \delta A_a^i A_b^m A_c^n = \tilde{\varepsilon}^{abc} k_{ij} \delta A_a^i [A_b, A_c]^j
\end{aligned} \tag{3.143}$$

Then,

$$\delta^{IG}S_{CS}[A] = \frac{1}{2}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc}k_{ij}\left[\delta A^i_a \underbrace{\left(2\partial_b A^j_c + [A_b, A_c]^j\right)}_{=:F^j_{bc}} + \partial_b(A^i_a \delta A^j_c)\right] \tag{3.144}$$

To obtain the Euler-Lagrange equations of motion, we need the action principle to be stationary under the appropiate boundary conditions, in this case asymptotically flat boundary condiions. But the fall-off conditions on e and ω are different, so we have to split the connection $A^i_a := (e_{aI}, \omega^I_a) = (e_a, \omega_a)^i$. Rembering that A is valued on the Lie algebra of $ISO(2,1)$, the Lie Algebra is a vector space and because δ and ∂ act linearly we can take,

$$\partial_b A^i_c = (\partial_b e_{cI}, \partial_b \omega^I_c) \ \text{ and } \ \delta A^i_c = (\delta e_{cI}, \delta\omega^I_c). \tag{3.145}$$

For the integrand in the bulk we use $F^j_{bc} := 2\partial_{[b}A^j_{c]} + [A_b, A_c]^j$ and (3.133),

$$F^j_{bc} = 2\partial_{[b}\left(e_{c]}, \omega_{c]}\right)^j + \left(-\{\omega_b, e_c\} + \{\omega_c, e_b\}, [\omega_b, \omega_c]\right)^j \tag{3.146}$$

which implies,

$$\tilde{\varepsilon}^{abc}k_{ij}\delta A^i_a F^j_{bc} = \tilde{\varepsilon}^{abc}k_{ij}\left[2\left(\partial_b e_c, \partial_b\omega_c\right)^j\left(\delta e_c, \delta\omega_c\right)^i + \left(-\{\omega_b, e_c\} + \{\omega_c, e_b\}, [\omega_b, \omega_c]\right)^j\left(\delta e_c, \delta\omega_c\right)^i\right] \tag{3.147}$$

using $k_{ij}(\alpha, v)^i(\beta, w)^j = \alpha_I w^I + \beta_I v^I$,

$$\begin{aligned} \tilde{\varepsilon}^{abc}k_{ij}\delta A^i_a F^j_{bc} &= \tilde{\varepsilon}^{abc}\left(2(\partial_b e_{cI}\delta\omega^I_a + \partial_b\omega_c\delta e_{aI}) + 2\{\omega_c, e_b\}_I\delta\omega^I_a + \delta e_{aI}[\omega_b, \omega_c]^I\right) \\ &= \tilde{\varepsilon}^{abc}\left[2\left(\partial_b e_{cI} + \{\omega_c, e_b\}_I\right)\delta\omega^I_a + \left(2\partial_b\omega^I_c + [\omega_b, \omega_c]^I\right)\delta e_{aI}\right] \end{aligned} \tag{3.148}$$

but $\tilde{\varepsilon}^{abc}\{\omega_c, e_b\}_I = \tilde{\varepsilon}^{abc}C^K{}_{JI}\omega^J_c e_{bK} = -\tilde{\varepsilon}^{abc}C^J{}_{IK}\omega^K_c e_{bJ} = \tilde{\varepsilon}^{abc}C^J{}_{IK}\omega^K_b e_{cJ}$, and $D_b e_{cI} + \varepsilon^J_{IK}\omega^K_b e_{cJ}$, and $C^J{}_{IK} = \varepsilon^J{}_{IK}$ since they are the structure constants of $SO(2,1)$. Then,

$$\tilde{\varepsilon}^{abc}k_{ij}\delta A^i_a F^j_{bc} = \tilde{\varepsilon}^{abc}\left[2D_b e_{cI}\delta\omega^I_a + F^I_{bc}\delta e_{aI}\right] \tag{3.149}$$

which are the Einstein equations of motion when the action principle is stationary under the boundary conditions.

Now we shall analyze the variations on the boundary from the Chern-Simons action (3.144) and from the additional boundary term ${}^{IG}S_B[A]$ in (3.131),

$$
\begin{aligned}
\delta^{IG} S_{CSB}[A]\Big|_{Boundary} &= -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{abc} k_{ij}\partial_b (A^i_a \delta A^j_c) \\
&\quad +\frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}} k_{ij}\delta A^i_a f^j_b \tilde{\varepsilon}^{ab} + \frac{\bar{\alpha}_1}{\kappa}\int_{\partial\mathcal{M}} k_{ij} A^i_a \delta f^j_b \tilde{\varepsilon}^{ab} \\
&= \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab} k_{ij}\left[A^i_a \delta A^j_b + 2\bar{\alpha}_1\left(A^i_a \delta f^j_b + \delta A^i_a f^j_b\right)\right] \\
&= \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab} k_{ij}\left[A^i_a\left(\delta A^j_b + 2\bar{\alpha}_1 \delta f^j_b\right) + 2\bar{\alpha}_1 \delta A^i_a f^j_b\right] \qquad (3.150)
\end{aligned}
$$

To see whether this boundary term vanishes under asymptotically flat boundary conditions, leading to a differentiable action, we have to decompose $A^i_a := (e_{aI}, \omega^I_a)$ and $f^j_b := (e_{bJ}, \frac{\bar{\alpha}_2}{n\cdot n} n_K \mathrm{d}_b n_L \varepsilon^{JKL})$. Thus since the variation acts linearly $\delta f^j_b := \left(\delta e_{bJ}, \delta\left[\frac{\bar{\alpha}_2}{n\cdot n} n_K \mathrm{d}_b n_L \varepsilon^{JKL}\right]\right)$. Hence

$$
\begin{aligned}
\delta^{IG} S_{CSB}[A]\Big|_{Boundary} &= \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab} k_{ij}\left[(e_{aI}, \omega^I_a)\left(\delta e_{bJ} + 2\bar{\alpha}_1 \delta e_{bJ}, \delta\omega^J_b + \delta\left[\frac{\bar{\alpha}_2}{n\cdot n} n_K \mathrm{d}_b n_L \varepsilon^{JKL}\right]\right)\right. \\
&\quad \left. + 2\bar{\alpha}_1\left(\delta e_{aI}, \delta\omega^I_a\right)\left(e_{bJ}, \frac{\bar{\alpha}_2}{n\cdot n} n_K \mathrm{d}_b n_L \varepsilon^{JKL}\right)\right] \\
&= \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab}\left[\left(e_{aI} - 2\bar{\alpha}_1 e_{aI}\right)\delta\omega^I_b + \delta\left(\frac{2\bar{\alpha}_1\bar{\alpha}_2}{n\cdot n} e_{aI} n_K \mathrm{d}_b n_L \varepsilon^{JKL}\right)\right. \\
&\quad \left. + \omega^I_a \delta\left(e_{bI} + 2\bar{\alpha}_1 e_{bI}\right)\right]. \qquad (3.151)
\end{aligned}
$$

Taking into account the value of the constants $\bar{\alpha}_1 = \frac{1}{2}$, $\bar{\alpha}_2 = -2\alpha$,

$$
\begin{aligned}
\delta^{IG} S_{CSB}[A]\Big|_{Boundary} &= \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab}\left[\underbrace{\delta\left(\frac{-\alpha}{n\cdot n} e_{aI} n_K \mathrm{d}_b n_L \varepsilon^{JKL}\right)}_{\delta(\text{const.})=0} + 2\underbrace{\omega^I_a \delta e_{bI}}_{-\omega^I_b \delta e_{aI}}\right] \\
&= -\frac{1}{\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab}\omega^I_b \delta e_{aI} \qquad (3.152)
\end{aligned}
$$

Note that the first term of the right hand side of the previous equation is the variation of the additional boundary term (3.8) add it to the Palatini action to make manifestly Lorentz invariant. But in appendix 3.7 we prove that this term is constant when evaluated on solutions compatible with the boundary conditions, then its variation vanishes on the boundary.

Summarizing,

Then the variation of the Chern-Simons action with the decomposition in $A^i_a := (e_{aI}, \omega^I_a) =$

$(e_a, \omega_a)^i$ leads, as expected, to a differentiable action with the Palatini equations of motion as LIP,

$$\delta\,^{IG}S_{CS}[A] = \frac{1}{2}\int_{\mathcal{M}} \tilde{\varepsilon}^{abc} k_{ij} \left[\delta A_a^i \underbrace{\left(2\partial_b A_c^j + [A_b, A_c]^j\right)}_{=:F_{bc}^j} + \partial_b (A_a^i \delta A_c^j) \right] \tag{3.153}$$

3.5.4 Covariant analysis

The symplectic potential can be read from the boundary contribution of the variation of Chern-Simons action with boundary term $^{IG}S_{CSB}[A]$ (3.150)

$$\begin{aligned}
\Theta \;&:=\; \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab} k_{ij} \left[A_a^i \left(\delta A_b^j + 2\bar{\alpha}_1 \delta f_b^j \right) + 2\bar{\alpha}_1 \delta A_a^i f_b^j \right] &(3.154)\\
&=\; \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \tilde{\varepsilon}^{ab} k_{ij} \left[A_a^i \delta A_b^j + \delta\left(2\bar{\alpha}_1 A_a^i f_b^j \right) \right] &(3.155)\\
&=\; \int_{\partial\mathcal{M}} \tilde{\theta} &(3.156)
\end{aligned}$$

Thus we can define the symplectic structure as,

$$\begin{aligned}
J(\delta_1, \delta_2) \;&:=\; \delta_1 \tilde{\theta}(\delta_2) - \delta_2 \tilde{\theta}(\delta_1)\\
&=\; \frac{\tilde{\varepsilon}^{ab} k_{ij}}{2\kappa} \left[\delta_1 A_a^i \delta_2 A_b^j - \delta_2 A_a^i \delta_1 A_b^j + \delta_1 \delta_2 (A_a^i f_b^j) - \delta_1 \delta_2 (A_a^i f_b^j) \right]\\
&=\; \frac{\tilde{\varepsilon}^{ab} k_{ij}}{2\kappa} \left[\delta_1 A_a^i \delta_2 A_b^j - \delta_2 A_a^i \delta_1 A_b^j \right]\\
&=\; \frac{\tilde{\varepsilon}^{ab} k_{ij}}{\kappa} \delta_1 A_a^i \delta_2 A_b^j &(3.157)
\end{aligned}$$

Note that, as expected from [27], the contribution to the symplectic current is only due to the Chern-Simons action, it is insensitive to the addition of an additional boundary term. In order to define a conserved pre-symplectic structure $\tilde{\Omega}$ we have to check that $\int_{\mathcal{J}} J(\delta_1, \delta_2) = 0$. But before that, note that if we decompose the symplectic current (3.157) into $A_a^i := (e_{aI}, \omega_a^I)$,

$$\begin{aligned}
J(\delta_1, \delta_2) \;&=\; \frac{\tilde{\varepsilon}^{ab} k_{ij}}{\kappa} \delta_1 A_a^i \delta_2 A_b^j &(3.158)\\
&=\; \frac{\tilde{\varepsilon}^{ab} k_{ij}}{\kappa} (\delta_1 e_a, \delta_1 \omega_a)^i (\delta_2 e_b, \delta_2 \omega_b)^j &(3.159)\\
&=\; \frac{\tilde{\varepsilon}^{ab}}{\kappa} \left[\delta_1 e_{aI} \delta_2 \omega_b^I + \delta_1 \omega_a^I \delta_2 e_{bI} \right] &(3.160)\\
&=\; -\frac{\tilde{\varepsilon}^{ab}}{\kappa} \left[\delta_2 e_{aI} \delta_1 \omega_b^I - \delta_1 e_{aI} \delta_2 \omega_b^I \right] &(3.161)
\end{aligned}$$

Note that this exactly the same expression as (3.44). But we already prove on section 3.3, that $\int_{\mathfrak{I}} J(\delta_1, \delta_2) = 0$ when evaluated on boundary conditions, so we can define a conserved pre-symplectic current $\tilde{\Omega}$.

$$\tilde{\Omega}(\delta_1, \delta_2) = \int_M \frac{\tilde{\varepsilon}^{ab} k_{ij}}{\kappa} \delta_1 A_a^i \delta_2 A_b^j \tag{3.162}$$

As discussed in subsection 3.3.0.3 if we want to compare results between the covariant and canonical schemes, we have to be sure we are working either with geometrodynamical o connection variables (in the first case the triad is the field variable and ω its canonical conjugate momenta and in the connection approach the other way around). Since in the canonical approach we chose the connection variables, and by the definition of the symplectic form $\bar{\Omega} = \mathrm{d}\Pi_A \wedge \mathrm{d}\phi^A$, we can see that the presymplectic form we are interested in is $\bar{\Omega} = -\tilde{\Omega}$.

Once we define the symplectic structure we can find the hamiltonian and the energy. As seen in section 3.3.1, the energy H_ξ can be found through the expression,

$$\bar{\Omega}(\delta, \delta_\xi) =: \delta H_\xi \tag{3.163}$$

where $\delta_\xi := \delta_\xi A_a^i = \pounds_\xi A_a^i = (\pounds_\xi e_a, \pounds_\xi \omega_a)^i$ and ξ generates asymptotic time translations of the space-time, which induces time evolution on the covariant phase space. Thus the variation of the energy H_ξ is,

$$\begin{aligned}
\delta H_\xi &= \bar{\Omega}(\delta, \delta_\xi) = -\tilde{\Omega}(\delta, \delta_\xi) \\
&= -\int_M \frac{\tilde{\varepsilon}^{ab} k_{ij}}{\kappa} \delta A_a^i \pounds_\xi A_b^j \\
&= -\int_M \frac{\tilde{\varepsilon}^{ab} k_{ij}}{\kappa} (\delta e_a, \delta \omega_a)^i (\pounds_\xi e_b, \pounds_\xi \omega_b)^j \\
&= -\int_M \frac{\tilde{\varepsilon}^{ab}}{\kappa} \left[\delta e_{aI} \pounds_\xi \omega_b^I + \delta \omega_a^I \pounds_\xi e_{bI}\right] \\
&= -\int_M \frac{\tilde{\varepsilon}^{ab}}{\kappa} \left[\delta \omega_b^I \pounds_\xi e_{aI} - \delta e_{aI} \pounds_\xi \omega_b^I\right]
\end{aligned} \tag{3.164}$$

Note that this is the same expression as (3.55), therefore when evaluated on the asymptotically flat boundary conditions (3.14) and (4.37) the energy will be the same as in section 3.3.1.1, that is,

$$\delta H_\xi = \frac{\delta\beta}{2(8\pi G)} 2\pi = \frac{\delta\beta}{8G}. \tag{3.165}$$

Since the previous expression only gives the variation, the energy will always be determined up to a constant,

$$E = \frac{\beta}{16G} + const \tag{3.166}$$

Following [11; 47], $\beta \in [0,2)$, so we can choose this constant to be zero for the energy of Minkowski space-time to be zero,

$$E \in \left[0, \frac{1}{4G}\right]. \tag{3.167}$$

3.5.5 Infinitesimal transformations

What happens to Chern-Simons action without boundary term, when we perform an infinitesimal transformation of the form $\tilde{A} = A + \delta A$?

We begin with the Chern-Simons action without boundary term,

$$^{IG}S_{CS}[A] = -\frac{1}{2\kappa}\int_M \tilde{\varepsilon}^{abc} k_{ij} \left(A^i_a \partial_b A^j_c + \frac{1}{3} A^i_a \left[A_b, A_c\right]^j \right). \tag{3.168}$$

After an infinitesimal transformation, $\tilde{A} = A + \delta A$, it becomes,

$$\begin{aligned} ^{IG}S_{CS}[\tilde{A}] &= -\frac{1}{2\kappa}\int_M \tilde{\varepsilon}^{abc} k_{ij} \left\{ \left(A^i_a + \delta A^i_a\right) \partial_b \left(A^j_c + \delta A^j_c\right) + \frac{1}{3}\left(A^i_a + \delta A^i_a\right) \left[\left(A_b + \delta A_b\right), \left(A_c + \delta A_c\right)\right]^j \right\} \\ &= -\frac{1}{2\kappa}\int_M \tilde{\varepsilon}^{abc} k_{ij} \left\{ A^i_a \partial_b A^j_c + A^i_a \partial_b \delta A^j_c + \delta A^i_a \partial_b A^j_c + \delta A^i_a \partial_b \delta A^j_c \right. \\ &\quad \left. + \frac{1}{3} A^i_a \left[\left(A_b + \delta A_b\right), \left(A_c + \delta A_c\right)\right]^j + \frac{1}{3}\delta A^i_a \left[\left(A_b + \delta A_b\right), \left(A_c + \delta A_c\right)\right]^j \right\}, \end{aligned} \tag{3.169}$$

but

$$\left[\left(A_b + \delta A_b\right), \left(A_c + \delta A_c\right)\right]^j = \left[A_b, A_c\right]^j + \left[A_b, \delta A_c\right]^j + \left[\delta A_b, A_c\right]^j + \left[\delta A_b, \delta A_c\right]^j. \tag{3.170}$$

Then

$$\begin{aligned} ^{IG}S_{CS}[\tilde{A}] &= -\frac{1}{2\kappa}\int_M \tilde{\varepsilon}^{abc} k_{ij} \left\{ A^i_a \partial_b A^j_c + A^i_a \partial_b \delta A^j_c + \delta A^i_a \partial_b A^j_c + \delta A^i_a \partial_b \delta A^j_c \right. \\ &\quad + \frac{1}{3} A^i_a \left(\left[A_b, A_c\right]^j + \left[A_b, \delta A_c\right]^j + \left[\delta A_b, A_c\right]^j + \left[\delta A_b, \delta A_c\right]^j \right) \\ &\quad \left. + \frac{1}{3} \delta A^i_a \left(\left[A_b, A_c\right]^j + \left[A_b, \delta A_c\right]^j + \left[\delta A_b, A_c\right]^j + \left[\delta A_b, \delta A_c\right]^j \right) \right\}. \end{aligned} \tag{3.171}$$

To first order in δA, the quadratic terms in δA and higher powers will vanish since we are considering infinitesimal transformations, so the Chern-Simons action can be written,

$$\begin{aligned}
{}^{IG}S_{CS}[\tilde{A}] &= -\frac{1}{2\kappa}\int_M \Big\{ \tilde{\varepsilon}^{abc}k_{ij}\left(A^i_a\partial_b A^j_c + \frac{1}{3}A^i_a\left[A_b,A_c\right]^j\right) + \boxed{\partial_b\left(\tilde{\varepsilon}^{abc}k_{ij}A^i_a\delta A^j_c\right)} \\
&\quad + \tilde{\varepsilon}^{abc}k_{ij}\left(-\partial_b A^i_a\delta A^j_c + \delta A^i_a\partial_b A^j_c + \frac{1}{3}A^i_a\left(\left[A_b,\delta A_c\right]^j + \left[\delta A_b,A_c\right]^j\right)\right) \\
&\quad + \frac{1}{3}\delta A^i_a\left[A_b,A_c\right]^j \Big\} \\
&= -\frac{1}{2\kappa}\int_M \Big\{ \tilde{\varepsilon}^{abc}k_{ij}\left(A^i_a\partial_b A^j_c + \frac{1}{3}A^i_a\left[A_b,A_c\right]^j\right) + \boxed{\partial_b\left(\tilde{\varepsilon}^{abc}k_{ij}A^i_a\delta A^j_c\right)} \\
&\quad + \tilde{\varepsilon}^{abc}k_{ij}\delta A^i_a\left(\partial_b A^j_c + \left[A_b,A_c\right]^j\right)\Big\} \\
&= {}^{IG}S_{CS}[A] - \boxed{\frac{1}{2\kappa}\int_M \partial_b\left(\tilde{\varepsilon}^{abc}k_{ij}A^i_a\delta A^j_c\right)} - \frac{1}{2\kappa}\int_M \tilde{\varepsilon}^{abc}k_{ij}\delta A^i_a F^j_{bc} \qquad (3.172)
\end{aligned}$$

Note that the Chern-Simons action would be invariant under infinitesimal transformations (modulo equations of motion) if either the space-time does not have boundaries or that somehow we find a way to cancel the boundary term in the previous equation. The answer to this questions is work in progress.

3.6 Discussion and remarks

In this chapter we have proposed a three dimensional manifestly Lorentz invariant Palatini action that is well posed under asymptotically flat boundary conditions. Note that the analog of the well posed Palatini action [7], that we called S_{SPB}, is not manifestly Lorentz invariant although it has a well posed action principle under the asymptotically flat boundary conditions. One can make a partial gauge fixing in the boundary to make it invariant under the residual gauge transformations. Although, introducing an extra appropriate boundary term 3.8, we can make the action manifestly Lorentz invariant and more over this action coincide with the three dimensional Einstein-Hilbert action with Gibbons Hawking term. We derive the asymptotically flat boundary conditions for the first order variables, and with these conditions we show that in fact the proposed action has a well posed action principle, i.e. finite and differentiable. Then using the covariant and canonical approaches we obtain an expression for the energy. In the first case, when we use the covariant formalism, our results coincide with those in [11] where they use Regge-Teitelboim method for the second order metric variables. In the second case using canonical formalism, our results coincide with those in [47] where they begin with the Einstein-Hilbert action with Gibbons-Hawking term, that is well posed under asymptotically flat boundary conditions. So the addition on the term 3.8, is crucial to recover the energy found by means of the metric variables, otherwise our results coincide up to a constant. We also propose a ISO$(2,1)$ valued Chern-Simons action with boundary term that reproduces exactly the manifestly Lorentz invariant Palatini action.

3.7 Appendix: On the new boundary term

As we commented on previous sections, particularly in section 3.2, the addition of the term (3.8),

$$\int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_I \wedge n_K \mathrm{d}n_L \tag{3.173}$$

has many advantages: its necessary for the action to be manifestly Lorentz invariant, it has a constant value when evaluated on solutions compatible with the asymptotically flat boundary conditions, so it does not spoil finiteness nor differentiability, the resulting well posed manifestly Lorentz invariant action is equivalent to the Einstein Hilbert action so we can fully recover previous results obtained by means of the metric formulation.

Here n_K is a spacetime scalar that is an internal vector. We can define it by $n_K/\sqrt{n\cdot n} := R^a e_{aK}$ where R^a is the spacetime unit normal to the boundary[1], that can either be n^a for the unit normal to the spacelike surfaces or r^a for the unit normal to the timeline boundary, we have introduced a normalization factor $\frac{1}{n\cdot n}$ to allow freedom in rescaling n_K, so we can use any multiple of n_K and the results will remain the same. Since n_K is a spacetime scalar $\mathrm{d}n_L$ is a one form as well as e_I then the previous boundary term is the integral of a two form over a two dimensional boundary.

For the more general case, when the boundary might become null we need to use densitized internal normals as discussed in [18], such that the expressions do not diverge. In the case treated here it is enough and more intuitive to use just the n_K.

3.7.1 New boundary term evaluated on Asymptotically flat boundary conditions

In this subsection we shall prove that the term (3.8) is constant when evaluated on the boundary conditions. On the boundary and in components, the term (3.8) can be written as,

$$\int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_I \wedge n_K \mathrm{d}n_L = \int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_{aI} n_K \mathring{\tilde{\mathcal{D}}}_b n_L \tilde{\varepsilon}^{ab} \tag{3.174}$$

$$= \left[-\int_{M_1} + \int_{M_2} + \int_{\mathcal{I}}\right] \frac{1}{n\cdot n}\varepsilon^{IKL} e_{aI} n_K \mathring{\tilde{\mathcal{D}}}_b n_L \tilde{\varepsilon}^{ab} \tag{3.175}$$

[1] Note that we have expanded the usual definition of $n_K = n^a e_{aK}$ for the Cauchy surfaces in the first order formalism to $n_k/\sqrt{n\cdot n} := R^a e_{aK}$ that allows, in principle, n_K to be rescaled, and now is extended also to include the timelike boundary.

Where we are considering the region $\mathcal{M}$ as bounded by $\partial_{\mathcal{M}} = M_1 \cup M_2 \cup \mathcal{I}$, M_1 and M_2 are spacelike slices and $\mathcal{I}$ an outer boundary. Remember we choose the torsion free flat connection $\mathring{\bar{\mathcal{D}}}_b$, such that $D = \mathring{\bar{\mathcal{D}}} + \omega$ and $\mathring{\bar{\mathcal{D}}}_b {}^0\bar{e}_a^I = 0$ and also that $n_k / \sqrt{n \cdot n} := R^a e_{aK}$ where R^a is the spacetime unit normal to the boundary, that can either be n^a for the unit normal to the spacelike surfaces or r^a for the unit normal to the timeline boundary. For the timelike part,

$$\int_{\mathcal{I}} \frac{1}{n \cdot n} \varepsilon^{IKL} n_K e_I \wedge \mathrm{d}n_L = \int_{\mathcal{I}} \underbrace{(\varepsilon^{IKL} \frac{n_K}{\sqrt{n \cdot n}})}_{-\varepsilon^{IL}} e_{aI} \mathring{\bar{\mathcal{D}}}_b \underbrace{\left(\frac{n_L}{\sqrt{n \cdot n}} \right)}_{r^a e_{aK}} \tilde{\varepsilon}^{ab} \tag{3.176}$$

$$= -\int_{\mathcal{I}} \varepsilon^{IL} e_{aI} \left(r^c \mathring{\bar{\mathcal{D}}}_b e_{cL} + e_{cL} \mathring{\bar{\mathcal{D}}}_b r^c \right) \tilde{\varepsilon}^{ab} \tag{3.177}$$

$$= \underbrace{-\int_{\mathcal{I}} \varepsilon^{IL} e_{aI} r^c \mathring{\bar{\mathcal{D}}}_b e_{cL}}_{B_1} \underbrace{- \int_{\mathcal{I}} \varepsilon^{IL} e_{aI} e_{cL} \mathring{\bar{\mathcal{D}}}_b r^c \tilde{\varepsilon}^{ab}}_{B_2}. \tag{3.178}$$

From the previous equation we have two terms, B_1 and B_2. We shall analyze first B_1, when evaluated on the boundary the term becomes,

$$B_1 = \lim_{r\to\infty} \left[-\frac{\alpha}{\kappa} \int_{\mathcal{I}} \varepsilon^{IL\,0} e_{aI} r^c \left(-\frac{\beta}{2r} r^{-\beta/2} \partial_b r\, {}^0\bar{e}_{\bar{c}L} \delta_c^{\bar{c}}\right) \tilde{\varepsilon}^{ab} + \mathcal{O}(r^{-1}) \right] \tag{3.179}$$

$$\begin{aligned}
&= \lim_{r\to\infty} \left[\frac{\alpha\beta}{2\kappa} \int_{\mathcal{I}} \underbrace{\varepsilon^{IL\,0} \bar{e}_{aI}\, {}^0\bar{e}_{\bar{c}L}}_{\bar{e}\tilde{\varepsilon}_{a\bar{c}}} \frac{1}{r} r^{-\beta/2} r^{\bar{c}} \partial_b r \tilde{\varepsilon}^{ab} + \mathcal{O}(r^{-1}) \right] \\
&= \lim_{r\to\infty} \left[\frac{\alpha\beta}{2\kappa} \int_{\mathcal{I}} \frac{1}{r} r^{-\beta/2} r^{\bar{c}} \partial_b r \underbrace{\tilde{\varepsilon}_{a\bar{c}} \tilde{\varepsilon}^{ab}}_{\delta_{\bar{c}}^b} r d\theta dt + \mathcal{O}(r^{-1}) \right] \\
&= \lim_{r\to\infty} \left[\frac{\alpha\beta}{2\kappa} \int_{\mathcal{I}} r^{-\beta/2} (+1) d\theta dt + \mathcal{O}(r^{-1}) \right] \\
&= \lim_{r\to\infty} \left[\frac{\alpha\beta}{2\kappa} r^{-\beta/2} 2\pi + \mathcal{O}(r^{-1}) \right] \\
&= \lim_{r\to\infty} \left[\mathcal{O}(r^{-\beta/2}) + \mathcal{O}(r^{-1}) \right] = 0 \;\; iff \;\; \beta > 0
\end{aligned} \tag{3.180}$$

and B_2 becomes,

$$
\begin{aligned}
B_2 &= \lim_{r\to\infty}\left[-\frac{\alpha}{\kappa}\int_{\mathcal{J}} \varepsilon^{IL\,0} e_{0I}\left({}^0e_{cL}\mathring{\tilde{\mathcal{D}}}_b r^c\right)\tilde{\varepsilon}^b\right] && (3.181)\\
&= -\frac{\alpha}{\kappa}\lim_{r\to\infty}\int_{\mathcal{J}} \underbrace{\varepsilon^{IL\,0}e_{0I}\,{}^0e_{cL}}_{\bar{e}\tilde{\varepsilon}^{0b}}\underbrace{\mathring{\tilde{\mathcal{D}}}_b r^c}_{\partial_b r^c}\tilde{\varepsilon}^{0c}\\
&= -\frac{\alpha}{\kappa}\lim_{r\to\infty}\int_{\mathcal{J}} \underbrace{\tilde{\varepsilon}^{0b}\tilde{\varepsilon}^{0c}}_{\delta^b_c}(\partial_b r^c) r d\theta dt\\
&= -\frac{\alpha}{\kappa}\lim_{r\to\infty}\int_{\mathcal{J}} \underbrace{(\partial_c r^c)}_{1/r} r d\theta dt = -\frac{\alpha}{2\kappa}\int_{\mathcal{J}} 2 d\theta dt && (3.182)
\end{aligned}
$$

Therefore the value of the boundary term (3.8) when evaluated in the timelike boundary and on the boundary conditions becomes,

$$
\begin{aligned}
\int_{\mathcal{J}} \frac{1}{n\cdot n}\varepsilon^{IKL} n_K e_I \wedge \mathrm{d}n_L &= B_1 + B_2 && (3.183)\\
&= \lim_{r\to\infty}\left[\mathcal{O}(r^{-\beta/2}) + \mathcal{O}(r^{-1})\right] - \frac{\alpha}{2\kappa}\int_{\mathcal{J}} 2d\theta && (3.184)\\
&= -\frac{\alpha}{2\kappa}\int_{\mathcal{J}} 2d\theta dt. && (3.185)
\end{aligned}
$$

Since we are integrating over a finite time interval with M_1 and M_2 asymptotically time-translated with respect to each other, the previous integral take a finite constant value.

Analogously, we can follow the same steps but for $R^a = n^a$ and check that the boundary term corresponding to the spacelike surfaces is also constant. Thus, the whole boundary term is constant when evaluated on the boundary conditions.

3.8 Appendix: On the equivalence between Einstein-Hilbert action with Gibbons Hawking term and the Palatini action with boundary term.

It has been shown for the three dimensional Einstein-Hilbert action that the Gibbons-Hawking term is the only term needed to make the variational principle well posed [47]. Taking $\kappa = 8\pi G$, the Einstein-Hilbert action with Gibbons-Hawking term is,

$$
S_{EH-GH}[g] = \frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{-g}R + 2\int_{\partial\mathcal{M}}\sqrt{-h}K \qquad (3.186)
$$

with R the Ricci scalar, g the determinant of the spacetime metric g_{ab}, h the determinant of the induced metric on the boundary $\partial\mathcal{M}$ and K the extrinsic curvature of the boudary.

We shall prove, on the other hand, that the Lorentz invariant well posed Palatini action with boundary term,

$$S_{LIP}[e,\omega] = -\frac{1}{\kappa}\int_{\mathcal{M}} e^I \wedge F_I \;\; -\frac{1}{\kappa}\int_{\partial\mathcal{M}} \frac{1}{n\cdot n}\varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L. \tag{3.187}$$

is in fact equivalent to the Einstein-Hilbert action with Gibbons-Hawking term.

We study first the Einstein-Hilbert term, $\frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{-g}R$, considering that $g^{ab} = \eta^{IJ}e^a_I e^b_J$, $\sqrt{-g} = e$, $2ee^{[a|}_I e^{|c]}_J = \tilde{\eta}^{acf}\varepsilon_{IJK}e^K_f$, $F^{IJ}_{ab} = e^{cI}e^{dJ}R_{acbd}$ and $F^{JK}_{ab} = F^L_{ab}\varepsilon^{KJ}{}_L$. The bulk term,

$$\begin{aligned}
\frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{-g}R &= \frac{1}{2\kappa}\int_{\mathcal{M}}\underbrace{\sqrt{-g}}_{e}\;\underbrace{g^{ab}}_{\eta^{IJ}e^a_I e^b_J}\;\underbrace{R_{ab}}_{R_{acbd}g^{cd}} \\
&= \frac{1}{2\kappa}\int_{\mathcal{M}} ee^{[a|}_I e^{bI} R_{acbd} e^{|c]}_J e^{dJ} \\
&= \frac{1}{2\kappa}\int_{\mathcal{M}}\frac{1}{2}\underbrace{2ee^{[a|}_I e^{|c]}_J}_{\tilde{\eta}^{acf}\varepsilon_{IJK}e^K_f} e^{bI}e^{dJ}R_{acbd} \\
&= \frac{1}{2\kappa}\int_{\mathcal{M}}\frac{1}{2}\tilde{\varepsilon}^{acf}\varepsilon_{IJK}e^K_f\underbrace{e^{bI}e^{dJ}R_{acbd}}_{F^{IJ}_{ac}} \\
&= \frac{1}{2\kappa}\int_{\mathcal{M}}\frac{1}{2}\tilde{\varepsilon}^{acf}\varepsilon_{IJK}e^I_f\;\underbrace{F^{JK}_{ac}}_{F^L_{ac}\varepsilon^{KJ}{}_L} \\
&= \frac{1}{2\kappa}\int_{\mathcal{M}}\frac{1}{2}\tilde{\varepsilon}^{acf}\underbrace{\varepsilon_{IJK}\varepsilon^{KJ}{}_L}_{-2\delta^L_I}e^I_f F^L_{ac} \\
&= -\frac{1}{2\kappa}\int_{\mathcal{M}}\tilde{\varepsilon}^{acf}e^I_f F_{acI} \qquad (3.188)\\
&= -\frac{1}{\kappa}\int_{\mathcal{M}} e^I\wedge F_I \qquad (3.189)
\end{aligned}$$

Note the change in sign when we write down the Palatini action defined over an arbitrary Lie group (see e.g. [60]).

Now we shall see the relation between the Lorentz invariant boundary term (3.10) introduced in section 3.2 and the Gibbons Hawking term. We begin with the Lorentz invariant boundary term,

$$\int_{\partial\mathcal{M}}\frac{1}{n\cdot n}\varepsilon^{IKL}e_I\wedge n_K\mathcal{D}n_L = \left[-\int_{M_1}+\int_{M_2}+\int_{\mathcal{I}}\right]\frac{1}{n\cdot n}\varepsilon^{IKL}e_I\wedge n_K\mathcal{D}n_L \tag{3.190}$$

where our integration region $\mathcal{M}$ is bounded by $\partial_{\mathcal{M}} = M_1\cup M_2\cup\mathcal{I}$, M_1 and M_2 are space-like slices and $\mathcal{I}$ a family of timelike cylinders we used to approach spatial infinity.

For the timelike boundary consider $n_L/\sqrt{n\cdot n} := r^a e_{aL}$, r^a the normal to the cylinder, $\mathcal{D}_c r^a = \nabla_c r^a$ where ∇ is the Levy Civita connection, $e=\sqrt{\gamma}$ where γ_{ab} is the induced metric on the timelike boundary and that $\varepsilon^{IKL} e_{bI} e_{dK} e_{aL} = e\tilde{\varepsilon}_{bda}$. The term on the timelike boundary is,

$$\begin{aligned}
\int_{\mathcal{J}} \frac{1}{n\cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L &= \int_{\mathcal{J}} \varepsilon^{IKL} e_{bI} \frac{n_K}{\sqrt{n\cdot n}} \mathcal{D}_c \left(\frac{n_L}{\sqrt{n\cdot n}}\right) \tilde{\varepsilon}^{bc} \\
&= \int_{\mathcal{J}} \varepsilon^{IKL} e_{bI} \frac{n_K}{\sqrt{n\cdot n}} \mathcal{D}_c (r^a e_{aL}) \tilde{\varepsilon}^{bc} \\
&= \int_{\mathcal{J}} \varepsilon^{IKL} e_{bI} \frac{n_K}{\sqrt{n\cdot n}} \left[r^a \underbrace{\mathcal{D}_c e_{aL}}_{=0\ by\ EOM} + e_{aL} \mathcal{D}_c r^a \right] \tilde{\varepsilon}^{bc} \\
&= \int_{\mathcal{J}} \varepsilon^{IKL} e_{bI} (r^d e_{dK}) e_{aL} \mathcal{D}_c r^a \tilde{\varepsilon}^{bc} \\
&= \int_{\mathcal{J}} \varepsilon^{IKL} e_{bI} e_{dK} e_{aL} r^d \nabla_c r^a \tilde{\varepsilon}^{bc} \\
&= \int_{\mathcal{J}} e \underbrace{(\tilde{\varepsilon}_{bda} r^d)}_{-\tilde{\varepsilon}_{ab}} \nabla_c r^a \tilde{\varepsilon}^{bc} \\
&= \int_{\mathcal{J}} \sqrt{-\gamma} \nabla_c r^a \underbrace{(-\tilde{\varepsilon}_{ab} \tilde{\varepsilon}^{bc})}_{-\delta_a^c} \\
&= -\int_{\mathcal{J}} \sqrt{-\gamma} \nabla_a r^a. \qquad (3.191)
\end{aligned}$$

Now we can rememeber that we define the extrinsic curvature, $\mathcal{K}$, of a surface (in this case the timelike cylinder) as the trace of $\mathcal{K}_a^b = \nabla_a r^b$ where r^b is the normal to the surface, then $\mathcal{K} = \gamma^{ab}\mathcal{K}_{ab} = \mathcal{K}_a^a = \nabla_a r^a$. With this at hand we can see that, in fact,

$$\int_{\mathcal{J}} \frac{1}{n\cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L = -\int_{\mathcal{J}} \sqrt{-\gamma} \nabla_a r^a = -\int_{\mathcal{J}} \sqrt{-\gamma} \mathcal{K}, \qquad (3.192)$$

where $\mathcal{K}$ is the extrinsic curvature of the timelike boundary. Following an analogous derivation for the spacelike surfaces M_1 and M_2, we can easily see that,

$$\int_{M_{1,2}} \frac{1}{n\cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L = -\int_{M_{1,2}} \sqrt{q} \nabla_a n^a = -\int_{M_{1,2}} \sqrt{q} k, \qquad (3.193)$$

again, with q the determinant of the induced metric on $M_{1,2}$, n^a and k its normal vector and extrinsic curvature respectively. With this at hand we can see that,

$$\int_{\partial\mathcal{M}} \frac{1}{n\cdot n} \varepsilon^{IKL} e_I \wedge n_K \mathcal{D} n_L = -\left[-\int_{M_1} + \int_{M_2}\right] \sqrt{q} k - \int_{\mathcal{J}} \sqrt{-\gamma} \mathcal{K} = -\int_{\partial\mathcal{M}} \sqrt{-h} K. \qquad (3.194)$$

From (3.10) in the section 3.2, we can see that,

$$
\begin{aligned}
\frac{1}{\kappa}\int_{\partial\mathcal{M}}\sqrt{-h}K &= -\frac{1}{\kappa}\int_{\partial\mathcal{M}}\frac{1}{n\cdot n}\varepsilon^{IKL}e_I\wedge n_K\mathcal{D}n_L && (3.195)\\
&= -\frac{1}{\kappa}\int_{\partial\mathcal{M}}e^I\wedge\omega_I-\frac{1}{\kappa}\int_{\partial\mathcal{M}}\frac{1}{n\cdot n}\varepsilon^{IKL}e_I\wedge n_K\mathrm{d}n_L && (3.196)
\end{aligned}
$$

This result coincides, appart from the second term of the right hand side of the last equation, with that given in [49] when the cosmological constant is zero. In [49] are used the Gaussian (normal) coordinates and also there are considered particular internal directions for the spin connection, this "fixing" of the internal directions is reflected in the fact that the second term of the RHS in (3.195) is not present in their action.

3.9 Appendix: Some consequences of the fall-off conditions

$$
\delta\,{}^0e_a^I=\delta\left({}^0\bar{e}_0^I\delta_a^0+r^{-\beta/2}\,{}^0\bar{e}_{\bar{a}}^I\delta_a^{\bar{a}}\right)=\delta(r^{-\beta/2})\,{}^0\bar{e}_{\bar{a}}^I\delta_a^{\bar{a}} \tag{3.197}
$$

but

$$
\begin{aligned}
\delta\left(r^{-\beta/2}\right) &= \frac{\partial\left(r^{-\beta/2}\right)}{\partial r}\delta r+\frac{\partial\left(r^{-\beta/2}\right)}{\partial\beta}\delta\beta\\
&= -\frac{\beta}{2}r^{-\beta/2-1}\delta r-\frac{r^{-\beta/2}}{2}\log(r)\delta\beta. && (3.198)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\delta\,{}^0\omega_c^M &= \delta\left(\frac{1}{2}\beta r^{-1}\partial_{\bar{a}}r\varepsilon_L{}^{KM}\,{}^0\bar{e}_K^{\bar{a}}\,{}^0\bar{e}_{\bar{c}}^L\delta_c^{\bar{c}}\right)\\
&= \left(\frac{1}{2}\delta\beta\; r^{-1}\partial_a r+\frac{1}{2}\beta\,\delta(r^{-1}\partial_a r)\right)\varepsilon_L{}^{KM}\,{}^0\bar{e}_K^{\bar{a}}\,{}^0\bar{e}_{\bar{c}}^L\delta_c^{\bar{c}} && (3.199)
\end{aligned}
$$

In the timeline boundary $\delta r=0$ so,

$$
\begin{aligned}
\delta\left(r^{-\beta/2}\right) &= -\frac{r^{-\beta/2}}{2}\log(r)\delta\beta && (3.200)\\
\delta\,{}^0\omega_c^M &= \left(\frac{1}{2}\delta\beta\; r^{-1}\partial_a r\right)\varepsilon_L{}^{KM}\,{}^0\bar{e}_K^{\bar{a}}\,{}^0\bar{e}_{\bar{c}}^L\delta_c^{\bar{c}} && (3.201)\\
\delta\,{}^0e_a^I &= -\frac{r^{-\beta/2}}{2}\log(r)\delta\beta\,{}^0\bar{e}_{\bar{a}}^I\delta_a^{\bar{a}} && (3.202)
\end{aligned}
$$

Chapter 4

Covariant analysis of the Generalized Holst action with topological in 4D terms

"The joy of discovery is certainly the liveliest that the mind of man can ever feel."
—Claude Bernard (1813-78) French physiologist.

This chapter is based on [27; 28]

4.1 The action for gravity in the first order formalism

As already mentioned in the introduction, we shall consider the most general action for four-dimensional gravity in the first order formalism. The choice of basic variables is the following: A pair of co-tetrads e_a^I and a Lorentz $SO(3,1)$ connection ω_{aIJ} on the spacetime $\mathcal{M}$, possibly with a boundary. In order for the action to be physically relevant, it should reproduce the equations of motion for general relativity and be: 1) differentiable, 2) finite on the configurations with a given asymptotic behaviour and 3) invariant under diffeomorphisms and local internal Lorentz transformations. The most general action that gives the desired equations of motion and is compatible with the symmetries of the theory is given by the combination of Palatini action, S_{P}, Holst term, S_{H}, and three topological terms, Pontryagin, S_{Po}, Euler, S_{E}, and Nieh-Yan, S_{NY}, invariants. As we shall see, the Palatini term contains the information of the ordinary Einstein-Hilbert 2nd order action, so it represents the backbone of the formalism. Since we are considering a spacetime region $\mathcal{M}$ with boundaries, one should pay special attention to boundary conditions. For instance, it turns out that the Palatini action, as well as Holst and Nieh-Yan terms are not differentiable for asymptotically flat spacetimes, and appropriate boundary terms should be provided. This section has four parts. In those subsections we are going to analyze, one by one, all of the terms of the action. We shall take the corresponding variation of the terms and identify both their contributions to the equations of motion and to the symplectic current. Since we are not considering yet any particular boundary conditions, the results of this section are generic.

4.1.1 Palatini action

Let us start by considering the Palatini action with boundary term is given by [7],

$$S_{\rm PB} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \Sigma^{IJ}\wedge F_{IJ} + \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \Sigma^{IJ}\wedge \omega_{IJ}\,. \tag{4.1}$$

where $\kappa = 8\pi G$, $\Sigma^{IJ} = \star(e^I\wedge e^J) := \frac{1}{2}\varepsilon^{IJ}{}_{JK}e^J\wedge e^K$, $F_{IJ} = \mathrm{d}\omega_{IJ} + \omega_{IK}\wedge\omega^K{}_J$ is a curvature two-form of the connection ω and, as before, $\partial\mathcal{M} = M_1\cup M_2\cup\Delta\cup\mathcal{I}$. The boundary term is not manifestly gauge invariant, but, as pointed out in [7], it is effectively gauge invariant on the spacelike surfaces M_1 and M_2 and also in the asymptotic region $\mathcal{I}$. This is due to the fact that the only allowed gauge transformations that preserve the asymptotic conditions are such that the boundary terms remain invariant. Let us consider first the behaviour of this boundary term on M_1 (or M_2). First we ask that the compatibility condition between the co-tetrad and connection should be satisfied on the boundary. Then, we partially fix the gauge on M, by fixing the internal time-like tetrad n^I, such that $\partial_a n^I = 0$ and we restrict field configurations such that $n^a = e^a_I n^I$ is the unit normal to M_1 and M_2. Under these conditions it has been shown in [7] that on M, $\Sigma^{IJ}\wedge\omega_{IJ} = 2K\mathrm{d}^3V$, where K is the trace of the extrinsic curvature of M. Note that this is the Gibbons-Hawking surface term that is needed in the Einstein-Hilbert action, with the constant boundary term equals to zero. On the other hand, at spatial infinity, $\mathcal{I}$, we fix the co-tetrads and only permit gauge transformations that reduce to identity at infinity. Under these conditions the boundary term is gauge invariant at M_1, M_2 and $\mathcal{I}$. We shall show later that it is also invariant under the residual local Lorentz transformations at a weakly isolated horizon, when such a boundary exists.

It turns out that at spatial infinity this boundary term does not reduce to Gibbons-Hawking surface term, the later one is divergent for asymptotically flat spacetimes, as shown in [7]. Let us mention that there have been other proposals for boundary terms for Palatini action, as for example in [54] and [18], that are equivalent to Gibbons-Hawking action and are obtained without imposing the time gauge condition. They are manifestly gauge invariant and well defined for finite boundaries, but they are not well defined for asymptotically flat spacetimes. In time gauge they reduce to (4.1).

The variation of (4.1) is,

$$\delta S_{\rm PB} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \left[\varepsilon^{IJ}{}_{KL}\delta e^K\wedge e^L\wedge F_{IJ} - D\Sigma_{IJ}\wedge\delta\omega^{IJ} - \mathrm{d}(\delta\Sigma^{IJ}\wedge\omega_{IJ})\right]\,, \tag{4.2}$$

where

$$D\Sigma_{IJ} = \mathrm{d}\Sigma_{IJ} - \omega_I{}^K \wedge \Sigma_{KJ} + \omega_J{}^K \wedge \Sigma_{KI}\,. \tag{4.3}$$

We shall show later that the contribution of boundary term $\delta\Sigma^{IJ} \wedge \omega_{IJ}$ vanishes at $\mathcal{I}$ and Δ, so that from (4.2) we obtain the following equations of motion

$$\varepsilon_{IJKL} e^J \wedge F^{KL} = 0\,, \tag{4.4}$$

$$\varepsilon_{IJKL} e^K \wedge De^L = 0\,, \tag{4.5}$$

where $T^L := De^L = \mathrm{d}e^L + \omega^L{}_K \wedge e^K$ is a torsion two-form. From (4.5) it follows that $T^L = 0$, and this is the condition of the compatibility of ω_{IJ} and e^I, that implies

$$\omega_{aIJ} = e^b_{[I}\partial_a e_{bJ]} + \Gamma^c_{ab} e_{c[I} e^b_{J]}\,, \tag{4.6}$$

where Γ^c_{ab} are the Christoffel symbols of the metric $g_{ab} = \eta_{IJ} e^I_a e^J_b$. Now, the equations (4.4) are equivalent to Einstein's equations $G_{ab} = 0$.

From equations (2.115) and (4.2), the symplectic potential for S_{PB} is given by

$$\Theta_{\text{PB}}(\delta) = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \delta\Sigma^{IJ} \wedge \omega_{IJ}\,. \tag{4.7}$$

Therefore from (2.119) and (4.92) the corresponding symplectic current is,

$$J_{\text{P}}(\delta_1, \delta_2) = -\frac{1}{2\kappa}\left(\delta_1\Sigma^{IJ} \wedge \delta_2\omega_{IJ} - \delta_2\Sigma^{IJ} \wedge \delta_1\omega_{IJ}\right)\,. \tag{4.8}$$

Note that the symplectic current is insensitive to the boundary term, as we discussed in Sec. 2.5.

As we shall discuss in the following sections, the Palatini action, in the asymptotically flat case, is not well defined, but it can be made differentiable and finite after the addition of the corresponding boundary term already discussed [7]. Furthermore, we shall also show that in the case when the spacetime has as internal boundary an isolated horizon, the contribution at the horizon to the variation of the Palatini action, either with a boundary term [24] or without it [8], vanishes.

4.1.2 Holst term

The first additional term to the gravitational action that we shall consider is the so called *Holst* term [35], first introduced with the aim of having a variational principle whose $3+1$ decomposition yielded general relativity in the Ashtekar-Barbero (real) variables [13]. It turns out that the Holst term, when added to the Palatini action, does not change the equations of motion (although it is not a topological term), so that in the Hamiltonian formalism its addition corresponds to a canonical transformation. This transformation leads to the Ashtekar-Barbero variables that are the basic ingredients in the loop quantum gravity approach. As we shall show in the next chapter, the Holst term is finite but not differentiable for asymptotically flat spacetimes, so an appropriate boundary term should be added in order to make it well defined. The result is [25],

$$S_{\mathrm{HB}} = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} \Sigma^{IJ}\wedge\star F_{IJ} + \frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}} \Sigma^{IJ}\wedge\star\omega_{IJ}\,, \tag{4.9}$$

where γ is the Barbero-Immirzi parameter. The variation of the Holst term, with its boundary term, is given by

$$\delta S_{\mathrm{HB}} = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} 2F_{IJ}\wedge e^I\wedge\delta e^J + D\Sigma^{IJ}\wedge\star(\delta\omega_{IJ}) - \mathrm{d}(\delta\Sigma^{IJ}\wedge\star\omega_{IJ})\,, \tag{4.10}$$

and it leads to the following equations of motion in the bulk: $D\Sigma^{IJ}=0$ and $e^I\wedge F_{IJ}=0$. The second one is just the Bianchi identity, and we see that the Holst term does not modify the equations of motion of the Palatini action. The contribution of the boundary term (that appears in the variation) should vanish at $\mathcal{I}$ and Δ, in order to have a well posed variational principle. In the following section we shall see that this is indeed the case.

On the other hand we should also examine the gauge invariance of the boundary term in (4.9). Under the same assumptions as in the case of the Palatini boundary term we obtain that [25]

$$\int_{M_1} \Sigma^{IJ}\wedge\star\omega_{IJ} = 2\int_{M_1} \varepsilon^{abc} e_b^J \partial_c e_{aJ}\mathrm{d}^3x = \int_{M_1} e^I\wedge\mathrm{d}e_I\,, \tag{4.11}$$

and this term is not gauge invariant at M_1 or M_2. As we shall see in the following section, at the asymptotic region it is gauge invariant, and also at Δ. In the analysis of differentiability of the action and the construction of the symplectic structure and conserved quantities there is no contribution from the spacial surfaces M_1 and M_2, and we can argue that the non-invariance of the boundary term in (4.9) is not important, but it would be desirable to have a boundary term that is compatible with all the symmetries of the theory. As we shall see later, the combination of the Holst and Neih-Yan terms *is* differentiable and gauge invariant.

It is easy to see that the symplectic potential for S_{HB} is given by [25]

$$\Theta_{\mathrm{HB}}(\delta) = \frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}} \delta\Sigma^{IJ}\wedge\star\omega_{IJ} = \frac{1}{\kappa\gamma}\int_{\partial\mathcal{M}} \delta e^I\wedge\mathrm{d}e_I\,, \tag{4.12}$$

where in the second line we used the equation of motion $De^I=0$. The symplectic current is given by

$$J_{\rm HB}(\delta_1,\delta_2)=\frac{1}{\kappa\gamma}\mathrm{d}\left(\delta_1 e^I\wedge\delta_2 e_I\right). \tag{4.13}$$

As we have seen in the subsection 2.5.1, when the symplectic current is a total derivative, the covariant Hamiltonian formalism indicates that the corresponding (pre)-symplectic structure vanishes. As we also remarked one could postulate a conserved two form $\tilde{\Omega}$ if $\int_{\mathcal{I}} J_{\rm H}=0$ and $\int_{\Delta} J_{\rm H}=0$, in which case this term defines a conserved symplectic structure. We shall, for completeness, consider this possibility in Sec. 4.3, after the appropiate boundary conditions have been introduced. There we shall also show that the Holst term modifies the Noether charge associated to diffeomorphisms.

4.1.3 Topological terms

In four dimensions there are three topological invariants constructed from e^I, F_{IJ} and De^I, consistent with diffeomorphism and local Lorentz invariance. They are exact forms and do not contribute to the equations of motion, but in order to be well defined they should be finite, and their variation on the boundary of the spacetime region $\mathcal{M}$ should vanish. The first two terms, the Pontryagin and Euler terms are constructed from the curvature F_{IJ} and its dual (in the internal space) $\star F_{IJ}$, while the third one, the Neih-Yan invariant, is related to torsion De^I.

These topological invariants can be thought of as 4-dimensional density lagrangians defined on a manifold $\mathcal{M}$, that additionally are exact forms, but they can also be seen as terms living on $\partial\mathcal{M}$. In that case it is obvious that they do not contribute to the equations of motion in the bulk. But a natural question may arise. If we take the lagrangian density in the bulk and take the variation, what are the corresponding equations of motion in the bulk? One can check that, for Pontryagin and Euler, the resulting equations of motion are trivial in the sense that one only gets the Bianchi identities, while for the Nieh-Yan term they vanish identically. Let us now see how each of this terms contribute to the variation of the action.

4.1.3.1 Pontryagin and Euler terms

The action corresponding to the Pontryagin term is given by,

$$S_{\rm Po}=\int_{\mathcal{M}}F^{IJ}\wedge F_{IJ}=2\int_{\partial\mathcal{M}}\left(\omega_{IJ}\wedge d\omega^{IJ}+\frac{2}{3}\omega_{IJ}\wedge\omega^{IK}\wedge\omega_K{}^J\right). \tag{4.14}$$

The boundary term is the Chern-Simons Lagrangian density, L_{CS}. We can either view the Pontryagin term as a bulk term or as a boundary term and the derivation of the symplectic structure in either case should render equivalent descriptions. The variation of $S_{\rm Po}$, calculated from the LHS expression in (4.14), is

$$\delta S_{\text{Po}} = -2\int_{\mathcal{M}} DF^{IJ}\wedge\delta\omega_{IJ} + 2\int_{\partial\mathcal{M}} F^{IJ}\wedge\delta\omega_{IJ}\,, \tag{4.15}$$

so it does not contribute to the equations of motion in the bulk, due to the Bianchi identity $DF^{IJ}=0$, and additionally the surface integral in (4.144) should vanish for the variational principle to be well defined. We will show later that this is indeed the case for boundary conditions of interest to us, namely asymptotically flat spacetimes possibly with an isolated horizon. In this case, the corresponding symplectic current is

$$J^{\text{bulk}}_{\text{Po}}(\delta_1,\delta_2) = 2(\delta_1 F^{IJ}\wedge\delta_2\omega_{IJ} - \delta_2 F^{IJ}\wedge\delta_1\omega_{IJ})\,. \tag{4.16}$$

On the other hand, if we calculate the variation of the Pontryagin term directly from the RHS of (4.14), we obtain

$$\delta S_{\text{Po}} = 2\int_{\partial\mathcal{M}}\delta L_{CS}\,. \tag{4.17}$$

The two expressions for δS_{Po} are, of course, identical since $F^{IJ}\wedge\delta\omega_{IJ} = \delta L_{CS} + \mathrm{d}(\omega^{IJ}\wedge\delta\omega_{IJ})$. The first one (4.144) is more convenient for the analysis of the differentiability of the Pontryagin term, but the second one (4.103) is more suitable for the definition of the symplectic current, which vanishes identically for any boundary contribution to the action, since

$$J^{\text{bound}}_{\text{Po}}(\delta_1,\delta_2) = 4\,\delta_{[2}\delta_{1]}L_{CS} = 0\,. \tag{4.18}$$

So, at first sight it would seem that there is an ambiguity in the definition of the symplectic current that could lead to different symplectic structures. Since the relation between them is given by

$$J^{\text{bulk}}_{\text{Po}}(\delta_1,\delta_2) = J^{\text{bound}}_{\text{Po}}(\delta_1,\delta_2) + 4\,\mathrm{d}(\delta_2\omega^{IJ}\wedge\delta_1\omega_{IJ})\,, \tag{4.19}$$

it follows that $J^{\text{bulk}}_{\text{Po}}(\delta_1,\delta_2)$ is a total derivative, that does not contribute in (2.123), and from the systematic derivation of the symplectic structure described in 2.5.1, we have to conclude that it does not contribute to the symplectic structure. This is consistent with the fact that $J^{\text{bound}}_{\text{Po}}$ and $J^{\text{bulk}}_{\text{Po}}$ correspond to the same action. As we have remarked in Sec. 2.5, a total derivative term in J, under some circumstances, could be seen as generating a non-trivial symplectic structure $\tilde{\Omega}$ on the boundary of M. But the important thing to note here is that one would run into an inconsistency if one choose to introduce that non-trivial $\tilde{\Omega}$. Thus, consistency of the formalism requires that $\tilde{\Omega}=0$.

Let us now consider the action for the Euler term, which is given by,

$$S_{\text{E}} = \int_{\mathcal{M}} F^{IJ}\wedge\star F_{IJ} = 2\int_{\partial\mathcal{M}}\left(\star\omega_{IJ}\wedge d\omega^{IJ} + \frac{2}{3}\star\omega_{IJ}\wedge\omega^{IK}\wedge\omega_K{}^J\right)\,, \tag{4.20}$$

with variation, calculated from the expression in the bulk, given by

$$\delta S_{\rm E} = -2\int_{\mathcal{M}} \star DF^{IJ}\wedge\delta\omega_{IJ} + 2\int_{\partial\mathcal{M}} \star F^{IJ}\wedge\delta\omega_{IJ}\,. \tag{4.21}$$

Again, the action will only be well defined if the boundary contribution to the variation (4.21) vanishes. In the following section we shall see that it indeed vanishes for our boundary conditions. Let us denote by L_{CSE} the boundary term on the RHS of (4.20), then we can calculate the variation of S_E from this term directly as

$$\delta S_{\rm E} = 2\int_{\partial\mathcal{M}} \delta L_{CSE}\,. \tag{4.22}$$

Finally, as before, the corresponding contribution to the symplectic current vanishes.

4.1.3.2 Nieh-Yan term

The Nieh-Yan topological invariant is of a different nature from the two previous terms. It is related to torsion and its contribution to the action is [51; 52],

$$S_{\rm NY} = \int_{\mathcal{M}} \left(De^I\wedge De_I - \Sigma^{IJ}\wedge\star F_{IJ}\right) = \int_{\partial\mathcal{M}} De^I\wedge e_I\,. \tag{4.23}$$

Note that the Nieh-Yan term can be written as

$$S_{\rm NY} = 2\kappa\gamma S_{\rm H} + \int_{\mathcal{M}} De^I\wedge De_I\,, \tag{4.24}$$

where $S_{\rm H}$ is the Holst term (4.9) without boundary term. The variation of the term $S_{\rm NY}$ is given by

$$\delta S_{\rm NY} = \int_{\partial\mathcal{M}} 2De_I\wedge\delta e^I - e^I\wedge e^J\wedge\delta\omega_{IJ}\,. \tag{4.25}$$

Contrary to what happens to the Euler and Pontryagin terms, the Nieh-Yan term has a different asymptotic behavior. In the next chapter we will show that the Nieh-Yan term is finite, but not differentiable, for asymptotically flat spacetimes. Thus, even when it is by itself a boundary term, it has to be supplemented with an appropriate boundary term to make the variational principle well defined. We shall see that this boundary term coincides precisely with the boundary term in (4.9) (up to a multiplicative constant), and the resulting well defined Neih-Yan action is given by

$$S_{\rm NYB} = S_{\rm NY} + \int_{\partial\mathcal{M}} \Sigma^{IJ}\wedge\star\omega_{IJ}\,. \tag{4.26}$$

It is straightforward to see that the symplectic potential and symplectic current for this action are the same as for (4.9) (up to a factor $2\kappa\gamma$).

This relation between $S_{\rm H}$ and $S_{\rm NY}$ points to another proposal for a boundary term for the Holst action, different from that in (4.9), that has the advantage of being manifestly gauge invariant. Namely, for asymptotically flat spacetimes, we can see that the surface term in the variation of Neih-Yan term cancels the surface term in the variation of the Holst term, and the action $S_{\rm HNY} := S_{\rm H} - \frac{1}{2\kappa\gamma} S_{\rm NY}$, given by

$$S_{\rm HNY} = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} \Sigma^{IJ}\wedge\star F_{IJ} - \frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}} De^I\wedge e_I = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} De^I\wedge De_I\,, \tag{4.27}$$

is well defined. This combination was proposed in [18], but since they were interested in finite boundaries the boundary conditions that they considered are different from the ones we use, namely they impose $\delta h_{ab} \equiv \delta(e^I_a e_{bI}) = 0$ (h_{ab} is the metric induced on the boundary) and leave $\delta\omega_{IJ}$ arbitrary, which is not compatible with our condition $De^I = 0$ on the boundary (and is not compatible with isolated horizons boundary conditions either).

To end this part, let us also comment that in the presence of fermions one has to generalize the Holst action to spacetimes with torsion, that naturaly leads to Neih-Yan topological term, instead of the Holst term, as shown in [48]. But, as we saw the Neih-Yan term is not well defined for our boundary conditions and should be modified as in (4.26).

4.1.4 The complete action

So far in this section we have introduced all the ingredients for the "most general" first order diffeomorphism invariant action that classically describes general relativity.

As we have already mentioned and we shall prove in the following section, both Pontryagin and Euler terms, $S_{\rm Po}$ and $S_{\rm E}$ respectively, are well defined in the case of asymptotically flat spacetimes with a weakly isolated horizon. This means that we can add them to the Palatini action with its boundary term, $S_{\rm PB}$, and the resulting action will be again well defined. As we foresaw in the previous section, the addition of the Nieh-Yan term, $S_{\rm NY}$, could lead to different possibilities for the construction of a well defined action. Therefore the complete action can be written as,

$$S[e,\omega] = S_{\rm PB} + S_{\rm H} + \alpha_1 S_{\rm Po} + \alpha_2 S_{\rm E} + \alpha_3 S_{\rm NY} + \alpha_4 S_{\rm BH}\,. \tag{4.28}$$

Here $\alpha_1,\ldots,\alpha_4$ are coupling constants. The coupling constants α_1 and α_2, are not fixed by our boundary conditions, while different choices for the Holst-Nieh-Yan sector of the theory, discussed in the previous part, imply particular combinations of α_3 and α_4. To see that, consider $S_{\rm BH}$ that represents the boundary term that we need to add to Holst term in order to make it well defined. As we have seen in the previous analysis, if $\alpha_3 = -\frac{1}{2\kappa\gamma}$ then the combination of the Holst and Nieh-Yan terms is well defined and no additional boundary term is needed, so $\alpha_4 = 0$ in that case. For every other value of α_3 we need to add a boundary term, and in that case $\alpha_4 = \frac{1}{2\kappa\gamma} + \alpha_3$. Other than these cases, there is no important relation between the different coupling constants.

This has to be contrasted with other asymptotic conditions studied in the literature (that we shall, however, not consider here). It turns out that the Palatini action with the negative cosmological constant term is not well defined for asymptotically anti-de Sitter (AAdS) spacetimes, but it can be made differentiable after the addition of an appropriate boundary term. In [9] it is shown that it can be the same boundary term as in the asymptotically flat case, with an appropriately modified coupling constant. On the other hand, as shown in [3] and [2], one can choose the Euler topological term as a boundary term and that choice fixes the value of α_2. In that case $\alpha_2 \sim \frac{1}{\Lambda}$, and the asymptotically flat case cannot be obtained in the limit $\Lambda \to 0$. The differentiability of Nieh-Yan term has been analyzed in [63]. The result is that this term is well defined, for AAdS space-times, only after the addition of the Pontryagin term, with an appropriate coupling constant. Let us also comment that the details of the asymptotic behaviour in [9] are different than in the other mentioned papers.

4.2 Boundary conditions

We have considered the most general action for general relativity in the first order formalism, including boundaries, in order to have a well defined action principle and covariant Hamiltonian formalism. We have left, until now, the boundary conditions unspecified, other that assuming that there is an outer and a possible inner boundary to the region $\mathcal{M}$ under consideration. In this section we shall consider specific boundary conditions that are physically motivated. For the outer boundary we will specify asymptotically flat boundary conditions that capture the notion of isolated systems. For the inner boundary we will consider isolated horizons boundary conditions. In this way, we allow for the possibility of spacetimes that contain a black hole. This section has two parts. In the first one, we consider the outer boundary conditions and in the second part, the inner horizon boundary condition. In each case, we study the finiteness of the action, its variation and its differentiability. Since this manuscript is to be self-contained, we include a detailed discussion of the boundary conditions before analysing the different contributions to the action.

4.2.1 Asymptotically flat spacetimes

We are interested in spacetimes that at infinity look like a flat spacetime, in other words, whose metric approaches a Minkowski metric at infinity (in some appropriately chosen coordinates). Here we will follow the standard definition of asymptotically flat spacetimes in the first order formalism (see e.g. [7], [25] and for a nice and pedagogical introduction in the metric formulation [6] and [65]). Here we give a brief introduction into asymptotically flat spacetimes, following closely [7].

In order to describe the behaviour of the metric at spatial infinity, we will focus on the region $\mathcal{R}$, that is the region outside the light cone of some point p. We define a 4−dimensional radial coordinate ρ given by $\rho^2 = \eta_{ab}x^a x^b$, where x^a are the Cartesian coordinates of the Minkowski metric η on $\mathbb{R}^4$ with origin at p. We will foliate the asymptotic region by timelike hyperboloids, $\mathcal{H}$, given by $\rho = \text{const}$, that lie in $\mathcal{R}$. Spatial infinity $\mathcal{I}$ corresponds to a limiting hyperboloid when $\rho \to \infty$. The standard angular coordinates on a hyperboloid are denoted by $\Phi^i = (\chi, \theta, \phi)$, and the relation between Cartesian and hyperbolic coordinates is given by: $x(\rho,\chi,\theta,\phi) = \rho\cosh\chi\sin\theta\cos\phi,\;\; y(\rho,\chi,\theta,\phi) = \rho\cosh\chi\sin\theta\sin\phi,\; z(\rho,\chi,\theta,\phi) = \rho\cosh\chi\cos\theta,\; t(\rho,\chi,\theta,\phi) = \rho\sinh\chi.$

We shall consider functions f that *admit an asymptotic expansion to order m* of the form,

$$f(\rho,\Phi) = \sum_{n=0}^{m} \frac{{}^n f(\Phi)}{\rho^n} + o(\rho^{-m}), \tag{4.29}$$

where the remainder $o(\rho^{-m})$ has the property that

$$\lim_{\rho\to\infty} \rho\, o(\rho^{-m}) = 0. \tag{4.30}$$

A tensor field $T^{a_1\ldots a_n}{}_{b_1\ldots b_m}$ will be said to admit an asymptotic expansion to order m if all its component in the *Cartesian* chart x^a do so. Its derivatives $\partial_c T^{a_1\ldots a_n}{}_{b_1\ldots b_m}$ admit an expansion of order $m+1$.

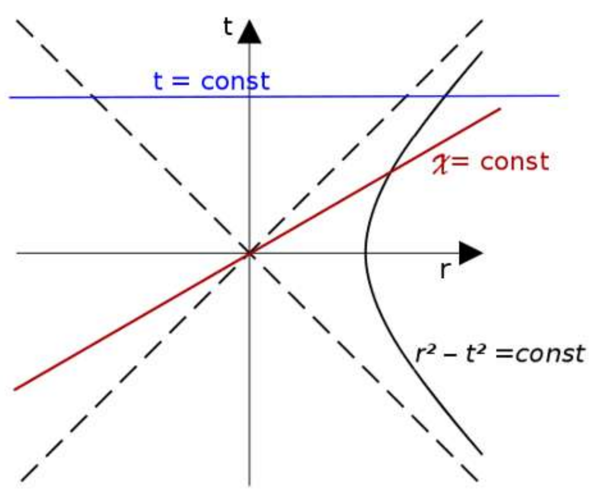

Figure 4.1: 2D visualization of slices at constant χ and t respectively.

With these ingredients at hand we can now define an asymptotically flat spacetime in terms of its metric: a smooth spacetime metric g on $\mathcal{R}$ is *weakly asymptotically flat at spatial infinity* if there exist a Minkowski metric η such that outside a spatially compact world tube $(g-\eta)$ admits an asymptotic expansion to order 1 and $\lim_{\rho\to\infty}(g-\eta) = 0$.

In such a space-time the metric in the region $\mathcal{R}$ takes the form,

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \left(1+\frac{2\sigma}{\rho}\right)\mathrm{d}\rho^2 + 2\rho\,\frac{\alpha_i}{\rho}\,\mathrm{d}\rho\,\mathrm{d}\Phi^i + \rho^2\left(h_{ij}+\frac{{}^1h_{ij}}{\rho}\right)\mathrm{d}\Phi^i\mathrm{d}\Phi^j + o(\rho^{-1}) \tag{4.31}$$

where σ, α_i and ${}^1h_{ij}$ only depend on the angles Φ^i and h_{ij} is the metric on the unit time-like hyperboloid in Minkowski spacetime:

$$h_{ij}\mathrm{d}\Phi^i\mathrm{d}\Phi^j = -\mathrm{d}\chi^2 + \cosh^2\chi\,(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{4.32}$$

Note that also we could have expanded the metric in a chart (r,Φ), associated with a timelike cylinder, or any other chart. But we chose the chart (ρ,Φ) because it is well adapted to the geometry of the problem and will lead to several simplifications. In the case of a $3+1-$decomposition a cylindrical chart could be a better choice.

For this kind of space-times, one can always find another Minkowski metric such that its off-diagonal terms α_i vanish in leading order. In [7] it is shown with details that the asymptotically flat metric can be written as

$$\mathrm{d}s^2 = \left(1+\frac{2\sigma}{\rho}\right)\mathrm{d}\rho^2 + \rho^2 h_{ij}\left(1-\frac{2\sigma}{\rho}\right)\mathrm{d}\Phi^i\mathrm{d}\Phi^j + o(\rho^{-1}), \tag{4.33}$$

with $\sigma(-\chi,\pi-\theta,\phi+\pi)=\sigma(\chi,\theta,\phi)$. We also see that ${}^1h_{ij}=-2\sigma h_{ij}$. These two conditions restrict the asymptotic behaviour of the metric, but are necessary in order to reduce the asymptotic symmetries to a Poincaré group, as demonstrated in [7].

From the previous discussion and the form of the metric one can obtain the fall-off conditions for tetrads. As shown in [7] in order to have a well defined Lorentz angular momentum one needs to admit an expansion of order 2, therefore we assume that in Cartesian coordinates we have the following behaviour

$$e_a^I = {}^oe_a^I + \frac{{}^1e_a^I(\Phi)}{\rho} + \frac{{}^2e_a^I(\Phi)}{\rho^2} + o(\rho^{-2}), \tag{4.34}$$

where ${}^0e^I$ is a fixed co-frame such that $g^0_{ab}=\eta_{IJ}\,{}^oe_a^I\,{}^oe_b^I$ is flat and $\partial_a\,{}^oe_b^I=0$.

The sub-leading term ${}^1e_a^I$ can be obtained from (4.33) and is given by [7],

$${}^1e_a^I = \sigma(\Phi)(2\rho_a\rho^I - {}^oe_a^I) \tag{4.35}$$

where

$$\rho_a = \partial_a\rho \quad \text{and} \quad \rho^I = {}^oe^{aI}\rho_a. \tag{4.36}$$

The asymptotic expansion for connection can be obtained from the requirement that the connection be compatible with tetrad on $\mathcal{I}$, to appropriate leading order. This leads to the asymptotic expansion of order 3 for the connection,

$$\omega_a^{IJ} = {}^o\omega_a^{IJ} + \frac{{}^1\omega_a^{IJ}}{\rho} + \frac{{}^2\omega_a^{IJ}}{\rho^2} + \frac{{}^3\omega_a^{IJ}}{\rho^3} + o(\rho^{-3})\,. \tag{4.37}$$

We require that De^I vanishes, to an appropriate order, more precisely, we ask that the term of order 0 in De^I vanishes

$$\mathrm{d}\,{}^o e^I + {}^o\omega^I{}_K \wedge {}^o e^K = 0\,, \tag{4.38}$$

and since $\mathrm{d}\,{}^o e^I = 0$ it follows that ${}^o\omega^{IK} = 0$. The term of order 1 should also vanish leading to ${}^1\omega^{IK} = 0$. We also ask that the term of order 2 in De^I vanishes, and we obtain

$$\mathrm{d}\Big(\frac{{}^1 e^I}{\rho}\Big) = -\frac{{}^2\omega^I{}_K}{\rho^2} \wedge {}^o e^K\,, \tag{4.39}$$

and we shall demand compatibility between e and ω only based on this condition. As a result, we obtain

$$\begin{aligned} {}^2\omega_a^{IJ}(\Phi) &= 2\rho^2\,\partial^{[J}(\rho^{-1}\,{}^1 e_a^{I]}) && (4.40)\\ &= 2\rho\,(2\rho^{[I}\rho_a\partial^{J]}\sigma - {}^o e_a^{[I}\partial^{J]}\sigma - \rho^{-1}\,{}^o e_a^{[I}\rho^{J]}\sigma)\,. && (4.41)\end{aligned}$$

Note that although ρ appears explicitly in the previous expression, it is independent of ρ.

Therefore, in the asymptotic region we have $De^I = O(\rho^{-3})$. This condition has its repercussions on the behaviour of the Holst and Neih-Yan terms, as we will show in what follows.

4.2.1.1 Palatini action with boundary term

Now we have all necessary elements in order to prove the finiteness of the Palatini action with boundary term, given by (4.1). This expression can be re-written as,

$$S_{\mathrm{PB}}(e,\omega) = \frac{1}{2\kappa}\int_{\mathcal{M}} \big(\mathrm{d}\Sigma^{IJ}\wedge\omega_{IJ} - \Sigma^{IJ}\wedge\omega_I{}^K\wedge\omega_{KJ}\big) \tag{4.42}$$

or in components

$$S_{\mathrm{PB}}(e,\omega) = \frac{1}{2\kappa}\int_{\mathcal{M}} \big(\partial_a\Sigma_{bc}^{IJ}\omega_{dIJ} - \Sigma_{ab}^{IJ}\omega_{cI}{}^K\omega_{dKJ}\big)\,\varepsilon^{abcd} \tag{4.43}$$

where ε^{abcd} is the metric compatible 4-form on $\mathcal{M}$. This volume element is related by $\varepsilon^{abcd} = \sqrt{g}\,\varepsilon^{abcd}\,d^4x$, to the Levi-Civita tensor density of weight +1, ε^{abcd}. We will prove that taking into account the boundary conditions (4.34) and (4.37), the integrand falls off as ρ^{-4}, while the volume element on any Cauchy slice in asymptotic region goes as ρ^2, so the action is manifestly finite always (even off-shell), if the two Cauchy surfaces are asymptotically time-translated with respect to each other.

From (4.32) and (4.33), we see that the volume element in the asymptotic region takes the form $\rho^3\cosh^2\chi\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi$. In order to prove finiteness we will consider the region bounded by Cauchy slices $t=$ const instead of $\chi=$ const, since in the second case for $\rho\to\infty$ the volumen of the region does not need to converge (see Fig. 4.1). Since $t(\rho,\chi,\theta,\phi)=\rho\sinh\chi$ at the surface with constant t we have $\rho\,\mathrm{d}\chi=-\tanh\chi\mathrm{d}\rho$. Substituting this into the metric we can see that the volume element is $\rho^2\cosh\chi\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$. As $\rho\to\infty$, the angle $\chi\to 0$ so $\cosh\chi\to 1$. It follows that in the limit $\rho\to\infty$ the volume of the region $\mathcal{M}$ behaves as ρ^2.

Now, we need to deduce the asymptotic behavior of $\mathrm{d}\Sigma^{IJ}\wedge\omega_{IJ}=\varepsilon_{IJKL}\mathrm{d}e^K\wedge e^L\wedge\omega^{IJ}$. Since $\mathrm{d}({}^oe^I)=0$ it follows that

$$\mathrm{d}e^K=\frac{1}{\rho}\mathrm{d}[{}^1e^K(\Phi)]+O(\rho^{-2})\,. \tag{4.44}$$

The partial derivative, with respect to cartesian coordinates, of any function $f(\Phi)$ is proportional to ρ^{-1},

$$\partial_a f(\Phi)=\frac{\partial\Phi^i}{\partial x^a}\frac{\partial f}{\partial\Phi^i}=\frac{1}{\rho}A_a^i(\Phi)\frac{\partial f}{\partial\Phi^i}\,, \tag{4.45}$$

where the explicit expression for $A_a^i(\Phi)$ can be obtained from the relation between Cartesian and hyperbolic coordinates. As a consequence $\mathrm{d}e^K=O(\rho^{-2})$, and since $\omega_{IJ}=O(\rho^{-2})$ it follows that $\mathrm{d}\Sigma^{IJ}\wedge\omega_{IJ}$ falls off as ρ^{-4}, and the Palatini action with boundary term is finite.

Now let us prove the differentiability of the action (4.1). As we have commented after (4.2), this action is differentiable if the boundary term that appears in the variation vanishes. This boundary term is

$$\frac{1}{2\kappa}\int_{\partial\mathcal{M}}\delta\Sigma^{IJ}\wedge\omega_{IJ}=\frac{1}{2\kappa}\left(-\int_{M_1}+\int_{M_2}+\int_{\mathcal{I}}-\int_{\Delta}\right)\delta\Sigma^{IJ}\wedge\omega_{IJ}\,, \tag{4.46}$$

where we decomposed the boundary as $\partial\mathcal{M}=M_1\cup M_2\cup\mathcal{I}\cup\Delta$, as in Fig.2.7. On the Cauchy slices, M_1 and M_2, we assume $\delta e_a^I=0$ so the integrals vanish, and in the following section we will prove that over Δ this integral also vanishes. Here we will focus on the contribution of the asymptotic region $\mathcal{I}$.

On a time-like hyperboloid $\mathcal{H}$, $\rho=$ const, so that its volume element is $\rho^3\cosh^2\chi\sin\theta\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi=\rho^3\mathrm{d}^3\Phi$ and the boundary term can be written as,

$$\frac{1}{2\kappa}\int_{\mathcal{I}}\delta\Sigma^{IJ}\wedge\omega_{IJ}=\frac{1}{2\kappa}\lim_{\rho\to\infty}\int_{\mathcal{H}}\delta\Sigma_{ab}^{IJ}\,\omega_{cIJ}\,\varepsilon^{abc}\rho^3\mathrm{d}^3\Phi\,, \tag{4.47}$$

Now we can use that,

$$\delta\Sigma_{abIJ}=\rho^{-1}\varepsilon_{IJKL}\,{}^oe_a^K\delta\sigma(\Phi)\left(2\rho_b\rho^L-{}^oe_b^L\right)+O(\rho^{-2})\,. \tag{4.48}$$

Since ρ is constant on a hyperboloid, it follows that ρ_a is orthogonal to it and $\rho_a \varepsilon^{abc} = 0$. Then, we obtain

$$\delta\Sigma^{IJ}_{ab}\,\omega_{cIJ}\,\varepsilon^{abc} = \frac{2}{\rho^3}\,\delta\sigma\,\varepsilon_{IJKL}\,{}^{o}e^{K}_{a}\,{}^{o}e^{L}_{b}\,{}^{o}e^{I}_{c}(\rho\,\partial^J\sigma + \rho^J\sigma)\varepsilon^{abc} + O(\rho^{-4})\,. \tag{4.49}$$

In this expression the term with a derivative of σ is proportional to $\partial_\rho\sigma = 0$, so that the variation (4.47) reduces to

$$\frac{1}{2\kappa}\int_{\mathcal{I}}\delta\Sigma^{IJ}\wedge\omega_{IJ} = \frac{3}{2\kappa}\delta\,(\int_{\mathcal{H}_1}\sigma^2\mathrm{d}^3\Phi)\,, \tag{4.50}$$

where $\mathcal{H}_1$ is the unit hyperboloid. So we see that the Palatini action with the boundary term is differentiable when we restrict to configurations that satisfy asymptotically flat boundary conditions, such that $C_\sigma := \int_{\mathcal{H}_1}\sigma^2\mathrm{d}^3\Phi$ has the same (arbitrary) value for all of them. In that case, the above expression (4.50) vanishes. This last condition is not an additional restriction to the permissible configurations, because every one of them (compatible with our boundary conditions) corresponds to some fixed value of C_σ.

Here we want to emphasize the importance of the boundary term added to the action given that, without it, the action fails to be differentiable. The contribution from the asymptotic region to the variation of the *Palatini action* is,

$$\frac{1}{2\kappa}\int_{\mathcal{I}}\Sigma^{IJ}\wedge\delta\omega_{IJ} = \frac{1}{2\kappa}\lim_{\rho\to\infty}\int_{\mathcal{H}}\Sigma^{IJ}_{ab}\,\delta\omega_{cIJ}\,\varepsilon^{abc}\rho^3\mathrm{d}^3\Phi\,. \tag{4.51}$$

Our boundary conditions imply that $\Sigma^{IJ}_{ab}\delta\omega_{cIJ} = O(\rho^{-2})$, so that the integral behaves as $\int_{\mathcal{I}}\rho\,\mathrm{d}^3\Phi$, and in the limit $\rho\to\infty$ is explicitly divergent.

4.2.1.2 Holst term

As we have seen earlier, in the asymptotic region we have $De^I = O(\rho^{-3})$. Furthermore, as $D(De^I) = F^{IK}\wedge e_K$, we have that $F^{IK}\wedge e_K = O(\rho^{-4})$. We can see that explicitly by calculating the term of order 3 in this expression

$$F^{IK}\wedge e_K = \mathrm{d}\big(\frac{{}^{2}\omega^{I}{}_{K}}{\rho^2}\big)\wedge{}^{o}e^{K} + O(\rho^{-4})\,. \tag{4.52}$$

The first term in the previous expression vanishes since $\mathrm{d}\big(\frac{{}^{2}\omega^{I}{}_{K}}{\rho^2}\big)\wedge{}^{o}e^{K} = \mathrm{d}\big(\frac{{}^{2}\omega^{I}{}_{K}}{\rho^2}\wedge{}^{o}e^{K}\big) = 0$, due to (4.39). So, we see that the Holst term

$$S_{\mathrm{H}} = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} e^I\wedge e^J\wedge F_{IJ}\,, \tag{4.53}$$

is finite under these asymptotic conditions, since $e^I \wedge e^J \wedge F_{IJ}$ goes as ρ^{-4}, while the volume element on every Cauchy surface goes as $\rho^2 \mathrm{d}\rho\, \mathrm{d}^2\Omega$.

The variation of the Holst term is well defined if the boundary term, obtained as a result of variation, vanishes. We will analyze the contribution of this term

$$\frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}} e^I \wedge e^J \wedge \delta\omega_{IJ}\,. \tag{4.54}$$

Let us examine the term of order 2 of the integrand, it is

$${}^o e^I \wedge {}^o e^J \wedge \frac{\delta({}^2\omega_{IJ})}{\rho^2} = \mathrm{d}\big[{}^o e^I \wedge \frac{\delta({}^1 e_I)}{\rho}\big]\,, \tag{4.55}$$

due to (4.39) and $\mathrm{d}\,{}^o e^I = \delta\,{}^o e^I = 0$, and this term does not contribute to (4.54). So, the leading term in $e^I \wedge e^J \wedge \delta\omega_{IJ}$ is of order 3, and it does not vanish, since it depends also on ${}^3\omega_{IJ}(\Phi)$, which is not fixed by our boundary conditions. Since the volume element on a hyperboloid $\mathcal{H}$ goes as $\rho^3 \mathrm{d}^3\Phi$, it follows that the boundary term (4.54) does not vanish at $\mathcal{I}$ (though it is finite).

As our analysis shows we should provide a boundary term for the Holst term, in order to make it differentiable. It turns out that this term should be [25]

$$S_{\mathrm{BH}} = \frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}} e^I \wedge e^J \wedge \omega_{IJ}\,. \tag{4.56}$$

Let us show first that this term is finite and for that we should prove that its term of order 2 vanishes. This term is

$${}^o e^I \wedge {}^o e^J \wedge \frac{{}^2\omega_{IJ}}{\rho^2} = \mathrm{d}\big({}^o e^I \wedge \frac{{}^1 e_I}{\rho}\big)\,, \tag{4.57}$$

due to the same arguments as in (4.55), and we see that it does not contribute to the boundary term (4.56). So, the leading term of the integrand is of order 3, and since the volume element at $\mathcal{H}$ goes as $\rho^3 \mathrm{d}^3\Phi$, it follows that (4.56) is finite.

The Holst term with its boundary term (4.9) can be written as

$$S_{\mathrm{HB}} = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} e^I \wedge e^J \wedge F_{IJ} + \frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}} e^I \wedge e^J \wedge \omega_{IJ}\,, \tag{4.58}$$

and also as an integral over $\mathcal{M}$

$$S_{\mathrm{HB}} = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} 2\,\mathrm{d}e^I \wedge e^J \wedge \omega_{IJ} - e^I \wedge e^J \wedge \omega_{IK} \wedge \omega^K{}_J\,. \tag{4.59}$$

As we have seen in (4.10), the variation of the Holst term with its boundary term is well defined provided that the following boundary contribution

$$\frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}}\delta\Sigma^{IJ}\wedge\star\omega_{IJ}=\frac{1}{2\kappa\gamma}\left(-\int_{M_1}+\int_{M_2}+\int_{\mathcal{I}}-\int_{\Delta}\right)\delta\Sigma^{IJ}\wedge\star\omega_{IJ}\,, \tag{4.60}$$

vanishes. We first note that ω_a^{IJ} and $\star\omega_a^{IJ}$ have the expansion of the same order, the leading term is $O(\rho^{-2})$. Using (4.48), the fact that ρ_a is orthogonal to $\mathcal{I}$ and $\eta_{ab}\varepsilon^{abc}=0$, one can see that the leading term in the integrand vanishes in the asymptotic region, so that $\delta\Sigma^{IJ}\wedge\star\omega_{IJ}=O(\rho^{-4})$ and the integral over $\mathcal{I}$ vanishes. In the next section we will prove that the integral over Δ vanishes, so that Holst action with boundary term is well defined.

4.2.1.3 Pontryagin and Euler terms

Since we are interested in a generalization of the first order action of general relativity, that includes topological terms, we need to study their asymptotic behaviour. We begin with Pontryagin and Euler terms, that turn out to be well defined.

It is straightforward to see that the Pontryagin term (4.14) is finite for asymptotically flat boundary conditions. Since

$$S_{\mathrm{Po}}[e,\omega]=\int_{\mathcal{M}}F_{ab}^{IJ}\wedge F_{cdIJ}\varepsilon^{abcd}\,, \tag{4.61}$$

the finiteness of this expression depends on the asymptotic behavior of F_{IJ}. Taking into account (4.37), we can see that the leading term of F_{abIJ} falls off as ρ^{-3}. Since the volume of any Cauchy slice is $\rho^2\sin\theta\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$, in the limit when $\rho\to\infty$ the integral goes to zero. As a result, the Pontryagin term is finite even off-shell. The same result holds for the Euler term (4.20), since the leading term in the asymptotic form of $\star F_{IJ}$ is of the same order as of F_{IJ}.

Now we want to prove that both terms are differentiable. As we have showed in (4.144), the variation of the Pontryagin term is,

$$\delta S_{\mathrm{Po}}=2\int_{\partial\mathcal{M}}F^{IJ}\wedge\delta\omega_{IJ}=2\left(-\int_{M_1}+\int_{M_2}+\int_{\mathcal{I}}-\int_{\Delta}\right)F^{IJ}\wedge\delta\omega_{IJ}\,. \tag{4.62}$$

In the following subsection we prove that on Δ the integral vanishes. For $\mathcal{I}$, we need to prove that the integral

$$\int_{\mathcal{I}}F^{IJ}\wedge\delta\omega_{IJ}=2\lim_{\rho\to\infty}\int_{\mathcal{H}}F_{ab}^{IJ}\delta\omega_{cIJ}\varepsilon^{abc}\rho^3\mathrm{d}^3\Phi\,, \tag{4.63}$$

vanishes. Taking into account (4.37) we can see that the leading term of $F_{abIJ}\omega_c{}^{IJ}$ goes as ρ^{-5}. Therefore the integral falls off as ρ^{-2} which in the limit $\rho\to\infty$ goes to zero. The same holds for the Euler term.

4.2.1.4 Nieh-Yan term

The Neih-Yan topological term is given by

$$S_{\mathrm{NY}} = \int_{\partial\mathcal{M}} De^I \wedge e_I \,. \tag{4.64}$$

and it is finite since the integrand is of order 3, and the volume element on $\mathcal{H}$ is $\rho^3 \mathrm{d}^3\Phi$, so the contribution at $\mathcal{I}$ is finite. The variation of S_{NY} is

$$\delta S_{\mathrm{NY}} = \int_{\partial\mathcal{M}} 2De_I \wedge \delta e^I - e^I \wedge e^J \wedge \delta\omega_{IJ} \,, \tag{4.65}$$

and we see that the first term vanishes, but the second one is exactly as in (4.54) and we have seen that it does not vanish, so we need to add a boundary term to the Nieh-Yan action in order to make it differentiable.

As we have seen in (4.1.3.2) the difference between the Nieh-Yan and Holst terms is given by an expression quadratic in torsion S_{T},

$$S_{\mathrm{T}} = \int_{\mathcal{M}} De^I \wedge De_I \,, \tag{4.66}$$

and we will analyze this term here. As we have seen before, our asymptotic boundary conditions imply that $De^I = O(\rho^{-3})$ and since the volume element on every Cauchy surface goes as ρ^2, it follows that S_{T} is finite off-shell.

The variation of S_{T} is given by

$$\delta S_{\mathrm{T}} = 2\int_{\mathcal{M}} \delta e^I \wedge F_{IK} \wedge e^K + \delta\omega^I{}_K \wedge e^K \wedge De_I + 2\int_{\partial\mathcal{M}} De^I \wedge \delta e^I \,, \tag{4.67}$$

and it is easy to see that it is well defined. Namely, the surface term vanishes since we demand $De^I = 0$ on an isolated horizon Δ, while at the spatial infinity the integrand behaves as $O(\rho^{-5})$ and the volume element goes as $\rho^3 \mathrm{d}^3\Phi$, and in the limit $\rho \to \infty$ the contribution of this term vanishes.

From (4.24) we see that the Neih-Yan term is not well defined due to the behaviour of the Holst term, so one need to provide a corresponding boundary term in order to make it well defined. The resulting action is given in (4.26), or equivalently

$$S_{\mathrm{NYB}} = 2\kappa\gamma S_{\mathrm{HB}} + S_{\mathrm{T}} \,. \tag{4.68}$$

4.2.2 Internal boundary: Isolated horizons

We shall consider the contribution to the variation of the action at the internal boundary, in this case a weakly isolated horizon. A weakly isolated horizon is a non-expanding null 3-dimensional hypersurface, with an additional condition that implies that surface gravity is constant on a horizon. Let us specify with some details its definition and basic properties [8].

Let Δ be a 3-dimensional null surface of $(\mathcal{M}, g_{ab})$, equipped with future directed null normal l. Let $q_{ab} \hat{=} g_{\underleftarrow{ab}}$ be the (degenerate) induced metric on Δ (we denote by $\hat{=}$ an equality which holds only on Δ and the arrow under a covariant index denotes the pullback of a corresponding form to Δ). A tensor q^{ab} that satisfies $q^{ab} q_{ac} q_{bd} \hat{=} q_{cd}$, is called an inverse of q_{ab}. The expansion of a null normal l is defined by $\theta_{(l)} = q^{ab} \nabla_a l_b$, where ∇_a is a covariant derivative compatible with the metric g_{ab}.

The null hypersurface Δ is called a *non-expanding horizon* if it satisfies the following conditions: (i) Δ is topologically $S^2 \times \mathbb{R}$, (ii) $\theta_{(l)} = 0$ for any null normal l and (iii) all equations of motion hold at Δ and $-T_{ab} l^b$ is future directed and causal for any l, where T_{ab} is matter stress-energy tensor at Δ. The second condition implies that the area of the horizon is constant 'in time', so that the horizon is isolated.

We need one additional condition in order to satisfy the zeroth law of black hole dynamics. In order to introduce it let us first specify some details of the geometry of the isolated horizon. It is convenient to use null-tetrads $(l, n, m, \bar{m})$, where a real, future directed null vector field n is transverse to Δ and a complex vector field m is tangential to Δ, such that $l \cdot n = -1$, $m \cdot \bar{m} = 1$ and all the other scalar products vanish.

Since l is a null normal to Δ it is geodesic and its twist vanishes. We define surface gravity $\kappa_{(l)}$ as the acceleration of l^a

$$l^a \nabla_a l^b \hat{=} \kappa_{(l)} l^b \,. \tag{4.69}$$

We note that $\kappa_{(l)}$ is associated to a specific null normal l, if we replace l by $l' = fl$ the acceleration changes $\kappa_{(l')} = f \kappa_{(l)} + \pounds_l f$.

The Raychaudhuri and Einstein's equations together with the condition on the stress-energy tensor imply that every l is also shear free and since its expansion and twist vanish there exists a one-form ω_a such that [23]

$$\nabla_{\underleftarrow{a}} l^b \hat{=} \omega_a l^b \,. \tag{4.70}$$

Under the rescaling of the null normal $l \to l' = fl$, ω transforms like a connection $\omega \to \omega' = \omega + \mathrm{d}(\ln f)$ (we see that ω is invariant under constant rescaling). It is also easy to see that the horizon is 'time' invariant, in the sense that $\pounds_l q_{ab} \hat{=} 0$. Furthermore, the area two-form on the cross-sections of Δ, ${}^2\varepsilon := i m \wedge \bar{m}$ is also preserved in 'time', $\pounds_l {}^2\varepsilon \hat{=} 0$.

Since l can be rescaled by an arbitrary positive function, in general $\kappa_{(l)}$ is not constant on Δ. If we want to establish the zeroth law of black hole dynamics $\mathrm{d}\kappa_{(l)} \hat{=} 0$ we need one additional condition, the 'time' invariance of ω,

$$\pounds_l \omega \hat{=} 0 \,. \tag{4.71}$$

Now, if we restrict to constant rescaling of l, $l \to l' = cl$ that leaves ω invariant, then the zeroth law of black hole dynamics follows, for every null normal l related to each other by constant rescaling.

All null normals related to each other by a constant rescaling form an equivalence class $[l]$. Now, we can define a *weakly isolated horizon* (WIH) $(\Delta, [l])$ as a non-expanding horizon equipped with an equivalence class $[l]$, such that $\pounds_l \omega \hat{=} 0$, for all $l \in [l]$.

In order to analyze the contribution to the variation of the action over the internal boundary, which is a WIH Δ, we equip Δ with a fixed class of null normals $[l]$ and fix an internal null tetrads $(l^I, n^I, m^I, \bar{m}^I)$ on Δ, such that their derivative with respect to flat derivative operator ∂_a vanishes.The permissible histories at Δ should satisfy two conditions: (i) the vector field $l^a := e^a_I l^I$ should belong to the fixed equivalence class $[l]$ (this is a condition on tetrads) and (ii) the tedrads and connection should be such that $(\Delta, [l])$ constitute a WIH.

The expression for tetrads on Δ is given by [24]

$$e^I_a \hat{=} -l^I n_a + \bar{m}^I m_a + m^I \bar{m}_a \,, \tag{4.72}$$

so that

$$\Sigma^{IJ} \hat{=} 2 l^{[I} n^{J]}\, {}^2\epsilon + 2i\, n \wedge (m\, l^{[I} \bar{m}^{J]} - \bar{m}\, l^{[I} m^{J]}) \,. \tag{4.73}$$

The expression for the connection on Δ is given by [24]

$$\omega_{IJ} \hat{=} -2\omega\, l_{[I} n_{J]} + 2U\, l_{[I} \bar{m}_{J]} + 2\bar{U}\, l_{[I} m_{J]} + 2V\, m_{[I} \bar{m}_{J]} \,, \tag{4.74}$$

where we have introduced two new one-forms, a complex one U and purely imaginary one V. In [24] the expression for these one forms is given in terms of Newman-Penrose (NP) spin coefficients and null tetrads. First we have

$$\omega_a = -(\varepsilon + \bar{\varepsilon}) n_a + (\bar{\alpha} + \beta) \bar{m}_a + (\alpha + \bar{\beta}) m_a \,, \tag{4.75}$$

where α, β and ε are NP spin coefficients. In what follows we do not need their explicit form, we will just write down the expression for ε since this coefficient will be of special importance, $\varepsilon = \frac{1}{2}(\bar{m}^a l^b \nabla_b m_a - n^a l^b \nabla_b l_a)$. Since $\kappa_{(l)} = l^a \omega_a$ it follows that $\kappa_{(l)} = \varepsilon + \bar{\varepsilon}$.

Also, it can be shown that [8]

$$\mathrm{d}\omega \hat{=} \mathcal{G}\, {}^2\epsilon \,, \tag{4.76}$$

where $\mathcal{G} = 2\,\mathrm{Im}\,[\Psi_2] =$, with $\Psi_2 = C_{abcd} l^a m^b \bar{m}^c n^d$, and C_{abcd} are the components of the Weyl tensor. Now, it is easy to see that the condition $\pounds_l \omega \hat{=} 0$ leads to $\mathrm{d}(\mathrm{Re}\,\varepsilon) \hat{=} 0$.

On the other hand we have

$$U_a \hat{=} -\bar{\pi} n_a + \bar{\mu} m_a + \bar{\lambda} \bar{m}_a \,, \tag{4.77}$$

and

$$V_a \hat{=} -(\varepsilon - \bar{\varepsilon})n_a + (\beta - \bar{\alpha})\bar{m}_a + (\alpha - \bar{\beta})m_a\,, \tag{4.78}$$

where π, μ and λ are additional NP spin coefficients. It has been also shown that

$$\mathrm{d}V \hat{=} \mathcal{F}\,{}^2\varepsilon\,, \tag{4.79}$$

where $\mathcal{F}$ is a function of the Riemann curvature and Weyl tensor. Then, we can calculate $\pounds_l V$,

$$\pounds_l V = l \cdot \mathrm{d}V + \mathrm{d}(l \cdot V) = 2\mathrm{d}(\mathrm{Im}\,\varepsilon)\,. \tag{4.80}$$

We shall also need the expression for the pull-back of the curvature two-form on the weakly isolated non-rotating horizon (the details are given in [44])

$$F^{IJ} \hat{=} -2i\mathcal{R}\,{}^2\varepsilon\, m^{[I}\,\bar{m}^{J]} + 2F^{KL}n_L\, l^{[I}(m^{J]}m_K + \bar{m}^{J]}\bar{m}_K)\,, \tag{4.81}$$

where $\mathcal{R}$ is the scalar curvature of the cross-section of Δ.

Now we have all necessary elements in order to calculate the contribution of the variation of the Palatini action, the Holst term and topological terms at isolated horizon. Before that, let us first examine the gauge invariance of the boundary terms of Palatini action and Holst term, given in (4.1) and (4.9), on a weakly isolated horizon Δ. The residual Lorentz transformations, compatible with the definition of Δ can be divided in two groups: ones that preserve the direction of a vector l and rotate m

$$l \to cl\,, \quad n \to \frac{1}{c}n\,, \quad m \to e^{i\theta}m\,, \tag{4.82}$$

and ones that leave l invariant, but change n and m

$$l \to l\,, \quad n \to n - um - \bar{u}\bar{m} + u\bar{u}l\,, \quad m \to m - \bar{u}l\,. \tag{4.83}$$

Note that $c = \mathrm{const}$, since $l \in [l]$. We shall also partially fixed θ by the condition $l^a\nabla_a\theta = -2\,(\mathrm{Im}\varepsilon)$. We will see below that this condition implies that $\mathrm{Im}\varepsilon = 0$ and then from (4.80) it follows that $\pounds_l V = 0$. There are no restrictions on u.

The Palatini boundary term on the horizon reduces to

$$\frac{1}{2\kappa}\int_\Delta \Sigma_{IJ} \wedge \omega^{IJ} = \frac{1}{\kappa}\int_\Delta {}^2\varepsilon \wedge \omega\,. \tag{4.84}$$

It was shown in [17] that ω is invariant under both classes of transformations. Two-form ${}^2\varepsilon = im \wedge \bar{m}$ is invariant under (4.82), and also under (4.83) since this transformation implies $m_a \to m_a$, due to $l_a \hat{=} 0$.

Similarly, the Holst boundary term on the horizon is

$$\frac{1}{2\kappa\gamma}\int_{\Delta}\Sigma_{IJ}\wedge\star\omega^{IJ}=\frac{i}{\kappa}\int_{\Delta}{}^{2}\varepsilon\wedge V\,. \tag{4.85}$$

It turns out that V is invariant under (4.83), and under (4.82) it transforms as $V\to V-i\mathrm{d}\theta$. Due to the restriction on θ, we have that $\nabla_a\theta\hat{=}wm_a+\bar{w}\bar{m}_a$, where w is arbitrary, so that ${}^2\varepsilon\wedge\mathrm{d}\theta\hat{=}0$. As a consequence ${}^2\varepsilon\wedge V$ is also invariant under gauge transformations on Δ.

4.2.2.1 Palatini action and isolated horizons

In this part we will analyze the variation of the Palatini action with boundary term (4.1), on an isolated horizon Δ

$$\delta S_{\mathrm{PB}}|_{\Delta}=\frac{1}{2\kappa}\int_{\Delta}\varepsilon_{IJKL}\omega^{IJ}\wedge e^{K}\wedge\delta e^{L}=\frac{1}{2\kappa}\int_{\Delta}\omega\wedge\delta\,{}^{2}\varepsilon\,. \tag{4.86}$$

Since Δ is a non-expanding horizon, $\pounds_l\,{}^2\varepsilon\hat{=}0$. Any other permissible configuration of tetrads, $(e^a_I)'$, should also satisfy $\pounds_{l'}{}^2\varepsilon'\hat{=}0$, where $l'^a\in[l]$ and ${}^2\varepsilon'={}^2\varepsilon+\delta\,{}^2\varepsilon$. For the null normals in the equivalence class $[l]$, $\pounds_{l'}{}^2\varepsilon'=c\,\pounds_l{}^2\varepsilon'\hat{=}0$, and it follows that $\pounds_l\,\delta\,{}^2\varepsilon\hat{=}0$. In the variational principle all fields are fixed on initial and final Cauchy surfaces, M_1 and M_2, in particular $\delta\,{}^2\varepsilon=0$ on two-spheres at the intersection of the initial and final Cauchy surface with the WIH, $S_{1,2}:=M_{1,2}\cap\Delta$ (see Fig. 1). Furthermore, $\delta^2\varepsilon$ does not change along any null normal l, so that $\delta\,{}^2\varepsilon\hat{=}0$ on the entire horizon (comprised between the two Cauchy surfaces) and the integral (4.86) vanishes. We should remark that, in the following parts, we *will* use the same argument whenever we have some field configuration whose Lie derivative along l vanishes on the horizon, to prove that its variation is zero on the horizon.

We note that the variation of the Palatini action, without boundary term, at Δ is

$$\delta S_{\mathrm{P}}|_{\Delta}=-\frac{1}{2\kappa}\int_{\Delta}\varepsilon_{IJKL}\delta\omega^{IJ}\wedge e^{K}\wedge e^{L}=-\frac{1}{2\kappa}\int_{\Delta}\delta\omega\wedge{}^{2}\varepsilon\,. \tag{4.87}$$

In this case, one can argue that the term on the RHS vanishes, because from $\pounds_l\omega\hat{=}0$ it follows that $\delta\omega\hat{=}0$ (a similar, but slightly different, argument was used in [8]). We see that the variational principle for the Palatini action is well defined even without boundary terms on the horizon. Nevertheless, for the reasons already mentioned in the previous chapter we shall keep the boundary terms in (4.1) on the whole boundary, including the internal one.

4.2.2.2 Holst term and isolated horizon

Let us now consider the contributions coming from the Holst term in the presence of an isolated horizon. The variation of the Holst term, with its boundary term, on an isolated horizon is given by

$$\delta S_{\mathrm{HB}}|_{\Delta} = \frac{1}{2\kappa\gamma}\int_{\Delta}\omega^{IJ}\wedge e_I\wedge\delta e_J = \frac{i}{2\kappa\gamma}\int_{\Delta}V\wedge\delta\,{}^2\epsilon\,, \tag{4.88}$$

and for the same reasons that we used before, after the equation (4.86), since $\pounds_l\,{}^2\epsilon\,\hat{=}\,0$, it follows that $\delta\,{}^2\epsilon\,\hat{=}\,0$ and the variation (4.88) vanishes.

On the other hand, the variation of the Holst term, without a boundary term, is also well defined on a horizon, since

$$\delta S_{\mathrm{H}}|_{\Delta} = -\frac{1}{2\kappa\gamma}\int_{\Delta}\delta\omega^{IJ}\wedge e_I\wedge e_J = -\frac{i}{2\kappa\gamma}\int_{\Delta}\delta V\wedge{}^2\epsilon\,. \tag{4.89}$$

Now we can not use the same argument as in the case of Palatini action since the Lie derivative of V does not vanish on Δ, as shown in (4.80), $\pounds_l V\,\hat{=}\,2\,\mathrm{d}\,(\mathrm{Im}\varepsilon)$. As we commented earlier, we have a freedom to perform local Lorentz transformations in order to make ε a real function. Namely, the rotation in the $(m,\bar{m})$ plane, given by $m\rightarrow e^{i\theta}m$, where θ is an arbitrary function, generates the following transformation of the NP spin coefficient ε [23]: $\varepsilon\rightarrow\varepsilon+\frac{i}{2}l^a\nabla_a\theta$. So, ε can be made real after the appropriate rotation that satisfies the condition $l^a\nabla_a\theta=-2\,(\mathrm{Im}\varepsilon)$. Due to this gauge freedom we can always choose a real ε, and as a result $\pounds_l V=0$. When we change the configuration of fields this condition could be violated, but then again one can perform a gauge transformation to obtain $\pounds_{l'}V'=0$. Then, using the same arguments as before we can conclude that $\delta V\,\hat{=}\,0$, in the variational principle and (4.89) also vanishes. Note that this argument is simpler and departs significantly from that in [24].

4.2.2.3 Topological terms and isolated horizon

Let us now consider the possible contributions coming from the topological terms. That is, we shall see whether the above conditions are sufficient to make the variation of the topological terms well defined at Δ. The variation of the Pontryagin term on the horizon is

$$\delta S_{\mathrm{Po}}|_{\Delta} = 2\int_{\Delta}F^{IJ}\wedge\delta\omega_{IJ} = 4i\int_{\Delta}\mathcal{R}\,{}^2\epsilon\wedge\delta V\,, \tag{4.90}$$

where we have used the expresions for the curvature at the horizon (4.81) and for the connection (4.74). The argument just presented in the previous part implies that $\delta V\,\hat{=}\,0$, so that the variation δS_{Po} vanishes at the horizon.

The variation of the Euler term on the horizon is

$$\delta S_{\rm E}|_{\Delta} = 2\int_{\Delta} \star F^{IJ} \wedge \delta\omega_{IJ} = 4\int_{\Delta} \mathcal{R}^2 \varepsilon \wedge \delta\omega\,, \tag{4.91}$$

and it vanishes since $\delta\omega \hat{=} 0$.

Note that the variational principle for the Neih-Yan term is also well defined, since it reduces to the variation of the Holst term plus an additional term, given in (4.67), that vanishes at Δ since part of the boundary conditions defining an isolated horizon require that the equation of motion $\mathrm{D}e_I = 0$, should also hold on Δ.

We can then conclude that the inclusion of the topological terms to the action is compatible with a well defined action principle, without the need of adding new boundary terms at the horizon.

4.3 Conserved charges

In this section we shall consider some of the information that comes from the covariant Hamiltonian formulation. In particular, we shall see how one can define conserved quantities. As we have discussed in Secs. 2.5.2 and 2.5.3 there are two classes of quantities, namely those that are generators of Hamiltonian symmetries and the so called Noether charges. We shall then analyze the relation between Hamiltonian and Noether charges for the most general first order gravitational action, focusing on the role that the boundary terms play. As one might anticipate, the fact that the boundary terms do not modify the symplectic structure implies that the Hamiltonian charges are insensitive to the existence of extra boundary terms. However, as we shall see in detail, the Noetherian quantities *do* depend on the boundary terms. Specifically, we are interested in the relation of the Noether charge with the energy at the asymptotic region and the energy of the horizon.

4.3.1 Hamiltonian charges

From equations (2.115) and (4.1), the symplectic potential for the well posed Palatini action $S_{\rm PB}$ is given by

$$\Theta_{\rm PB}(\delta) = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \delta\Sigma^{IJ} \wedge \omega_{IJ}\,. \tag{4.92}$$

Therefore from (2.119) and (4.92) the corresponding symplectic current is,

$$J_{\rm P}(\delta_1,\delta_2) = -\frac{1}{2\kappa}\left(\delta_1\Sigma^{IJ} \wedge \delta_2\omega_{IJ} - \delta_2\Sigma^{IJ} \wedge \delta_1\omega_{IJ}\right)\,. \tag{4.93}$$

Note that the symplectic current is insensitive to the boundary term, as we discussed in Sec. 2.5.1. From the equation (2.123) one can obtain a conserved pre-symplectic structure, as an integral of $J_{\rm P}$ over a spatial surface, if the integral of the symplectic current over the asymptotic region vanishes and if the integral over an isolated horizon behaves appropriately. As shown in [7], for asymptotically flat spacetimes, $\int_{\mathcal{I}} J_{\rm P} = 0$, and on a WIH we have $J_{\rm P} \hat{=} \mathrm{d}j$, [8]. As a result, the conserved pre-symplectic structure for the Palatini action, for asymptotically flat spacetimes with weakly isolated horizon, takes the form [8]

$$\bar{\Omega}_P(\delta_1, \delta_2) = -\frac{1}{2\kappa}\int_M \left(\delta_1 \Sigma^{IJ} \wedge \delta_2 \omega_{IJ} - \delta_2 \Sigma^{IJ} \wedge \delta_1 \omega_{IJ}\right) - \frac{1}{\kappa}\int_{S_\Delta} \delta_1 \psi\, \delta_2({}^2\varepsilon) - \delta_2 \psi\, \delta_1({}^2\varepsilon)\,, \quad (4.94)$$

where S_Δ is a 2-sphere at the intersection of a Cauchy surface M with a horizon and ψ is a potential defined as

$$\pounds_l \psi = \kappa_{(l)}\,, \quad \psi = 0 \ \text{ on } \ S_{1\Delta}\,, \quad (4.95)$$

with $S_{1\Delta} = M_1 \cap \Delta$. We see that the existence of an isolated horizon modifies the symplectic structure of the theory.

Let us now see what is the contribution of the well posed Holst term, $S_{\rm HB}$. In this case the symplectic potential is given by [25]

$$\Theta_{\rm HB}(\delta) = \frac{1}{2\kappa\gamma}\int_{\partial\mathcal{M}} \delta\Sigma^{IJ} \wedge \star\omega_{IJ} = \frac{1}{\kappa\gamma}\int_{\partial\mathcal{M}} \delta e^I \wedge \mathrm{d}e_I\,, \quad (4.96)$$

where in the second line we used the equation of motion $De^I = 0$. The symplectic current in this case is a total derivative and is given by

$$J_{\rm H}(\delta_1, \delta_2) = \frac{1}{\kappa\gamma}\mathrm{d}\left(\delta_1 e^I \wedge \delta_2 e_I\right). \quad (4.97)$$

As we have seen in the Sec. 2.4, when the symplectic current is a total derivative, the covariant Hamiltonian formalism indicates that the corresponding (pre)-symplectic structure vanishes.

As we also remarked in Sec. 2.4, one could postulate a conserved two form $\tilde{\Omega}$ if $\int_{\mathcal{I}} J_{\rm H} = 0$ and $\int_\Delta J_{\rm H} = 0$, in which case this term defines a conserved symplectic structure. Let us, for completeness, consider this possibility. In [25] it has been shown that the integral at $\mathcal{I}$ vanishes, so here we shall focus on the integral over Δ

$$\int_\Delta J_{\rm H} = \frac{1}{\kappa\gamma}\int_{\partial\Delta} \delta_1 e^I \wedge \delta_2 e_I = \frac{1}{\kappa\gamma}\int_{\partial\Delta} \delta_1 m \wedge \delta_2 \bar{m} + \delta_1 \bar{m} \wedge \delta_2 m\,. \quad (4.98)$$

We can perform an appropriate Lorentz transformation at the horizon in order to get a foliation of Δ spanned by m and $\bar{m}$, that is Lie dragged along l [8], that implies $\pounds_l m^a \hat{=} 0$. At the other hand, $\partial\Delta = S_{\Delta 1} \cup S_{\Delta 2}$, so it is sufficient to show that the integrand in (4.98) is Lie dragged along l. The variations in (4.98) are tangential to S_Δ, hence we have $\pounds_l \delta_1 m = \delta_1 \pounds_l m = 0$, so that the integrals over $S_{\Delta 1}$ and $S_{\Delta 2}$ are equal and $\int_\Delta J_{\rm HB} = 0$. So we can define a conserved pre-symplectic structure corresponding to the Holst term

$$\tilde{\Omega}_H(\delta_1,\delta_2) = \frac{1}{\kappa\gamma}\int_{\partial M} \delta_1 e^I \wedge \delta_2 e_I\,, \tag{4.99}$$

where the integration is performed over $\partial M = S_\infty \cup S_\Delta$. As shown in [25], the integral over S_∞ vanishes, due to asymptotic conditions, and the only contribution comes from S_Δ. Finally, we see that the quantity

$$\tilde{\Omega}_H(\delta_1,\delta_2) = \frac{1}{\kappa\gamma}\int_{S_\Delta} \delta_1 e^I \wedge \delta_2 e_I\,. \tag{4.100}$$

defines a conserved two-form. Note that this is precisely the symplectic structure for the Holst term defined in [45], though there the authors did not explicitly show that it is independent of M (this result depends on the details of the boundary conditions).

As we have seen in (2.119) the boundary terms in the action (that is, the topological terms) do not contribute to the symplectic current J, so that the only contributions in our case come from the Palatini action and possibly, as we have just seen, from the Holst term[1]. In order to illustrate how some possible inconsistencies arise when one postulates the existence of a symplectic structure for the topological terms, let us see, with some detail, what happens in the case of the Pontryagin term (as suggested, for instance, in [50]. Similar results follow for the other topological terms.) Recall that this term can be written as a total derivative, which means that we can either view it as a bulk term or as a boundary term. Considering the derivation of the symplectic structure in either case should render equivalent descriptions. Let us consider the variation of $S_{\rm Po}$, calculated from the LHS (bulk expression) in (4.14), is

$$\delta S_{\rm Po} = -2\int_{\mathcal{M}} DF^{IJ}\wedge\delta\omega_{IJ} + 2\int_{\partial\mathcal{M}} F^{IJ}\wedge\delta\omega_{IJ}\,, \tag{4.101}$$

so it does not contribute to the equations of motion in the bulk, due to the Bianchi identity $DF^{IJ} = 0$. In this case, the corresponding symplectic current is

$$J^{\rm bulk}_{\rm Po}(\delta_1,\delta_2) = 2(\delta_1 F^{IJ}\wedge\delta_2\omega_{IJ} - \delta_2 F^{IJ}\wedge\delta_1\omega_{IJ})\,. \tag{4.102}$$

On the other hand, if we calculate the variation of the Pontryagin term directly from the RHS (boundary expression) of (4.14), we obtain

$$\delta S_{\rm Po} = 2\int_{\partial\mathcal{M}} \delta L_{CS}\,. \tag{4.103}$$

The two expressions for $\delta S_{\rm Po}$ are, of course, identical since $F^{IJ}\wedge\delta\omega_{IJ} = \delta L_{CS} + \mathrm{d}(\omega^{IJ}\wedge\delta\omega_{IJ})$.

[1] Note that there have been some statements in the literature claiming that the topological terms do contribute to the symplectic structure when there are boundaries present [43; 50; 59].

The corresponding symplectic current in this case is

$$J_{\rm Po}^{\rm bound}(\delta_1,\delta_2) = 4\,\delta_{[2}\delta_{1]}L_{CS} = 0\,, \tag{4.104}$$

as we have obtained in the Sec. 2.4. So, at first sight it would seem that there is an ambiguity in the definition of the symplectic current that could lead to different symplectic structures. Since the relation between them is given by

$$J_{\rm Po}^{\rm bulk}(\delta_1,\delta_2) = J_{\rm Po}^{\rm bound}(\delta_1,\delta_2) + 4\,{\rm d}(\delta_2\omega^{IJ}\wedge\delta_1\omega_{IJ})\,, \tag{4.105}$$

it follows that $J_{\rm Po}^{\rm bulk}(\delta_1,\delta_2)$ is a total derivative, that does not contribute in (2.123), and from the systematic derivation of the symplectic structure described in Sec. 2.4, we have to conclude that it does not contribute to the symplectic structure. This is consistent with the fact that $J_{\rm Po}^{\rm bound}$ and $J_{\rm Po}^{\rm bulk}$ correspond to the same action. As we have remarked in Sec. 2.4, a total derivative term in J, under some circumstances, could be seen as generating a non-trivial symplectic structure $\tilde{\Omega}$ on the boundary of M. But the important thing to note here is that one would run into an inconsistency if one choose to introduce that non-trivial $\tilde{\Omega}$. Thus, consistency of the formalism requires that $\tilde{\Omega}=0$.

Let us now construct the conserved charges for this theory, and from the previous reasons we shall only consider the Palatini and Holst terms in this part. We shall consider the Hamiltonian H_ξ that is a conserved quantity corresponding to asymptotic symmetries and symmetries on the horizon of a spacetime. Our asymptotic conditions are chosen in such a way that the asymptotic symmetry group be the Poincaré group. The corresponding conserved quantities, for the well posed Palatini action, energy-momentum and relativistic angular momentum, are constructed in [7]. The contribution to the energy from a weakly isolated horizon has been analyzed in [8], where the first law of mechanics of non-rotating black holes was deduced. Rotating isolated horizons have been the topic of [5], where the contribution from the angular momentum of a horizon has been included. In this section we restrict our attention to energy and give a review of the principal results presented in [8].

Let us consider a case when ξ is the infinitesimal generator of asymptotic time translations of the spacetime. It induces time evolution on the covariant phase space, generated by a vector field $\delta_\xi := (\pounds_\xi e, \pounds_\xi \omega)$. At infinity ξ should approach a time-translation Killing vector field of the asymptotically flat spacetime. On the other hand, if we have a non-rotating horizon Δ, then ξ, at the horizon, should belong to the equivalence class $[l]$. In order that δ_ξ represents a phase space symmetry the condition $\pounds_{\delta_\xi}\bar{\Omega}=0$ should be satisfied. As we have seen in Sec. 2.4.2, δ_ξ is a Hamiltonian vector field iff the one-form

$$X_\xi(\delta) = \bar{\Omega}(\delta,\delta_\xi)\,, \tag{4.106}$$

is closed, and the Hamiltonian H_ξ is defined as

$$X_\xi(\delta) = \delta H_\xi \,. \tag{4.107}$$

In the presence of the isolated horizon, the symplectic structure (4.94) has two contributions, one from the Cauchy surface M and the other one from the two-sphere S_Δ. This second term does not appear in $\bar{\Omega}_P(\delta, \delta_\xi)$, it is equal to

$$-\frac{c}{\kappa}\int_{S_\Delta} {\delta_l}^2\varepsilon\,\delta\psi - \delta^2\varepsilon\,\delta_l\psi\,, \tag{4.108}$$

since $\xi = cl$ at Δ. We will show that this integral vanishes. When acting on fields $\delta_l = \pounds_l$, so that ${\delta_l}^2\varepsilon = {\pounds_l}^2\varepsilon \hat{=} 0$. On the other hand, as pointed out in [5], ψ is a potential, a given function of the basic variables. If we define $\delta_l\psi = \pounds_l\psi$, as in the case of our basic fields, then the boundary condition in (4.95) cannot be fulfilled. So we need to define $\delta_l\psi$ more carefully. Let ψ' denote a potential that corresponds to a null vector $l' = cl$ in the sense that $\pounds_{l'}\psi' = \kappa_{(l')}$. Since $\kappa_{(l')} = c\kappa_{(l)}$, it follows that $\pounds_l\psi' = \kappa_{(l)}$. We define $\delta_l\psi = \psi' - \psi$, then

$$\pounds_l(\delta_l\psi) = 0\,. \tag{4.109}$$

We ask ψ to be fixed at $S_{1\Delta}$, as a result $\delta_l\psi \hat{=} 0$. Then, it follows that the integral (4.108) vanishes and the only contribution to $\bar{\Omega}_P(\delta, \delta_\xi)$ comes from the integral over the Cauchy surface M in (4.94).

On the other hand, the symplectic structure for the Holst term $\tilde{\Omega}_H$ (4.100) is restricted to S_Δ, but it turns out that

$$\tilde{\Omega}_H(\delta, \delta_\xi) = \frac{c}{\kappa\gamma}\int_{S_\Delta} \delta m \wedge \pounds_l\bar{m} + \delta\bar{m}\wedge\pounds_l m = 0\,. \tag{4.110}$$

As a result $\delta H_\xi := \bar{\Omega}(\delta, \delta_\xi) = \bar{\Omega}_P(\delta, \delta_\xi)$ only has a contribution from the Palatini action

$$\delta H_\xi = -\frac{1}{2\kappa}\int_{\partial M} (\xi\cdot\omega^{IJ})\delta\Sigma_{IJ} - (\xi\cdot\Sigma_{IJ})\wedge\delta\omega^{IJ}\,, \tag{4.111}$$

where the integration is over the boundaries of the Cauchy surface M, the two-spheres S_∞ and S_Δ, since the integrand in $\bar{\Omega}_P(\delta, \delta_\xi)$ is a total derivative, as shown in [8].

The asymptotic symmetry group is the quotient of the group of spacetime diffeomorphisms which preserve the boundary conditions by its subgroup consisting of asymptotically identity diffeomorphisms. In our case this is the Poincaré group and its action generates canonical transformations on the covariant phase space whose generating function is H_ξ^∞. The situation is similar at the horizon Δ and infinitesimal diffeomorphisms need not be in the kernel of the symplectic structure unless they vanish on Δ and the horizon symmetry group is the quotient of the Lie group of all infinitesimal spacetime diffeomorphisms which preserve the horizon structure by its subgroup consisting of elements which are identity on the horizon [5].

The surface term at infinity in the expression (4.111) defines the gravitational energy at the asymptotic region, whose variation is given by

$$\delta E_\infty^\xi := -\frac{1}{2\kappa}\int_{S_\infty}(\xi\cdot\omega^{IJ})\delta\Sigma_{IJ} - (\xi\cdot\Sigma_{IJ})\wedge\delta\omega^{IJ} = \frac{1}{2\kappa}\int_{S_\infty}(\xi\cdot\Sigma_{IJ})\wedge\delta\omega^{IJ}\,, \tag{4.112}$$

since, due to the asymptotic behaviour of the tetrad and connection, the first term in the above expression vanishes. As shown in [7], after inserting the asymptotic form of the tetrad (4.34) and connection (4.37), this integral represents the variation of the ADM energy, $\delta E^\xi_{\mathrm{ADM}}$, associated with the asymptotic time-translation defined by ξ

$$E_\infty^\xi = E^\xi_{\mathrm{ADM}} = \frac{2}{\kappa}\int_{S_\infty}\sigma\,\mathrm{d}^2S_o\,, \tag{4.113}$$

where d^2S_o is the area element of the unit 2-sphere.

On the other hand, the surface term at the horizon in the expression (4.111) represents the horizon energy defined by the time translation ξ, whose variation is given by

$$\delta E_\Delta^\xi := \frac{1}{2\kappa}\int_{S_\Delta}(\xi\cdot\omega^{IJ})\delta\Sigma_{IJ} - (\xi\cdot\Sigma_{IJ})\wedge\delta\omega^{IJ} = \frac{1}{2\kappa}\int_{S_\Delta}(\xi\cdot\omega^{IJ})\delta\Sigma_{IJ}\,, \tag{4.114}$$

since the second term in the above expression vanishes at the horizon. The remaining term is of the form

$$\delta E_\Delta^\xi = \frac{1}{\kappa}\,\kappa_{(\xi)}\delta a_\Delta\,, \tag{4.115}$$

where a_Δ is the area of the horizon.

Now we see that the expression (4.111) encodes the first law of mechanics for non-rotating black holes, since it follows that

$$\delta H_\xi = \delta E^\xi_{\mathrm{ADM}} - \frac{1}{\kappa}\,\kappa_{(\xi)}\delta a_\Delta\,. \tag{4.116}$$

We see that the necessary condition for the existence of H_ξ is that surface gravity, $\kappa_{(\xi)}$, be a function only of a horizon area a_Δ. In that case

$$H_\xi = E^\xi_{\mathrm{ADM}} - E^\xi_\Delta\,. \tag{4.117}$$

In the following section we want to calculate the Noether charge that corresponds to time translation for every term of the action (4.28). We have just seen that δH_ξ is an integral over a Cauchy surface of the symplectic current $J(\delta,\delta_\xi)$. In section 2.5.3 we displayed the relation between the symplectic and Noether currents, given in (2.146), and using the definition of Noether charge Q_ξ (2.139), we obtain the following relation

$$\delta H_\xi = \int_M J(\delta, \delta_\xi) = \int_{\partial M} \delta Q_\xi - \xi \cdot \theta(\delta)\,. \tag{4.118}$$

There are two contributions to the above expression, one at S_∞ and the other one at S_Δ. As before, δE^ξ_∞, is the integral at the RHS of (4.118) calculated over S_∞, and δE^ξ_Δ the same integral calculated over S_Δ. Note that the necessary and sufficient condition for the existence of H_ξ is the existence of the form B such that

$$\int_{\partial M} \xi \cdot \theta(\delta) = \delta \int_{\partial M} \xi \cdot B\,. \tag{4.119}$$

Let us now consider how the different terms appearing in the action contribute to the Noether charges.

4.3.2 Noether charges

In this part we consider the Noether charges that appear as conserved quantities associated to diffeomorphisms generated by vector fields ξ. There are two parts. In the first one we consider in detail the consistent Palatini action with a boundary term, and compare it to the case without a boundary term. In the second part we consider the Holst and topological terms.

4.3.2.1 Palatini action

Let us start by considering the case of Palatini action with boundary term. We have seen in the section 4.3.1 that the symplectic potential current in this case is given by

$$\theta_{\rm PB}(\delta) = \frac{1}{2\kappa}\,\delta\Sigma^{IJ} \wedge \omega_{IJ}\,. \tag{4.120}$$

In order to calculate the Noether current 3-form (2.138), $J_N(\delta_\xi) = \theta(\delta_\xi) - \xi \cdot \mathbf{L}$, we need the following two expressions

$$\theta_{\rm PB}(\delta_\xi) = \frac{1}{2\kappa}\pounds_\xi \Sigma^{IJ} \wedge \omega_{IJ} = \frac{1}{2\kappa}[\mathrm{d}(\xi \cdot \Sigma^{IJ}) + \xi \cdot \mathrm{d}\Sigma^{IJ}] \wedge \omega_{IJ}\,, \tag{4.121}$$

and

$$\xi \cdot \mathbf{L}_{\rm PB} = -\frac{1}{2\kappa}[\xi \cdot (\Sigma^{IJ} \wedge F_{IJ}) - \xi \cdot \mathrm{d}(\Sigma^{IJ} \wedge \omega_{IJ})]\,. \tag{4.122}$$

From these expressions we obtain the following result for the Noether current 3-form

$$J_{\rm NPB}(\delta_\xi) = \frac{1}{2\kappa}\{(\xi \cdot \Sigma^{IJ}) \wedge F_{IJ} + (\xi \cdot \omega_{IJ})\mathrm{D}\Sigma^{IJ} + \mathrm{d}[(\xi \cdot \Sigma^{IJ}) \wedge \omega_{IJ}]\}\,. \tag{4.123}$$

We see that on-shell ($e^I \wedge F_{IJ} = 0$ and $\mathrm{D}\Sigma^{IJ} = 0$), we have $J_N(\delta_\xi) = \mathrm{d}Q_\xi$, where the corresponding Noether charge is given by

$$Q_{\xi \mathrm{PB}} = \frac{1}{2\kappa}(\xi \cdot \Sigma^{IJ}) \wedge \omega_{IJ} \,. \tag{4.124}$$

We shall show that the contribution of the second term in (4.118) over S_∞ vanishes. Namely,

$$\int_{S_\infty} \xi \cdot \theta_{\mathrm{PB}}(\delta) = \frac{1}{2\kappa} \int_{S_\infty} \xi \cdot (\delta\Sigma^{IJ} \wedge \omega_{IJ}) = 0 \,, \tag{4.125}$$

since $\delta\Sigma = O(\rho^{-1})$, $\omega = O(\rho^{-2})$ and the volume element goes as ρ^2. It follows that $B = 0$ on S_∞ and the Hamiltonian at infinity exists. The remaining term at infinity in (4.118) is

$$\delta \int_{S_\infty} Q_{\xi \mathrm{PB}} = \frac{1}{2\kappa} \delta \int_{S_\infty} (\xi \cdot \Sigma^{IJ}) \wedge \omega_{IJ} \,, \tag{4.126}$$

and since $\int_{S_\infty} \delta(\xi \cdot \Sigma^{IJ}) \wedge \omega_{IJ} = 0$, due to asymptotic behaviour of the fields, the above expression is equal to δE_∞^ξ given in (4.112), so in this case

$$E^\xi_{\mathrm{ADM}} = \int_{S_\infty} Q_{\xi \mathrm{PB}} \,, \tag{4.127}$$

up to an additive constant that we choose to be zero. Note that a similar result is obtained in the second order formalism for the Einstein-Hilbert action with the Gibbons-Hawking term, as shown in [58].

On the other hand, at the horizon the situation is different. In fact,

$$\int_{S_\Delta} Q_{\xi \mathrm{PB}} = \frac{1}{2\kappa} \int_{S_\Delta} (\xi \cdot \Sigma^{IJ}) \wedge \omega_{IJ} = 0 \,, \tag{4.128}$$

because $\xi^a = cl^a$ on the horizon, and due to the expressions for Σ (4.73) and ω (4.74) at the horizon. Then, δE_Δ is determined by the remaining term

$$\int_{S_\Delta} \xi \cdot \theta_{\mathrm{PB}}(\delta) = \frac{1}{2\kappa} \int_{S_\Delta} [(\xi \cdot \delta\Sigma^{IJ}) \wedge \omega_{IJ} + \delta\Sigma^{IJ}(\xi \cdot \omega_{IJ})] \,. \tag{4.129}$$

The first term vanishes since on Δ we have $(\xi \cdot \delta\Sigma^{IJ}) \wedge \omega_{IJ} \hat{=} 2c(l \cdot {}^2\varepsilon) \wedge \omega \hat{=} 0$, because $l \cdot {}^2\varepsilon \hat{=} 0$. We are then left with the expression given in (4.114), and the necessary condition for the existence of E_Δ is that the surface gravity $\kappa_{(\xi)}$ depends only on the area of the horizon [8]. It follows also that there exists a form B such that (4.119) is satisfied.

We see that in this case

$$\delta E_\infty = \delta \int_{S_\infty} Q_{\xi \mathrm{PB}} \,, \qquad \delta E_\Delta = \int_{S_\Delta} \xi \cdot \theta_{\mathrm{PB}}(\delta) \,. \tag{4.130}$$

In globally stationary spacetimes, $\pounds_\xi e = \pounds_\xi \omega = 0$, so that $\delta H_\xi = \bar{\Omega}(\delta, \delta_\xi) = 0$, and from the first law (4.116) it follows $\delta E_\infty = \delta E_\Delta$. For Palatini action with boundary term this implies that

$$\delta \int_{S_\infty} Q_{\xi \mathrm{PB}} = \int_{S_\Delta} \xi \cdot \theta_{\mathrm{PB}}(\delta)\,. \tag{4.131}$$

This result depends on the particular form of the action, and it is sensitive to the presence of boundary terms.

Let us briefly comment the case of Palatini action *without* boundary term. We know that this action is not well defined, its symplectic potential $\Theta_{\mathrm{P}}(\delta)$ diverges, but we can *formally* calculate its Noether charge and compare it to the previous example. As we showed in the sections 2.5.1 and 2.5.3, the addition of the total derivative to the action changes its Noether charge (2.142), but leaves the symplectic structure (2.119) unaltered. In the previous example the ADM energy was determined completely by the integral of the Noether charge over the two-sphere at infinity. Now, the situation is different and both terms in (4.118) contribute to δE_∞. We first note that

$$\theta_{\mathrm{P}}(\delta) = -\frac{1}{2\kappa}\Sigma^{IJ} \wedge \delta \omega_{IJ}\,, \quad Q_{\xi \mathrm{P}} = -\frac{1}{2\kappa}\Sigma^{IJ}(\xi \cdot \omega_{IJ})\,, \tag{4.132}$$

where θ_{P} and $Q_{\xi \mathrm{P}}$ denote the corresponding quantities for Palatini action (without boundary term). It turns out that

$$\int_{S_\infty} \xi \cdot \theta_{\mathrm{P}}(\delta) = -\frac{1}{2\kappa}\,\delta \int_{S_\infty} \xi \cdot (\Sigma^{IJ} \wedge \omega_{IJ})\,, \tag{4.133}$$

since $\int_{S_\infty} \xi \cdot (\delta \Sigma^{IJ} \wedge \omega_{IJ}) = 0$ due to our asymptotic conditions. On the other hand,

$$\int_{S_\infty} \delta Q_{\xi \mathrm{P}} = -\frac{1}{2\kappa}\,\delta \int_{S_\infty} (\xi \cdot \omega_{IJ})\Sigma^{IJ} = -\frac{1}{2\kappa}\int_{S_\infty} \delta(\xi \cdot \omega_{IJ})\Sigma^{IJ}\,, \tag{4.134}$$

and the combination of the above expressions, as in (4.118) gives the previous expression for δE_∞^ξ (4.112). Thus, we see that the Hamiltonian generator at infinity is not given by the integral of the Noether charge, as in the case of the Palatini action with boundary term.

At the horizon both terms contribute, again. The results are

$$\int_{S_\Delta} \xi \cdot \theta_{\mathrm{P}}(\delta) = -\frac{1}{\kappa}\,\delta \kappa_{(\xi)} a_\Delta\,, \tag{4.135}$$

where we used the fact that $l \cdot {}^2\varepsilon = 0$ and $\xi \cdot \delta \omega = \delta(\xi \cdot \omega) = \delta \kappa_{(\xi)}$. We see again, that in order to satisfy the condition (4.119), $\kappa_{(\xi)}$ should be a function of a_Δ only. We also obtain

$$\delta \int_{S_\Delta} Q_{\xi \mathrm{P}} = \frac{1}{\kappa}\,\delta(\kappa_{(\xi)} a_\Delta)\,, \tag{4.136}$$

and the combination of the above expressions, as in (4.118) gives the previous result for δE_{Δ}^{ξ} (4.114).

Finally, let us compare these results for the Noether charge with the results of [36], and to that end we shall recall one of the principal results in [36], referring to the variations of a stationary black hole solution, that states that in diffeomorphism invariant theories, in the second order formalism, the Noether charge relative to a bifurcate Killing horizon Σ_0 is proportional to the entropy of a black hole S. The result is the following

$$\delta \int_{\Sigma_0} Q_{\xi_0} = \frac{\kappa_{(\xi_0)}}{2\pi} \delta S\,, \tag{4.137}$$

where ξ_0 is the Killing field that vanishes on Σ_0 and at infinity tends to a stationary time-like Killing vector field with unit norm and $\kappa_{(\xi_0)}$ is the corresponding surface gravity of a stationary black hole. In the proof of this result it is assumed that $\delta\kappa_{(\xi_0)} = 0$. Furthermore, it has been shown that in the case of stationary variations the integral is independent of the choice of horizon cross-section. Our analysis, based on the IH formalism [7] and [8], is different in various aspects: (1) we consider the first order formalism; (2) in our case the existence of the internal boundary is consistently treated, as, for example, in the expression for δH that involves integration over the whole boundary, not only over the asymptotic region, as in [36]; (3) our results are valid also for nonstationary configurations, and; (4) in our approach the integration is performed over an arbitrary 2-sphere cross section of a weakly isolated horizon, and not restricted to a preferred bifurcation surface.

Taking this into account let us now see whether, in our approach, the Noether charge can be related to the black hole energy (or entropy). We already know that in general this is not the case, since neither the Holst term nor the topological terms contribute to the energy of the black hole, though they modify the Noether charge.

We can formally compare the expression (4.137) with our result (4.136), taking into account all differences between the two approaches. We see that, if we impose that $\delta\kappa_{(\xi)} = 0$, then the result in (4.136) would look like (4.137). But this restriction is not consistent with the result of [8] that shows that, as we saw in the previous subsection, the surface gravity is a function of the area of the horizon, and that this is a necessary condition to have a well defined Hamiltonian. As we have seen in this subsection, in neither of the cases, namely Palatini action with or without surface term, is the variation of a corresponding Noether charge relative to an isolated horizon proportional to $(\kappa_{(\xi)}\delta a_{\Delta})$.[1] Note that this fact poses a challenge to the generality of the result relating Noether charge and energy (or entropy) derived in [36].

[1]nor to δa_{Δ}, for that matter.

4.3.2.2 Holst and topological terms

To end this section, let us calculate the Noether charges for the Holst term and the topological terms. We shall see that in all of these cases the integrals of the corresponding Noether charge 2-form over S_∞ vanish. For S_Δ, there is one case where the charge is non vanishing. Let us first consider the Holst term with its boundary term $S_{\rm HB}$, given by (4.9). We know that this term does not contribute to the energy. As we have seen in section 4.1.2, the symplectic potential current of $S_{\rm HB}$ is given by

$$\theta_{\rm HB}(\delta) = \frac{1}{2\kappa\gamma}\delta\Sigma^{IJ}\wedge\star\omega_{IJ} = \frac{1}{\kappa\gamma}\delta e^I\wedge {\rm d}e_I\,, \tag{4.138}$$

where in the second line we used the equation of motion ${\rm D}e^I = 0$. The corresponding Noether charge 2-form is given by

$$Q_{\xi{\rm HB}} = \frac{1}{\kappa\gamma}(\xi\cdot\Sigma^{IJ})\wedge\star\omega_{IJ} = \frac{1}{\kappa\gamma}(\xi\cdot e^I)\,{\rm d}e_I\,. \tag{4.139}$$

Now, one can show that

$$\int_{S_\infty} Q_{\xi{\rm HB}} = 0\,. \tag{4.140}$$

Namely

$$\int_{S_\infty}(\xi\cdot e^I)\,{\rm d}e_I = \int_{S_\infty}(\xi\cdot {}^oe^I)\,{\rm d}\Big(\frac{{}^1e_I}{\rho}\Big) = \int_{S_\infty}{\rm d}\Big[(\xi\cdot {}^oe^I)\frac{{}^1e_I}{\rho}\Big] = 0\,, \tag{4.141}$$

since ξ is constant on S_∞ and ${\rm d}\,{}^oe^I = 0$. On the other hand, it is also easy to show that

$$\int_{S_\Delta} Q_{\xi{\rm HB}} = 0\,. \tag{4.142}$$

since

$$\int_{S_\Delta}(\xi\cdot e^I)\,e^J\wedge\omega_{IJ} = \int_{S_\Delta} cl^I(e^J\wedge\omega_{IJ}) = 0\,, \tag{4.143}$$

due to the expressions for the tetrad (4.73) and connection (4.74) on the horizon.

The variation of $S_{\rm Po}$, calculated from the LHS expression in (4.14), is

$$\delta S_{\rm Po} = -2\int_{\mathcal M} DF^{IJ}\wedge\delta\omega_{IJ} + 2\int_{\partial\mathcal M} F^{IJ}\wedge\delta\omega_{IJ}\,, \tag{4.144}$$

so it does not contribute to the equations of motion in the bulk, due to the Bianchi identity $DF^{IJ} = 0$, and additionally the surface integral in (4.144) should vanish for the variational principle to be well defined. We will show later that this is indeed the case for boundary conditions of interest to us, namely asymptotically flat spacetimes possibly with an isolated horizon.

The symplectic potential current and the corresponding Noether charge 2-form for the Pontryagin term $S_{\rm Po}$, calculated from the LHS expression in (4.14) is

$$\theta_{\rm Po}(\delta) = 2F^{IJ}\wedge\delta\omega_{IJ}\,, \quad Q_{\xi\,\rm Po} = 2(\xi\cdot\omega_{IJ})F^{IJ}\,. \tag{4.145}$$

We will show that the integrals of the Noether charge 2-form $Q_{\xi\,\rm Po}$ over S_∞ and S_Δ vanish. For the first one we have

$$\int_{S_\infty} Q_{\xi\,\rm Po} = 2\int_{S_\infty}(\xi\cdot\omega_{IJ})F^{IJ} = 0\,. \tag{4.146}$$

since $\omega_{IJ} = O(\rho^{-2})$, $F^{IJ} = O(\rho^{-3})$ and the volume element goes as ρ^2.

Since the pull-back of the connection on S_Δ is given by (**??**), we obtain that the integral of the Noether 2-form over S_Δ is

$$\int_{S_\Delta} Q_{\xi\,\rm Po} = 4ic\int_{S_\Delta}(l\cdot V)\mathcal{R}^2\varepsilon = 0\,, \tag{4.147}$$

where we have used the form of the connection on the horizon given by (4.74), and since we can use the remaining gauge transformation to fix $l\cdot V \hat{=} 0$, as shown in [27].

At the other hand, we can calculate the symplectic potential current and the Noether charge 2-form from the RHS of (4.14), and obtain $\tilde{\theta}_{\rm Po}(\delta) = \theta_{\rm Po}(\delta) - 2{\rm d}(\omega^{IJ}\wedge\delta\omega_{IJ})$ and, as we have seen in (2.142), this produces a following change in the Noether charge 2-form

$$\tilde{Q}_{\xi\,\rm Po} = Q_{\xi\,\rm Po} - 2\omega^{IJ}\wedge\pounds_\xi\,\omega_{IJ}\,. \tag{4.148}$$

It is easy to see that the integrals of the last term in the above equation over S_∞ and S_Δ vanish, due to our boundary conditions, hence the Noether charges remain invariant.

Similarly, for the Euler term, from the variation of the LHS of (4.20), we obtain

$$\theta_{\rm E}(\delta) = 2\star F^{IJ}\wedge\delta\omega_{IJ}\,, \quad Q_{\xi\,\rm E} = 2(\xi\cdot\omega_{IJ})\star F^{IJ}\,. \tag{4.149}$$

Then, as in case of the Pontryagin term it is easy to see that

$$\int_{S_\infty} Q_{\xi\,\rm E} = 2\int_{S_\infty}(\xi\cdot\omega_{IJ})\star F^{IJ} = 0\,, \tag{4.150}$$

due to the asymptotic behaviour of the fields.

At the horizon the situation is different since the dual of the pull-back (**??**) is given by

$$\star F^{IJ}|_{S_\Delta} = -2\mathcal{R}^2\varepsilon\, l^{[I}n^{J]}\,, \tag{4.151}$$

the corresponding Noether charge is non vanishing

$$\int_{S_\Delta} Q_{\xi\,\rm E} = 4c\int_{S_\Delta}(l\cdot\omega)\,\mathcal{R}^2\varepsilon = 16\pi c\,\kappa_{(l)} \tag{4.152}$$

since $l\cdot\omega = \kappa_{(l)}$ is constant on the horizon and the remaining integral is a topological invariant.

This result is consistent with the expression for the entropy of the Euler term in [37], obtained in the second order formalism for stationary black holes. Though the Noether charge of the Euler term over a WIH is non-zero, the corresponding contribution to the Hamiltonian energy is nonetheless vanishing. As we have seen previously, in Section 4.3.1, the variation of the energy at the horizon is

$$\delta H^{\xi}_{\Delta} = \int_{S_\Delta} \delta Q_{\xi} - \xi \cdot \theta(\delta)\,, \tag{4.153}$$

with $\xi = cl$. For the Euler term we obtain

$$\int_{S_\Delta} cl \cdot \theta_{\mathrm{E}}(\delta) = 4c \int_{S_\Delta} l \cdot ({}^2\varepsilon \wedge \delta\omega)\,\mathcal{R} = 16\pi c\,\delta\kappa_{(l)}\,, \tag{4.154}$$

since $l \cdot {}^2\varepsilon \hat{=} 0$ and $l \cdot \delta\omega = \delta\kappa_{(l)}$. We see that this term cancels the variation of (4.152) in the expression for the energy at the horizon.

Similarly as for the Pontryagin term, the variation of the RHS of (4.20), leads to a change in the symplectic potential current and the Noether charge 2-form, but the Noether charges stay invariant.

Finally, we have seen in Section 4.1.3 that the variation of the Neih-Yan term on shell is proportional to the variation of the Holst term, so all the results for the Noether charge of the Holst term apply directly here. Namely, for the Neih-Yan term, with its boundary term, given in (4.26), we obtain that its Noether charge 2-form is

$$Q_{\xi \mathrm{NYB}} = 2\kappa\gamma Q_{\xi \mathrm{HB}}\,, \tag{4.155}$$

so that its integrals over S_∞ and S_Δ vanish as well.

Let us end this section with a remark. One should note that the Noether charges at infinity of all the topological terms vanish for asymptotically flat boundary conditions, but this is not the case for locally asymptotically anti-de Sitter (AAdS) space-times. In [3] and [2], AAdS asymptotic conditions are considered and the Noether charge at infinity of the Palatini action with negative cosmological constant term turns out to be divergent. In that case the Euler term is added in order to make the action well defined and finite. With this modification, the non vanishing (infinite) Noether charge becomes finite for the well defined action. This illustrates that, in several respects, asymptotic AdS and asymptotically flat gravity behave in qualitatively different manners.

4.4 Discussion and remarks

Let us start by summarizing the main results that we have here presented.

1. We have analyzed whether the most general first order action for general relativity in four dimensions has a well posed variational principle in spacetimes with boundaries. We showed

that it is necessary to introduce additional boundary terms in order to have a differentiable action, which is finite for the field configurations that satisfy our boundary conditions: asymptotically flat spacetimes with an isolated horizon as an internal boundary.

2. We discussed the impact of the topological terms and boundary terms needed to have a well defined variational principle for any well posed field theory, on the symplectic structure and the conserved Hamiltonian and Noether charges of the theory. We showed, in particular, that for generic theories, no boundary term can modify the symplectic structure.

3. In the case of first order gravity, we showed that the topological terms do not modify the symplectic structure. In the case of the Holst term (that is *not* topological), there is a particular instance in which it could modify the symplectic structure. Thus, the Hamiltonian structure of the theory remains unaffected by the introduction of boundary and topological terms. In particular, all Hamiltonian conserved quantities, that are generators of asymptotic symmetries, remain unaffected by such terms. We have also shown that for our boundary conditions the contribution from the Holst term to the Hamiltonian charges is always trivial. It is important to note that this simple result proves incorrect several assertions that have repeatedly appeared in the literature.

4. We have shown that even when the Hamiltonian conserved charges remain insensitive to the addition of boundary and topological terms, the corresponding Noetherian charges *do* depend on such choices. This has as a consequence that the identification of Noether charges with, say, energy depends on the details of the boundary terms one has added. For instance, if one focuses on the asymptotic region, then it is only for the well defined Palatini with boundary action (of [7]) that the Noether charge coincides with the Hamiltonian (ADM) energy. Any other choice, including Palatini without a boundary term, would yield a different conserved quantity. Furthermore, if one only had an internal boundary (and no asymptotic region), several possibilities for the action are consistent (compare [8] and [27]), and the relation between energy and Noether charge depends on such choices. We have also made some comments regarding the relation between our analysis and others based on Noether charges for stationary spacetimes [36].

In this chapter, our focus was on first order gravity, but our analysis can be taken over to more general diffeomorphism invariant theories. Our results indicate that there is an interesting interplay between symmetries and conserved quantities that depends on the formalism used; Hamiltonian and Noether charges that have very different interpretations within the theory, in general do not coincide. As we have seen, for the boundary conditions we considered most of the Noether charges associated to topological terms vanished – while the Noether currents were non-vanishing–, but for generic boundary conditions this might not be the case (such as in AADS asymptotics, for instance), indicating that generically these two sets of charges do not coincide. A deeper understanding of this issue is certainly called for.

Our analysis was done using the covariant Hamiltonian formalism, that has proved to be economical and powerful to unravel the Hamiltonian structure of classical gauge field theories. It should be interesting to see whether a parallel analysis, using a 3+1 decomposition of spacetime and taking special care on the effects of boundaries, yields similar results. This work is in progress.

Chapter 5

Conclusions

In this thesis we analyzed first order gravity in spacetimes with boundaries. We begin by defining what it means to have a well posed action principle, that is finite and differentiable, and discussing what is the effect of adding additional boundary terms to the action. Then we discuss the covariant hamiltonian formalism taking enough care when boundaries are present. In particular we found that when we add a boundary term to the original action it will not change the symplectic current nor the symplectic structure. Also within the covariant formalism we study symmetries and conserved charges; in particular we discuss the relation between the Hamiltonian and Noether charges, while the former is insensible to the addition of boundary terms, the latter depends on the boundary terms. So far our results are generic and can be applied to a variety of theories. In particular, we use these results to analyze first order gravity in three and four dimensions in spacetimes with boundaries.

In the three dimensional case we propose a manifestly Lorentz invariant well posed Palatini action. We find the fall-off conditions for the first order variables and with these conditions we prove that in fact the action is well posed, that is finite and differentiable. Moreover we follow the covariant formalism and we find an expression for the energy, this expression coincide with that obtained by Regge Teitelboim methods in the second order formulation in the metric variables, that is the expression for the energy is determined up to a constant. Then taking two different $2+1$ decompositions we find an expression for the energy, this is completely determined by the canonical hamiltonian. This results coincide with those in the second order formalism, but this time beginning with the Einstein-Hilbert action with Gibbons-Hawking term that is well posed under asymptotically flat boundary conditions, and the expression for the energy is given by the canonical hamiltonian. This also proves that the proposed action is the one corresponding to the Einstein-Hilbert with Gibbons-Hawking. From this we can see that even though the covariant formalism is powerful and economical, it only determines the conserved quantities up to a constant. In the case of the energy this constant may shift the region in which the energy is bounded and pass from positive to negative values for the energy. Contrary, the hamiltonian method determines completely the value of the energy.

In the four dimensional case we have analyzed whether the most general first order action for general relativity in four dimensions has a well posed variational principle in spacetimes with boundaries. We showed that it is necessary to introduce additional boundary terms in order to have a differentiable action, which is finite for the field configurations that satisfy our boundary conditions: asymptotically flat spacetimes with an isolated horizon as an internal boundary. We discussed the impact of the topological terms and boundary terms added in order to have a well defined variational principle, to the symplectic structure and the conserved Hamiltonian and Noether charges of the theory. We showed that the topological terms do not modify the symplectic structure. In the case of the Holst term (that is *not* topological), there is a particular instance in which it could modify the symplectic structure. We have also shown that for our boundary conditions the contribution from the Holst term to the Hamiltonian charges is always trivial. Thus, the Hamiltonian structure of the theory remains unaffected by the introduction of boundary and topological terms. It is important to note that this result proves incorrect several assertions that have repeatedly appeared in the literature. We have shown that even when the Hamiltonian conserved charges remain insensitive to the addition of boundary and topological terms, the corresponding Noetherian charges *do* depend on such choices. This has as a consequence that the identification of Noether charges with, say, energy depends on the details of the boundary terms one has added. For instance, if one only had an internal boundary (and no asymptotic region), several possibilities for the action are consistent, and the relation between energy and Noether charge depends on such choices. We have also seen that one particular topological terms, namely the Euler term, contributes non-trivially to the Noether charge at the horizon. Our analysis was done using the covariant Hamiltonian formalism, that has proved to be economical and powerful to unravel the Hamiltonian structure of classical gauge field theories. It should be interesting to perform a similar analysis using a 3+1 decomposition of spacetime and taking special care on the effects of boundaries, extending the analysis of [38; 39], where the boundaries were ignored. A natural question is whether the two Hamiltonian approaches to treat this system are equivalent and, if not, understand the underlying reasons for that discrepancy. This work is in progress and will be reported elsewhere.

Chapter 6

Perspectives and forthcoming work

"Never make a calculation until you know the answer: make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every puzzle. Courage: no one else needs to know what the guess is. Therefore make it quickly, by instinct. A right guess reinforces this instinct. A wrong guess brings the refreshment of surprise. In either case life as a spacetime expert, however long, is more fun!"
—Wheeler, John A. and Edwin F. Taylor. Spacetime Physics, Freeman, 1966. Page 60.

There is almost an endless chain of questions, but following Wheeler's counselling, there are some few immediate questions I want to address:

- Inspired by the success of the three dimensional manifestly Lorentz invariant well posed Palatini action. We want to extend this results and see how this works in the four dimensional case and compare it with the already known results [7]. Using both the covariant and canonical formalisms, we want to find the ADM energy.

- We can think of adding matter to the four dimensional Palatini action and see what happens if we consider spacetimes that include a boundary at infinity and/or an internal boundary satisfying isolated horizons boundary conditions. In particular if we add Skyrme field the fall-off conditions at infinity are quasi-asymptotically flat [20], so instead of the ADM mass we expect to recover the Sudarsky-Nucamendi mass [53]. We want to extend the results also for isolated horizons boundary conditions. (This work is in collaboration with Alejandro Corichi and Ulises Nucamendi).

- Also I have a proposal for a $n-$dimensional manifestly Lorentz invariant Palatini action, that can be seen as an $n-$dimensional BF theory with boundary. I want to see whether this action will be well posed under asymptotically flat boundary conditions [10].

- We want to understand the effects of the addition of boundary terms to a well posed action, in the context of a complete or pure hamiltonian analysis. How the canonical pair corresponding to the original action relates with the canonical pair of the action with the added boundary terms. How this boundary terms affects the structure of constraints. We want to begin with a manageable example to begin with: Maxwell plus Pontryagin and/or BF plus Pontryagin. The goal is to fully understand the action 1.1 in this canonical context.

I would love to extend many of this results to isolated horizons, asymptotically AdS, and null infinity contexts. And the rest is history...

Appendix A

Tensor densities and the volume element

Although tensors have such an irresistible beauty, sometimes is necessary to consider non-tensorial objects. Take for example the *Levi-Civita symbol* defined as,

$$\varepsilon_{a_1 a_2 \cdots a_n} = \begin{cases} +1 & \text{if } a_1 a_2 \cdots a_n \text{ is an even permutation of } 01\cdots(n-1), \\ -1 & \text{if } a_1 a_2 \cdots a_n \text{ is an odd permutation of } 01\cdots(n-1), \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

This definition is for any coordinate system (right-handed coordinate system, otherwise we have an overall minus sign). That is what we call it symbol, because it is not a tensor and it is defined not to change under coordinate transformations, so we use it as a *constant*. We are only able to treat it as a tensor in flat inertial reference frames since the Lorentz transformations will leave it unchanged. If we want to allow its components to change in a 'nice' way under any coordinate transformations in an arbitrary geometry we have to extend our notion of tensor. We shall define a *tensor density* $\pi^{a_1 a_2 \cdots a_n}{}_{b_1 b_2 \cdots b_n}$ of *weight* $n \in \mathbb{R}$ as an object on a differentiable manifold whose components transform under changes of coordinates $x^a \mapsto x'^{a'}$ by

$$\pi'^{a'_1 a'_2 \cdots a'_n}{}_{b'_1 b'_2 \cdots b'_n} = \left| \det\left(\frac{\partial x^c}{\partial x'^{c'}}\right) \right|^n \pi^{a_1 a_2 \cdots a_n}{}_{b_1 b_2 \cdots b_n} \frac{\partial x'^{a'_1}}{\partial x^{a_1}} \cdots \frac{\partial x'^{a'_k}}{\partial x^{a_k}} \frac{\partial x^{b_1}}{\partial x'^{c'_1}} \cdots \frac{\partial x^{b_l}}{\partial x'^{c'_l}} \tag{A.2}$$

Also we can extend the covariant derivative for tensor to tensor densities,

$$\begin{aligned} \nabla_a \pi^{a_1 a_2 \cdots a_n}{}_{b_1 b_2 \cdots b_n} &= |\det g|^{n/2} \nabla_a \left(|\det g|^{n/2} \pi^{a_1 a_2 \cdots a_n}{}_{b_1 b_2 \cdots b_n} \right) \\ &= \partial_a \pi^{a_1 a_2 \cdots a_n}{}_{b_1 b_2 \cdots b_n} + \Gamma^{a_1}_{ac} \pi^{c a_2 \cdots a_n}{}_{b_1 b_2 \cdots b_n} + \cdots + \Gamma^{a_n}_{ac} \pi^{a_1 a_2 \cdots a_{n-1} c}{}_{b_1 b_2 \cdots b_n} \\ &\quad - \Gamma^c_{a b_1} \pi^{a_1 a_2 \cdots a_n}{}_{c b_2 \cdots b_n} - \cdots - \Gamma^c_{a b_n} \pi^{a_1 a_2 \cdots a_n}{}_{b_1 b_2 \cdots c} \\ &\quad - n \Gamma^b_{ba} \pi^{a_1 a_2 \cdots a_n}{}_{b_1 b_2 \cdots b_n} \end{aligned} \tag{A.3}$$

for a tensor density $\pi^{a_1a_2\cdots a_n}{}_{b_1b_2\cdots b_n}$ of weight n. The last term arises from $\partial_a \log \mathrm{set} g = 2\Gamma^b_{ba}$. In a similar fashion we can extend this results to the Lie derivative. See e.g. [19; 22] for further explanation and derivations.

In physics tensor densities arises naturally in canonical formulations of field theories where the space-time metric is considered one of the physical fields. To make a Legendre transform, a Liouville term of the form $\int \dot{\phi} p_\phi d^3x$ is needed, but on the other hand we need this term to include a measure factor to have coordinate independent integrations. We can not insert $\sqrt{|g|}$ because in that case ϕ and p_ϕ will not longer be a canonical pair. This is fixed by requiring the momenta be a density of weight one. As we can see in a particular example in 3.4.

A.1 Some properties

- Taking $\tilde{\varepsilon}$ as the Levi-Civita tensor density and ε the Levi-Civita permutation symbol,

$$\tilde{\varepsilon}_{a_1a_2\cdots a_n} = \frac{1}{\sqrt{|g|}}\varepsilon_{a_1a_2\cdots a_n} \tag{A.4}$$

$$\tilde{\varepsilon}^{a_1a_2\cdots a_n} = \sqrt{|g|}\varepsilon^{a_1a_2\cdots a_n} \tag{A.5}$$

- When we have internal indices additionally to the space-time indices we have the following relations,

  - Takinking $\varepsilon_{I_1,\cdots,I_n}$ as the Levi-Civita permutation symbol on the internal indices, $\tilde{\varepsilon}^{b_1b_2a_3\ldots a_n}$ the Levi-Civita tensor density, and $({}^n e) = \sqrt{|g|}$ with $|g|$ the $n-$dimensional metric,

$$\varepsilon_{I_1,\cdots,I_n} e^{I_1}_{a_1}\cdots e^{I_n}_{a_n} = ({}^n e)\tilde{\varepsilon}_{a_1,\cdots,a_n} \tag{A.6}$$

  - From the previous equation by contracting both sides with two $e^{a_i}_{J_i}$ we have,

$$\frac{1}{(n-2)!}\varepsilon_{J_1J_2I_3\ldots I_n}\tilde{\varepsilon}^{b_1b_2a_3\ldots a_n} e^{I_3}_{a_3}\cdots e^{I_n}_{a_n} = 2({}^n e) e^{[b_1}_{J_1} e^{b_2]}_{J_2} \tag{A.7}$$

    In three and four dimensions this expression becomes

$$\varepsilon_{IJK}\tilde{\varepsilon}^{abc} e^K_c = 2\,({}^3 e) e^{[a}_I e^{b]}_J \tag{A.8}$$

    and

$$\varepsilon_{IJKL}\tilde{\varepsilon}^{abcd} e^K_c e^L_d = 4\,({}^4 e) e^{[a}_I e^{b]}_J \tag{A.9}$$

    respectively.

Appendix B

Appendix: On the derivation of the spin connection

From the compatibility condition of the connection with the tetrad, which makes the Lorentz connection a spin connection,

$$\mathrm{d}e^I + \omega^I{}_J \wedge e^J = 0 \tag{B.1}$$

we have

$$\partial_a e^I_b - \partial_a e^J_b + \omega^I_{a\,J} e^J_b - \omega^I_{b\,J} e^J_b = 0 \tag{B.2}$$

or equivalently

$$\omega^I_{a\,J} e^J_b - \omega^I_{b\,J} e^J_b = -2\partial_{[a} e^I_{b]}. \tag{B.3}$$

Multiplying by e_{cI} and permitting spaciotemporal indices

$$\begin{aligned}
\omega^I_{a\,J} e^J_b e_{cI} - \omega^I_{b\,J} e^{Je_{cI}}_b &= -2e_{cI}\partial_{[a} e^I_{b]} && \text{(B.4)}\\
\omega^I_{c\,J} e^J_a e_{bI} - \omega^I_{a\,J} e^{Je_{bI}}_c &= -2e_{bI}\partial_{[c} e^I_{a]} && \text{(B.5)}\\
\omega^I_{b\,J} e^J_c e_{aI} - \omega^I_{c\,J} e^{Je_{aI}}_b &= -2e_{aI}\partial_{[b} e^I_{c]} && \text{(B.6)}
\end{aligned}$$

Then by summing (B.5) and (B.6), and subtracting (B.4),

$$\omega^I_{c\,J} e^J_a e_{bI} = e_{cI}\partial_{[a} e^I_{b]} - e_{bI}\partial_{[c} e^I_{a]} - e_{aI}\partial_{[b} e^I_{c]} \tag{B.7}$$

multiplying by $e^a_K e^{bL}$,

$$\omega^L_{c\,K} = e^a_K e^{bL} e_{cI}\partial_{[a} e^I_b - e^a_K \partial_{[c} e^L_{a]} - e^{bL}\partial_{[b} e_{c]K} \tag{B.8}$$

In the 3−dimensional case, by using $\omega_c^M = -\frac{1}{2}\varepsilon_L{}^{KM}\omega_c^L{}_K$ and $\tilde{\eta}^{abc}\varepsilon_{IJK}e_c^K = 2ee_I^{[a}e_J^{b]}$ where $\tilde{\eta}^{abc}$ is the 3−dimensional tensor density with weight one.

$$\omega_c^M = -\frac{1}{2}\left(\varepsilon_L{}^{KM}e_K^a e^{bL}e_{cI}\partial_{[a}e_{b]}^I - \varepsilon_L{}^{KM}e_K^a\partial_{[c}e_{a]}^L - \varepsilon_L{}^{KM}e^{bL}\partial_{[b}e_{c]K}\right) \tag{B.9}$$

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